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O.C.S.E.I.P. SYLLABUS

Grade 7

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U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
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PREFACE

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

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I. The decimal numeration system

A. Roman numerals

For interest, the student should be introduced to the historical development of our numeration system via the Egyptian, Babylonian and Roman antecedents. Our present system comes directly from the Hindu-Arabic system and is called decimal, referring to a base (group) ten numeration system.

Teaching Hint: There are many references on history of number systems available in libraries.

Students may acquire insight into our system by learning to perform routine computations in some of the earlier systems, but this should not be overdone. The average student can learn to read Roman numerals with relative ease, and this facility will heighten an appreciation for the increased utility found in our present system.

Stress that the Roman system used position as an additive and subtractive concept. ie. IV ≠ VI. Teach by using this concept in writing numerals in sequence to 100.

Teaching Hint: XXXX is 40 by XL is simpler. Look for other simplifications: IIII or IV, LXXXX or XC.

Teaching Hint: Stimulate interest by having students write numbers in different numeral systems.

B. Expanded numerals

The importance of place value cannot be overemphasized. This is the source of many difficulties in understanding number systems. Students must be able to demonstrate proficiency and acquire insight into this factor.

	Teaching Hint: Use expanded numerals (polynomial form). Example: 923 = (9 x 100) + (2 x 10) + (3 x 1). Extensive practice of number expansion is suggested.
C. Place value	Teaching Hint: Use a chart to demonstrate place values, digits, and periods.
D. Reading and writing numerals	Each group of 3 places (or digits) is called a period.
1. Grouping into periods	Teaching Hint: Emphasize there are only three names, which recur in each period, but an infinite number of periods.
2. Reading place value	Read the digits in each period and add the period name. For initial learning include the reading of the word "units" as a period name. In England, different periods are used. Teaching Hint: Emphasize oral reading practice.
3. Using "and"	In reading numerals, the word "and" is used to separate whole numbers from fractions, in both types of number, i.e., common and decimal fractions. Teaching Hint: Illustrate the use of "and" as in writing checks.
E. Rounding numerals	"Rounding up," stems from monetary use. Teaching Hint: From a mathematical standpoint, rounding to the even number is correct for later learning, but may be a particularly complex idea for the student to grasp at this level, where two different systems of rounding are in general practice, the teacher should utilize his or her own judgment as to the most useful path to be followed. Reference: Keedy, *Exploring Modern Math*, BK 2, p. 153.

F. Exponential notation

1. Purpose

The purpose of the exponent is to avoid the writing of a factor many times.

Example: $10 \times 10 \times 10 = 10^3$

Teaching Hint: At this level, first define "factors." The exponent tells how many times the base is used as a factor. Develop pattern from known (10^2) to unknown, i.e. right to left, then complete unknown on right (10^1 and 10^0).

10,000	1000	100	10	1
10^4	10^3	10^2	10^1	10^0

Teaching Hint: Children confuse the number of "times the base is used as a factor" with "times the base."

Example: $10^2 = 10 \times 10$, and not 2×10

Also: $2^3 \neq 3^2$

2. Exponents 1 and 0

For faster groups, the teacher may explain exponents 1 and 0 by showing how exponents are subtracted in division. When the exponent "one" is used, it means the number to the first power, which is the same as the base numeral. In general the exponent "one" is "understood," when no other exponent is indicated.

Example: $10^3 \div 10^2 = 10^1$

$1000 \div 100 = 10$ (base numeral)

Any number (except zero) with the exponent "zero" is "one."

Example: $10^1 \div 10^1 = 10^0 = 1$

$10 \div 10 = 1$

But $0^0 \neq 1$, since $\frac{0}{0}$ is not defined.

G. Base numerals

Stress the point that bases are for a better understanding of the decimal system, and are in current use in business and science.
Reference: Keedy, Exploring Modern Math, Bk. 1, pp. 30-31.

1. Writing a base numeral

A system of base numerals will utilize the number of digits in the base including 0.

Teaching Hint: In base 10 there is no specific single numeral for 10. If this concept is realized the children will understand the idea of grouping which is necessary for all number systems.

Example:

Base	Groups of base	
10	10 = 1	10 elements - 1 group
5	5 = 1	5 elements = 1 group
2	2 = 1	2 elements = 1 group

Just as we carry 10 as a group to the next column as 1 so we also carry 5 as a group to the next column as 1 in base five or 2 as a group to the next column as 1 in base two, etc.

Much time has been unprofitably used in attempting to make a huge distinction between the use of a word versus a numeral in denoting the base used (e.g., 124_5 versus 124_{five}, with the purist insisting that the first is improper since, after all, there is no "5" in "base five." The teacher may wish to point these two diverging views out to the student, but it needs to be remembered that the study of bases is important only as an aid to understanding base ten, and minor distinctions of this type can thus be overdone.

2. Expanding a base numeral

The easiest way to expand a base numeral is to use a chart similar to the following base value chart:

b^4	b^3	b^2	b^1	b^0
5^4	5^3	5^2	5^1	5^0
625	125	25	5	1
		3	2	4

Example: $324_5 = (3 \times 5^2) + (2 \times 5^1) + (4 \times 5^0)$

$324_5 = (3 \times 25) + (2 \times 5) + (4 \times 1)$

3. Exponential notation

Teaching Hint: Use above chart, and teach base "b" for understanding that all bases follow the same grouping pattern.

4. Changing base numeral to a decimal numeral

Use the base chart and simplify the above example:

$$324_5 = (3 \times 25) + (2 \times 5) + (4 \times 1)$$
$$75 + 10 + 4 = 89_{10}$$
$$324_5 = 89_{10}$$

5. Changing a decimal numeral to a base numeral

Using short division (in the above example), continue dividing 5 into 89 until further division is impossible. The remainders are read for the base numeral.

$$\begin{array}{r|l} 5 & 89 - \\ 5 & 17 - 4 \\ & 3 - 2 \\ & 0 - 3 \end{array} \qquad 324_5$$

Note: Dividing by a sequence of 5's give multiples or powers of 5. Using long division, show the remainder as below:

a. $\begin{array}{r} 17 \\ 5\,\overline{)89} \\ \underline{5} \\ 39 \\ \underline{35} \\ 4 \end{array}$ b. $\begin{array}{r} 3 \\ 5\,\overline{)17} \\ \underline{15} \\ 2 \end{array}$ c. $\begin{array}{r} 0 \\ 5\,\overline{)3} \\ \underline{0} \\ 3 \end{array}$

base numeral - 324_5

Other bases may be obtained by dividing by the base as above.

6. Operations in bases by using addition and multiplication charts

Construction and use of base numeral charts for addition and multiplication strengthens the student's concept of place value. Students should have practice in construction and use of these charts, but only for this understanding.
Reference: Keedy, Exploring Modern Math, Bk. 1 pp 15-31 for a particularly excellent explanation of bases.

H. Sample test

1. Write base five numerals for the decimal number 103. $\underline{403_5}$
2. How many basic symbols are used in base b numeration? $\underline{b}$
3. What is the place value of the digit symbol 3 in 231458_{10}? $\underline{10{,}000 \text{ or } 10^4}$
4. What is the place value of the digit symbol 3 in 2310_5? $\underline{5^2 \text{ or } 25}$
5. What base is our money system? $\underline{\text{10 or decimal}}$
6. In what base, other than decimal, would you most easily speak of a large number of eggs? $\underline{12}$

7.

$$\begin{array}{r} 132_5 \\ +\ 231_5 \\ \hline 413_5 \end{array}$$

8.

$$\begin{array}{r} 41_5 \\ -32_5 \\ \hline 4_5 \end{array}$$

9.

$$\begin{array}{r} 21_5 \\ \times\ 4_5 \\ \hline 134_5 \end{array}$$

10. Fill in this table:

base 10	base 2	base 3	base 4	base 5
1	1	1	1	1
2	10	2	2	2
3	11	10	3	3
4	100	11	10	4
5	101	12	11	10

II. Properties of addition and subtraction

A. The inverse operation

Inverse operation means "undoing what was done." The inverse operation chart for addition and subtraction should be presented to the class.

Teaching Hint: All operations are binary, i.e., an operation is performed upon two numbers which results in a third number, the answer. Students should be required to demonstrate the relationships and various patterns which exist between these three numbers on an inverse operation chart.

1. Inverse operation chart for addition and subtraction

The student should make his own chart, which can be found in any good text.

Teaching Hint: The teacher should be able to demonstrate the inverse relationships as follows:

$$5 + 4 = 9 \quad \text{and} \quad 9 - 4 = 5$$

$$4 + 5 = 9 \quad \text{and} \quad 9 - 5 = 4$$

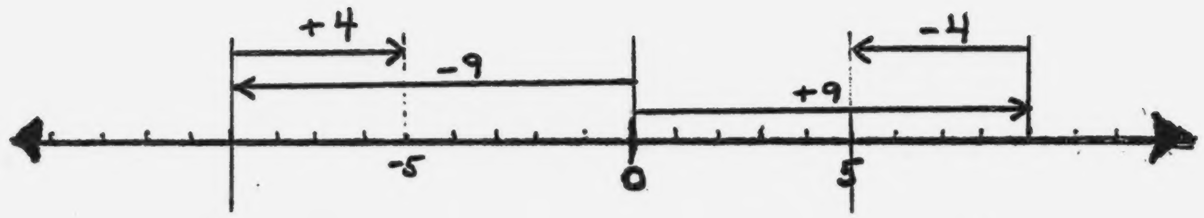
The teacher may wish to explain the derivation of the above relationship as follows:

$$5 + (4 - 4) = 9 - 4 \qquad 4 + (5 - 5) = 9 - 5$$

$$5 \qquad = 9 - 4 \qquad 4 \qquad = 9 - 5$$

2. The number line

The number line may be used to diagram the inverse operation for addition and subtraction. It may be displayed permanentlyover the chalkboard, on a chart or with beads on wire. It may be displayed vertically, as with the thermometer, etc.



$(-9) - (-4) = (-5) \qquad (+9) - (+4) = (+5)$

$-9 + 4 = -5 \qquad 9 - 4 = 5$

Teaching Hint: Some students will realize the need to extend the number line to include the negative numbers (the set of integers) at this point. This may be done for information only, at the seventh level. Some students will discover the algebraic rules for signed numbers.

$(\underline{+}) + (\underline{+}) = (\underline{+})$

$(-) \underline{+} (-) = (-)$

$(\underline{+}) \underline{-} (\underline{+}) = (\underline{+})\ (-)$ subtract; give sign of larger number

$(-) - (-) = (-)\ (=)$

Refer to eighth level text for integers on a number line.

Teaching Hint:

Introduce all problems with diagrams (arrows) on the number line.

Work many problems both with algorism (vertically) and equation (horizontally).

Discover patterns.

Teaching Hint: Remember integers are called positive and negative numbers. - (-8) This is read "minus a negative 8."

Teaching Hint: Addition and subtraction on the number line of positive numbers is to the right and left respectively. Addition and subtraction of negative numbers is to the left and right respectively. (opposite or inverse)

B. Oral vocabulary

Students should be able to spell and identify the following vocabulary: addition, addends, sum, subtraction, minuend, subtrahend, difference.

C. Properties of addition and subtraction

Emphasize that the underlying concepts of these properties are familiar to the students and that they are being identified by name now to avoid repetition of explanation: commutative property, associative property and identity for addition.

Teaching Hint: Students should begin to accept the use of letters (variables) in place of numerals in statements of equations.

III. Properties of multiplication and division

A. Meaning of inverse operation of multiplication and division shown diagramatically

The teacher can use the following diagram showing that multiplication and division are inverse operations:

$4 \times 5 = 20$ $\qquad$ $20 \div 5 = 4$

Dividend
Product

Divisor
Factor

Quotient
Factor

Teaching Hint: Students must understand set terminology; the symbols for operations, particularly division (÷ , , -); the relationships between operations, such as division and subtraction, and multiplication and addition.

The students should reproduce the Inverse-operation chart for multiplication and division, similar to the one for addition and subtraction, which may be found in any text.

Teaching Hint: The overhead projector can be used for discovery of patterns. Specific practice should be given in oral reading of division problems.

B. Oral vocabulary

Children must understand the language of mathematics, therefore proper spelling and usage of mathematical terms should be expected. Spelling of terms may be included as test items. Students should be able to spell and identify the following expressions: multiplication, multiplicand, multiplier, factors, product, division, divisor, dividend quotient.

Teaching Hint: Students should be able to begin to verbalize using proper mathematical expressions.

C. Properties of multiplication

The following basic concepts should be identified by name for ready reference: closure, commutative, associative, distributive property of multiplication over addition.

Teaching Hint: Distribution over addition is sometimes confusing but simply means spreading out the multiplication across an addition sign.

Example: $3 \times 26 = n$

$3 \times (20 + 6) = n$

$(3 \times 20) + (3 \times 6) = n$

$60 + 18 = n$

$78 = n$

D. Multiplying decimals by 10, 100, 1000

The following plan for multiplying a decimal numeral is recommended for teacher explanation:

Example: Expand the numeral

$147 \times 10 = n$

$(100 + 40 + 7) \times 10 = n$

Using the distributive property multiply by 10 (100;1000) $(100 \times 10) + (40 \times 10) + (7 \times 10) = n$

Perform indicated operations: collect terms

$(1000) + (400) + (70) = n$

$1470 = n$

If students understand place value the above demonstrations may be held to a minimum. Students should realize that in multiplying by ten, for example, each place has increased its value ten times.

E. Forms to show division

1. The scaffold, for the more able student

Teaching Hint: Capable students should be taught to round the divisor and estimate quotients.

The scaffold method.

Example:

```
48 )1313 |
     960 | 20 x 48
     353 |
     336 |  7 x 48
      17 | 27
```

After a little use of this method the student will drop the right hand numerals and place the quotient at the top.

Teaching Hint: For the more able student the teacher may explain why we start in the high digits place in division as opposed to the other operations. To be useful to us, remainders must be changed to smaller digit values. In the above example: one thousand cannot be divided in 48 groups, neither then can thirteen hundreds. One hundred thirty-one tens, however, can be divided into 48 groups, with thirty-five tens left over. Changing the remainder into ones, we have 353 ones which can be divided into 48 groups with 17 one left. The convenience of starting in the high digit place should be apparent.

2. A short method for one and two number division

A short division may be used when concepts are understood.

Example:

```
   33 R1
8 )265
```

Think: How many 8's in 26 - 3?

(3 x 8 = 24; 26 - 24 = 2)

Place the 2 remainder next to the 5 and continue to divide, repeating the above calculation Write quotient and remainder as shown.

3. For the less able student

The less capable student should work in terms he can understand, such as 10's, 100's and very small numbers. He must understand how to compare "larger than" and "less than."

Example:

$$\begin{array}{r|l}
4\,\overline{)513} & \\
\underline{400} & 100 \times 4 < 513 \\
113 & \\
\underline{40} & 10 \times 4 < 113 \\
73 & \\
\underline{40} & 10 \times 4 < 73 \\
33 & \\
\underline{20} & 5 \times 4 < 33 \\
13 & \\
\underline{12} & 3 \times 4 < 13 \\
1 & - 128 \qquad \text{or } 128 \text{ R } 1
\end{array}$$

Think: "What multiple of 4 that you know is less than 513? (100) What multiple of 4 of each succeeding difference is less than the difference, etc. --" Add the mulitples (arranged vertically) to obtain the quotient; the remainder completes the quotient.

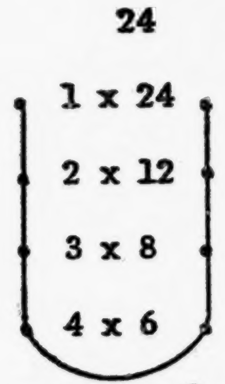
Teaching Hint: The average 7th grade student is generally inefficient in mental calculations. To improve this situation short daily mental exercises using addition, subtraction, multiplication and division are recommended.

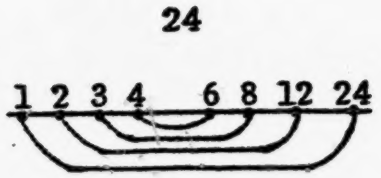
IV. Factoring and Primes

A. Factoring

Natural numbers can be expressed as the product of two or more factors. One is understood to be a factor of all numbers so it is not necessary to use it more than once.

Teaching Hint: Two ways to show factorization patterns are the factor "horseshoe" and factor "line".





When pattern repeats, factorization is complete.

B. Odd and even numbers

Students should be given a simple definition of even and odd numbers to clarify the concept: A number is even if and only if it is a multiple of 2. If a number is not even, it is odd. (Zero is considered even)

C. Prime numbers

Prime numbers are numbers with only two factors. The student should construct a sieve of Eratosthenes, as it is a convenient way to display prime numbers. Numbers that are not prime are called composite.

Teaching Hint: The set of natural numbers consists of three sets: 1, prime, and composite.

D. Smallest factors

Smallest factors are the result of complete factorization until prime numbers only are reached. Students should be told to write final factors in their natural order for convenience. After the concept of complete factorization is understood, exponential notation may be used.

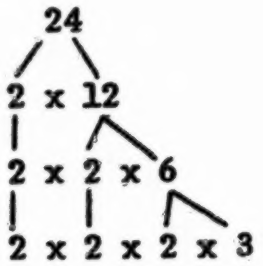
Teaching Hint: Use prime numbers as divisors, starting with 2.

$72 = 2^3 \times 3^2$

$$\begin{array}{r|l} 2 & \underline{72} \\ 2 & \underline{36} \\ 2 & \underline{18} \\ 3 & \underline{9} \\ 3 & \underline{3} \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & \underline{120} \\ 2 & \underline{60} \\ 2 & \underline{30} \\ 3 & \underline{15} \\ 5 & \underline{5} \\ & 1 \end{array}$$

$120 = 2^3 \times 3 \times 5$

A "factor tree" for 24. If "limb" can grow, it is not prime.



E. Greatest common factor (G.C.F.) and methods of finding G.C.F.

Teacher should explain a few practical uses for G.C.F. (architecture, building, etc.). Then students may follow the teacher's explanation for finding G.C.F. in the following systematic steps:

1. Using power notation

Using power notation, write the prime factors of each number involved.

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$$
$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$

Select the smallest power of each common prime factor: 2^2 and 3.

The product of the smallest powers of the common prime factors is the G.C.F.

$$2^2 \times 3 = 2 \times 2 \times 3 = 4 \times 3 = 12$$

Teaching Hint: Students may be confused by the words "greatest" and "least" G.C.F. and L.C.M. The teacher may wish to omit these words and use the concepts as "common factor" (small number) and "common multiple" (larger number) in the beginning.

2. Listing all factors of numbers

An alternate method for finding the G.C.F. does not use exponential notation but lists all factors of each number.

Example:

Factors of 40: 2, 4, 5, ⑧ 10, 20, 40

Factors of 24: 2, 3, 4, 6, ⑧ 12, 24

Factors of 32: 2, 4, ⑧ 16, 32

Upon examination, 8 is seen to be the G.C.F. common to all 3 numbers: 40, 24, 32.

Teaching Hint: For advanced students, see: Keedy, Exploring Modern Mathematics, Book I, p. 166, for additional method.

F. Least common multiple (L.C.M.)

For class presentation the teacher may suggest a plan for finding L.C.M. by two alternate methods.

Teaching Hint: L.C.M. is least common denominator in fractions (rational numbers).

1. Prime Factorization.

Example: Find L.C.M. for 12 and 15.

Write prime factors for each number.

12 = 2 x 2 x 3 15 = 3 x 5

Select each prime factor the most number of times it appears in any one of the factorizations.

(2 x 2) x (3) x (5)

The product of the selected prime factors is the L.C.M.

2 x 2 x 3 x 5 = 4 x 3 x 5 = 60

2. Sets of multiples methods

An alternate method is to find the multiples of each number and select the least common multiple:

Example: Find L.C.M. for 4 and 6

4, 8, ⑫, 16, 20 -----
6, ⑫, 18, 24, 30 -----

Teaching Hint: For advanced students, see: Keedy, *Exploring Modern Mathematics*, Book I, p. 180, for additional method.

G. Relatively prime numbers

Relatively prime numbers are those having no common factor, or their G.C.F. is one (only common divisor is one).

Example: 15 and 32

Factorizations are: (1) x 3 x 5 and (1) x 2^5
They have no common prime. G.C.F. is (1).

Teaching Hint: Divisibility of numbers should be emphasized. Students should know at least the tests for the small primes (2,3,5), but should also be familiar with the others. (See 8th grade syllabus for these tests in detail.)

V. Fractions (arithmetic numbers) (rational numbers)	Understanding the basic concepts of manipulation of fractions is one of the major parts of 7th grade mathematics and should be stressed. Fractions are used for many practical purposes (1 3/4 c. sugar in a recipe; 7 1/2 yds. of cloth; 4 3/4 miles of freeway, comparison or ratio as 3 to 4 is 3/4, etc.)
A. Why fractions:	
B. What is a fractional numeral?	A fractional numeral is the indicated quotient of a whole number divided by a counting or natural number. Example: 1/3, 4/2, 1/1, 2/5, 0/2, etc. The whole numbers are a part (subset) of the numbers of arithmetic (non-negative rational numbers). e.g. 4/2 = 8/4 = 2, but they are still fractional numerals. Teaching Hint: Emphasize that the "fraction bar" means divide (ratio). Teaching Hint: $\frac{\text{Numero - number}}{\text{Denote - to name}} = \frac{\text{number of parts}}{\text{name of parts}}$ Latin: $\frac{\text{numerare}}{\text{denominare}}$
C. Oral vocabulary review for fractional numerals	The student should be able to spell and identify the following: numerator, denominator, fractions, terms.
D. Zero in division	The use of zero in the denominator is not defined: If 3/0 = 0, then 3 = 0 x 0 0 x 0 ≠ 3, therefore 3/0 is not defined.
E. Renaming fractional numerals	Two methods for finding different names for fractional numbers: (1) using the identity element: multiply or divide the fraction by one in any fractional form $\frac{3}{4} \times \frac{4}{4} = \frac{12}{16}$ $\frac{30}{35} \div \frac{5}{5} = \frac{6}{7}$ (2) factoring the numerator and denominator and finding a common factor, one, in

fractional form. $\frac{8}{24} = \frac{2 \times 2 \times 2 \times 1}{2 \times 2 \times 2 \times 3} = 1 \times 1 \times 1 \times \frac{1}{3} = \frac{1}{3}$

Teaching Hint: Reinforce the student's concept of reduction of fractions by dividing numerator and denominator by the GCF.

F. Multiplication of fractions

Students may be shown the following method of simplification of multiplication of fractions by factoring so they will understand the process:

$$n = \frac{9}{12} \times \frac{8}{21} = \frac{9 \times 8}{12 \times 21} \times \frac{(3x3) \times (2x2x2)}{(2x2x3) \times (3x7)} = \frac{(2x2x3x3)}{(2x2x3x3)} \times \frac{2}{7}$$

$$n = 1 \times \frac{2}{7} = \frac{2}{7}$$

After understanding the method students will be ready for multiplying fractions in this way:

renaming the numerals using commutative property

$$\frac{\overset{\cancel{3}}{\cancel{9}}}{\underset{\cancel{4}}{\cancel{12}}} \times \frac{\overset{2}{\cancel{8}}}{\underset{7}{\cancel{21}}} = \frac{2}{7} \text{ or } \frac{9 \times 8}{12 \times 21} = \frac{8 \times 9}{12 \times 21} = \frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{12}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{7}{\cancel{21}}} = \frac{6}{21} = \frac{2}{7}$$

(reduce)

G. Addition of fractions

1. If denominators are alike

In the addition of fractions having like denominators, add the numerators and keep the same denominator

$$\frac{2}{5} + \frac{4}{5} + \frac{1}{5} = \frac{2 + 4 + 1}{5} = \frac{7}{5}$$

2. If denominators are not alike

If the denominators are not the same, find the L.C.M. for the denominators. Example: $\frac{5}{12} + \frac{7}{18}$; L.C.M. = 36

Change fractions to their equivalents using the L.C.M., 36, as a denominator.

Teaching Hint: Current mathematical practices emphasize understanding. Students using "shortcuts" get confused because they do not understand the mathematical development.

$$\frac{5}{12} + \frac{7}{18} = \left(\frac{5}{12} \times \frac{3}{3}\right) + \left(\frac{7}{18} \times \frac{2}{2}\right) =$$

$$\frac{15}{36} \quad + \quad \frac{14}{36} \quad = \quad \frac{29}{36}$$

Another form for the same addition is the traditional vertical method. However, encourage the use of the horizontal pattern, as it is preferred in later mathematics.

H. Addition of fractional numbers

1. A method

The addition of fractional numbers may be taught as follows:

Example: $15\frac{1}{2} + 18\frac{3}{4} + 22\frac{1}{3} = n$

Rewrite: $\left(15 + \frac{1}{2}\right) + \left(18 + \frac{3}{4}\right) + \left(22 + \frac{1}{3}\right) = n$

Addition: Add whole numbers. Find L.C.M. for denominators, change to equivalent fractions and add

$$55 + \left(\frac{6}{12} + \frac{9}{12} + \frac{4}{12}\right) = n$$

$$55 + \left(\frac{6 + 9 + 4}{12}\right) = n$$

$$55 + \frac{19}{12} = n$$

2. Simplify the answer

It is more convenient to give the answer as the sum of a whole number and a fraction less than 1.

$55 + \frac{19}{12} = n$

$55 + \left(\frac{12 + 7}{12}\right) = n$ by regrouping

$55 + \left(\frac{12}{12} + \frac{7}{12}\right) = n$ form units (ones)

$55 + \left(1 + \frac{7}{12}\right) = n$ rewrite as whole numbers

$(55 + 1) + \frac{7}{12} = n$ use associative property

$56 + \frac{7}{12} = n$ combine

$56 \frac{7}{12} = n$ simplest form

I. Comparison of fractional numbers

In comparison of fractional numbers the inequality symbols, $<$ and $>$, are used to denote the smaller or larger of two fractions.

1. Method of comparison

Change to equivalent fractions and compare numerators.

Example: Compare $\frac{3}{7}$ with $\frac{1}{3}$:

$\frac{3}{7} = \frac{9}{21}$ and $\frac{1}{3} = \frac{7}{21}$

comparing numerators, $7 < 9$ therefore, $\frac{1}{3} < \frac{3}{7}$.

J. Subtraction of fractions

1. Inverse of addition

Subtraction is the inverse of addition. We are given a sum and one addend, and asked to find the other addend.

Example: $\frac{19}{20} = \frac{3}{4} + n$

$$\frac{19}{20} - \frac{3}{4} = \left(\frac{3}{4} - \frac{3}{4}\right) + n$$

therefore: $\frac{19}{20} - \frac{3}{4} = n$

2. Subtraction of fractional numbers

As students know that subtraction is the opposite of addition, they can understand that if the denominators of two fractions are the same they may be subtracted by subtracting the numerators. If the denominators of two fractions are not the same, rename the fractions as in addition and subtract the numerators. Keep the common denominator. The difference should normally be expressed in lowest terms.

3. Subtraction involving whole numbers with fractions

Three patterns for number combinations found in subtraction are as follows:

Example: $n = 5\frac{1}{2} - 3\frac{3}{7}$ (problem)

$n = \left(5 + \frac{1}{2}\right) - \left(3 + \frac{3}{7}\right)$ (rewrite)

$n = (5 - 3) + \left(\frac{1}{2} - \frac{3}{7}\right)$ (associative)

$n = 2 + \left(\frac{7}{14} - \frac{6}{14}\right)$ (subtract whole numbers; find L.C.M. for fractional denominators; write the equivalent fractions and subtract)

$n = 2\frac{1}{14}$

Example: $n = 7\frac{1}{6} - 4\frac{3}{4}$

$n = \left(7 + \frac{1}{6}\right) - \left(4 + \frac{3}{4}\right)$ (rewrite)

$n = \left(6 + 1 + \frac{1}{6}\right) - \left(4 + \frac{3}{4}\right)$ (renaming "7")

$n = \left(6 + \frac{6}{6} + \frac{1}{6}\right) - \left(4 + \frac{3}{4}\right)$ (renaming one)

$n = \left(6 + \frac{7}{6}\right) - \left(4 + \frac{3}{4}\right)$ (perform indicated fractional addition)

$n = \left(6 - 4\right) + \left(\frac{7}{6} - \frac{3}{4}\right)$ (associative)

$n = 2 + \left(\frac{14}{12} - \frac{9}{12}\right)$ (subtract whole numbers; find the L.C.M. and subtract fractions)

$n = 2 + \frac{5}{12} = 2\frac{5}{12}$

Example: $n = 15 - 4\frac{7}{9}$

$n = \left(14 + \frac{9}{9}\right) - \left(4 + \frac{7}{9}\right)$

$n = (14 - 4) + \left(\frac{9}{9} - \frac{7}{9}\right)$

$n = 10 + \frac{2}{9}$

$n = 10\frac{2}{9}$

Teaching Hint: Teacher may wish to use vertical form.

Example:

$$\begin{array}{rcl} 5\frac{1}{2} & = & 5\frac{7}{14} \\ -\ 3\frac{3}{7} & = & 3\frac{6}{14} \\ \hline & = & 2\frac{1}{14} \end{array}$$

Example:

$$\begin{array}{rcccl} 7\frac{1}{6} & = & 6\frac{7}{6} & = & 6\frac{14}{12} \\ -\ 4\frac{3}{4} & = & 4\frac{3}{4} & = & 4\frac{9}{12} \\ \hline & & & = & 2\frac{5}{12} \end{array}$$

Example:

$$\begin{array}{rcl} 15 & = & 14\frac{9}{9} \\ -\ 4\frac{7}{9} & = & 4\frac{7}{9} \\ \hline & = & 10\frac{2}{9} \end{array}$$

K. Multiplication of fractions

Teaching Hint: Teacher should require correct form i.e. $5\frac{1}{2} = 4\frac{3}{2}$ rather than $5\frac{1}{2} = \frac{3}{2}$

Example: $n = 2\frac{1}{2} \times 3\frac{1}{4}$

$n = \frac{5}{2} \times \frac{13}{4}$

$n = \frac{65}{8}$

$n = 8\frac{1}{8}$

Students may be shown the following distributive property method for multiplication

Example: $2\frac{1}{2} \times 3\frac{1}{4} = n$

$$\left(2 + \frac{1}{2}\right) \times 3\frac{1}{4} = n$$

$$\left(2 \times 3\frac{1}{4}\right) + \left(\frac{1}{2} \times 3\frac{1}{4}\right) = n$$

$$\left[2 \times \left(3 + \frac{1}{4}\right)\right] + \left[\frac{1}{2} \times \left(3 + \frac{1}{4}\right)\right] = n$$

$$\left[(2 \times 3) + \left(2 \times \frac{1}{4}\right)\right] + \left[\left(\frac{1}{2} \times 3\right) + \left(\frac{1}{2} \times \frac{1}{4}\right)\right] = n$$

$$\left(6 + \frac{2}{4}\right) + \left(\frac{3}{2} + \frac{1}{8}\right) = n$$

$$6 + \frac{4 + 12 + 1}{8} = n$$

$$6 + \frac{17}{8} = n$$

$$6 + \frac{16 + 1}{8} = n$$

$$(6 + 2) + \frac{1}{8} = n \qquad 8\frac{1}{8} = n$$

L. Reciprocals

If the product of two fractional numbers is 1, the fractions are called reciprocals of each other.

Examples: $\frac{5}{9} \times \frac{9}{5} = 1$; $4 \times \frac{1}{4} = 1$ $\qquad n \times \frac{1}{n} = 1 \ (n \neq 0)$

The reciprocal of one is one.

All fractional numbers except zero have reciprocals.

Teaching Hint: Write a whole number as a fraction with 1 as a denominator, and invert to form the reciprocal.

Example: $4 = \frac{4}{1}$, so its reciprocal is $\frac{1}{4}$

$$\frac{4}{1} \times \frac{1}{4} = 1$$

Teaching Hint: Write a whole number and a fraction in fractional form, then write its reciprocal:

$$7\frac{1}{2} = \frac{15}{2} \qquad \frac{15}{2} \times \frac{2}{15} = 1$$

M. Division of fractional numbers

Multiplication and division are inverse operations.

Since: $n \times 3 = 15$

$n = 15 \div 3$

It follows:

$$n \times \frac{3}{4} = \frac{5}{8}$$

$$n = \frac{5}{8} \div \frac{3}{4}$$

$$n \times \left(\frac{3}{4} \times \frac{4}{3}\right) = \frac{5}{8} \times \frac{4}{3}$$

$$n \times 1 = \frac{5}{8} \times \frac{4}{3}$$

$$n = \frac{5}{6}$$

Or:

$$n = \frac{\frac{5}{8}}{\frac{3}{4}}$$

$$n = \frac{\frac{5}{8} \times \left(\frac{4}{3}\right)}{\frac{3}{4} \times \left(\frac{4}{3}\right)}$$

$$n = \frac{5}{8} \times \frac{4}{3} \qquad n = \frac{5}{6}$$

Also: $n = 7\frac{1}{2} \div 1\frac{7}{8}$

$$n \quad \left[\left(7 \times \frac{2}{2}\right) + \frac{1}{2}\right] \div \left[\left(1 \times \frac{8}{8}\right) + \frac{7}{8}\right]$$

$$n = \frac{15}{2} \div \frac{15}{8}$$

$$n = \frac{15}{2} \times \frac{8}{15}$$

$$n = 4$$

N. Simplifying equations

Teaching Hint: Since the term "cancelling" has been misused at this level, the following is suggested.

In the above example:

$$n = \frac{15}{2} \times \frac{8}{15}$$

$$n = \frac{8}{2} \times \frac{15}{15} \qquad \text{commutative property}$$

$$n = \left(\frac{8}{2} \quad \frac{2}{2}\right) \times \left(\frac{15}{15} \div \frac{15}{15}\right) \qquad \text{divide by identity}$$

$$n = \frac{4}{1} \times \frac{1}{1} \qquad \text{reduced}$$

$$n = 4$$

O. Reducing to simplest terms

Teaching Hint: Pupils should be taught that it is not always necessary to reduce fractions to lowest terms. However, for ease in comparison and understanding it is best that the answer be reduced.

VI. Decimal numerals

Teaching Hint: Children must understand place value and their relationships, names for numbers, and renaming numbers prior to undertaking the work with decimal numerals. Decimal numerals, or "decimals" are names for fractional numbers whose denominators are powers of ten ("decem," Latin, means "ten). i.e. $\frac{3}{10}$, $\frac{6}{100}$, $\frac{200}{10}$, etc.

A. Addition and subtraction with decimal numerals

Computation is carried out as with whole numbers, but is explained by using the fractional form of the numbers. Teaching Hint: In addition and subtraction of decimals by the column method all decimal points must be in the same vertical column.

1. Addition

Decimal

$$\begin{array}{r} 2.3 \\ +\ 2.8 \\ \hline 5.1 \end{array}$$

Fractional

$$\begin{array}{r} 2\,\frac{3}{10} \\ +\ 2\,\frac{8}{10} \\ \hline 4\,\frac{11}{10} \end{array} = 4 + \left(\frac{10}{10} + \frac{1}{10}\right) = (4 + 1) + \frac{1}{10} =$$

$$5 + \frac{1}{10} = 5\,\frac{1}{10}$$

2. Subtraction

Decimal

$$\begin{array}{r} 5.26 \\ -\ 4.78 \\ \hline .48 \end{array}$$

Fractional

$$\begin{array}{rcl} 5\,\frac{26}{100} = 4 + \left(\frac{100}{100} + \frac{26}{100}\right) & = & 4 + \frac{126}{100} \\ -\ 4\,\frac{78}{100} = 4 + \frac{78}{100} & = & 4 + \frac{78}{100} \\ \hline & & \frac{78}{100} \end{array}$$

3. If minuend or subtrahend is a whole number

If either the minuend or subtrahend is a whole number, rename the whole number as a decimal numeral. Align the decimal points and subtract as with whole numbers.

Examples:

$$\begin{array}{r} 8.0000 \\ -\ \underline{7.2651} \\ .7349 \end{array} \qquad \begin{array}{r} 254.604 \\ -\ \underline{38.000} \\ 216.604 \end{array}$$

Teaching Hint: Encourage use of zeros as place holders.

B. Multiplication with decimal numerals

Multiplication also is carried out as with whole numbers but is explained by using fractional numbers.

Examples: $2 \times .3 = .6$ $\qquad 2 \times \frac{3}{10} = \frac{6}{10}$

$.3 \times .15 = .045$ $\qquad \frac{3}{10} \times \frac{15}{100} = \frac{45}{1000}$

1. Number of decimal places in the product

Teaching Hint: Students should be able to discover a pattern for placing the decimal point.

Factors	Number of decimal places in product
$\frac{1}{10} \times 3 = .1 \times 3 = .3$	1
$\frac{1}{10} \times \frac{7}{10} = .1 \times .7 = .07$	2
$\frac{1}{100} \times 4 = .01 \times 4 = .04$	2
$\frac{1}{100} \times \frac{7}{10} = .001 \times 9 = .009$	3
$\frac{1}{1000} \times 9 = .001 \times 9 = .009$	3

$\frac{1}{100} \times \frac{6}{100} = .01 \times .06 = .006$ 4

$\frac{1}{10000} \times 11 = .0001 \times 11 = .0011$ 4

From the above examples it may be concluded that the number of decimal places in the product is equal to the sum of the number of decimal places in the decimal factors.

2. Estimating the product

Teaching Hint: Students should be able to compute a reasonable estimate. To do this in the case of decimal numerals, students should round off to a convenient whole number for easy multiplication.

Example: $82.7 \times 5.4 = n$

$80 \times 5 = n$

$400 = n$

Obviously, as the decimal parts were not included in the multiplication, $n > 400$. It should be noted that the answer will be expressed in hundreds plus a decimal numeral.

D. Division of decimal

Decimal	Fractional
$\begin{array}{r} .06 \\ 8\overline{).48} \\ \underline{.48} \end{array}$	$\frac{48}{100} \div \frac{8}{1} = n$
	$\frac{48}{100} \times \frac{1}{8} = n$
	$\frac{48}{8} \times \frac{1}{100} = n$ (commutative)
	$\frac{48}{8} \div \left(\frac{8}{8}\right) \times \frac{1}{100} = n$ (reduce)

$$\frac{6}{1} \times \frac{1}{100} = n$$

$$\frac{6}{100} \text{ or } .06 = n$$

Teaching Hint: The student may perform the indicated division, and multiply the quotient by $\frac{1}{10}$, $\frac{1}{100}$, etc. according to the number of decimal places in the dividend. Write the quotient as a decimal.

2. Division with decimals

If the fractional form of division is clearly understood, change any decimal divisor to a whole number by multiplying the elements of the equation by $\frac{10}{10}$, $\frac{100}{100}$, $\frac{1000}{1000}$ (identity) and proceed with division as in preceding subsection (1).

Decimal	Fractional
$.45\overline{)5.715}$	$\frac{5.715}{.45} \times \left(\frac{100}{100}\right) = n$
Multiply divisor and dividend by 100 and divide.	$\frac{571.5}{45} = n$
$45\overline{)571.5}$ with quotient 12.7: 45 121 90 315 315	$12.7 = n$
	$\frac{5715}{1000} \div \frac{45}{100} = n$
	$\frac{5715}{1000} \times \frac{100}{45} = n$
	$\frac{5715}{45} \times \frac{100}{1000} = n$
	$\frac{5715}{45} \times \frac{1}{10} = n$

$$127 \times \frac{1}{10} = n$$

$$12.7 = n$$

Teaching Hint: Don't "move" decimal point. Multiply by the identity. After this concept is understood the student will be able to apply it to all possible division cases. Reinforce.

3. Terminating decimals

Any fraction whose denominator has prime factors of only 2's or 5's (or both 2's and 5's) may be written as a terminating decimal, or a decimal that repeats 0.

Example: $\frac{3}{25} = \frac{3}{5^2} = .12$

4. Non-terminating decimals

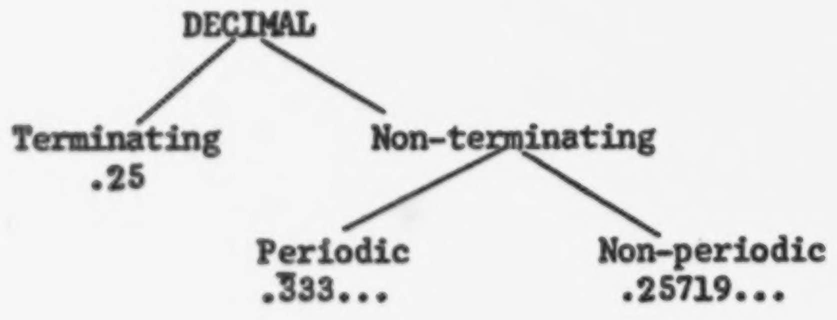
Any fraction whose denominator has prime factors other than 2, and/or 5, are non-terminating or periodic decimals.

Example: $\frac{8}{33} = .\overline{24}2424...$

5. Irrational numbers

Fractions or decimals which are non-terminating and also not repeating, or periodic, are called irrational numbers.
Example: $\pi = 3.14159...$

$\sqrt{2} = 1.4141...$



VII. Equations, inequalities, and open sentences.	Any sentence having the equal (=) sign as its verb is an equation. Example: (3+ 1) = 4 and 4 = (3 + 1). The number named on one side of the equation must be the same number as named on the other side if the equation is true.
A. Variables	Letters of the alphabet are often used as variables. Any symbol may be used except the name for a number. Emphasize that when a variable appears more than once in an equation we must agree to name the same number each time the variable appears.
B. Basic properties	The basic properties of equality between numbers are the following: a = a (reflexive property) If a = b, b = a (symmetric property) a = b, b = c, then a = c (transitive property) If the same number is added to or subtracted from both sides of an equation, the relationship between the members of an equation by the same number, and the relationship remains the same. (Division by zero is not defined).
C. Rules of order	1. For simplifying expressions 2. For simplifying properties of equations Division and multiplication take precedence over addition and subtraction. Operations shown in parentheses should be done **first.** Operations shown in brackets should be done after those in parentheses. Braces, if used, are done **last.**

The multiplication operation is also indicated in the following ways:

Teaching Hint: XY, $X(Y)$, $(X)(Y)$, $X \cdot Y$

Teaching Hint: In simplifying expressions, work from inner parentheses to outer bracket.

Example:

$H = 4 + \{5 \; [3(2+4)] - 8\}$	Clear parentheses
$H = 4 + \{5 \; [3 \cdot 6] - 8\}$	Clear brackets
$H = 4 + \{5 \cdot 18 - 8\}$	Multiply
$H = 4 + \{90 - 8\}$	Clear braces
$H = 4 + 82$	Add
$H = 86$	

Teaching Hint: Review the following symbols of relationship.

Equal	$=$	
Not equal	$\neq$	
Less than	$<$	(point is to lesser quantity)
Greater than	$>$	(point is to lesser quantity)
Greater than or equal to	$>$	
Not equal to nor greater than	$\not>$	

D. Closed and open sentences

Closed sentences are mathematical statements or equations which express a complete thought. They do not have to contain only numerals. The statement can be either true or false. This is closed because it is true for all replacements of a and b.

Example: $a \cdot b = b \cdot a$, $a \cdot b \neq b + a$.

Open sentences are mathematical statements or equations which are incomplete until the value of the missing numeral or variable is known.

Example: $3 + 5 = x$ $3X - 4 = 8$

The solution set or truth set contains the elements which make an equation true. A root of the equation is a number which belongs to the truth set.

Teaching Hint: Test questions are examples of open sentences. In working out the solutions, a student closes the sentence if his solution is correct.

Teaching Hint: In checking equations, the variable must be replaced in the original equation.

E. Reading problems and setting up equations

Analyze a written problem to determine the relationship involved and to discover the proper operations for solving the problem.

Determine:

1. What is given?
2. What are we looking for? (the question)
3. What method (operation) could be used to solve the problem?

When the above analysis has been completed an equation or mathematical statement should be set up.

Teaching Hint: If students cannot set up an equation in association with a specific problem, they do not understand the problem itself.

VIII. Measurement

A. Standard units of Measure

1. History

Measurement is defined by the unit of measure in use. The size of a unit of measure is purely arbitrary and becomes a standard unit of measure when people agree to its size and usage.

Early systems of measurement were based on use of parts of the body, such as the hand, foot, cubit (elbow to fingertips), pace, and fathom (tip of fingers when arms are spread horizontally). Because of the possibility of inconsistency, standarization has developed two systems, the British American and the Metric system.

New units of measure, very small and very large, have been demanded by modern technology. The unit of measure must be suited to the property characteristics or quantity being measured.

Teaching Hint: Measure various objects using non-standardized units of measure. Employ estimating.

2. Types of units of measure

The seventh level is concerned with the following types of measurement: (a) counting (b) linear (c) area (d) volume (e) weight and mass (f) time (g) temperature (h) liquid and dry capacity.

Teaching Hint: Many kinds of measuring instruments should be available for illustration.

Examples: Rulers, tapes, balance scales, measuring cup, pints, quarts, meterstick, measuring spoons and cups, ice cream cartons, gallon, calipers, protractors, stop watch, yard stick, bushel and peck, tin cans, geometric models.

3. Measurement is an approximation

Since measurements are never exact it is important to know how accurate the measurements really are.

Teaching Hint: Discuss clothing sizes for exactness.

a. Greatest possible error (g.p.e.)

The greatest possible error is one half the unit of measure being used.

Example: If the unit being used is $\frac{1}{4}$ inch, g.p.e. is 1/8 inch.

If the unit being used is 1/8 inch, g.p.e. is 1/16 inch.

b. Relative error

The relative error of a measurement is the quotient of the g.p.e. and the measurement itself.

Example: A door measures 6'8". Find the relative error.

$$\text{Relative error} = \frac{\text{gpe}}{\text{measure}}$$

$$\text{Relative error} = \frac{.5 \text{ inch}}{80 \text{ inches}}$$

Relative error = .06 inch per inch

Teaching Hint: Introduce with a discussion on tolerance. Will the piston fit the cylinder?

4. The same unit of measure

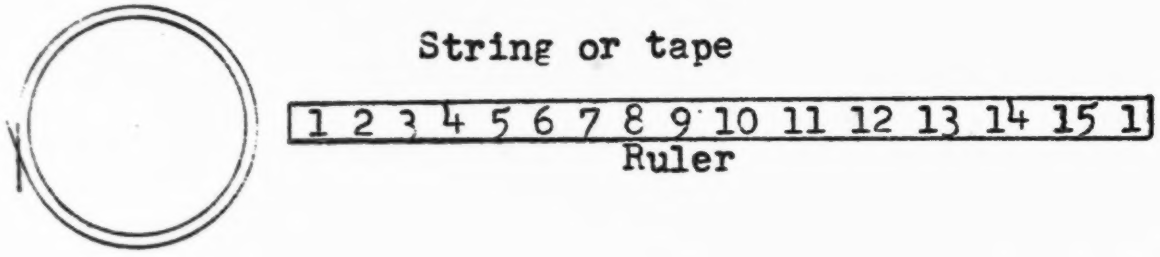
Measurements must be expressed in the same units when adding or subtracting.

Example:

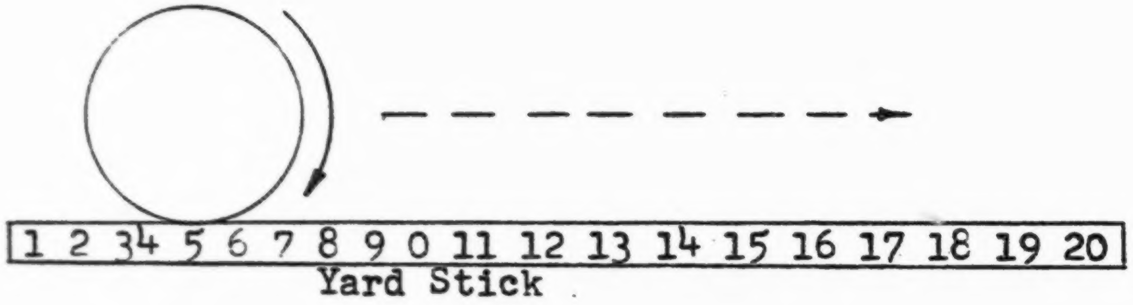
$$\begin{array}{lrrr} & 1 \text{ yd.} & & 6 \text{ in.} \\ + & & 1 \text{ ft.} & 7 \text{ in.} \\ \hline & 1 \text{ yd.} & 1 \text{ ft.} & 13 \text{ in.} \end{array} \qquad \begin{array}{lrrr} & 2 \text{ ft. } 6 \text{ in.} & = 1 \text{ ft.} & 18 \text{ in.} \\ - & 9 \text{ in.} & & 9 \text{ in.} \\ \hline & & 1 \text{ ft.} & 9 \text{ in.} \end{array}$$

B. Linear measure

Teaching Hint: Practice measuring objects in the classroom.
Teaching Hint: Measure a circular object by wrapping a string or flexible tape around it and later measure the length of the tape or string.



Another way to understand the distance around is to roll an object on a yard stick so that it completes one turn.

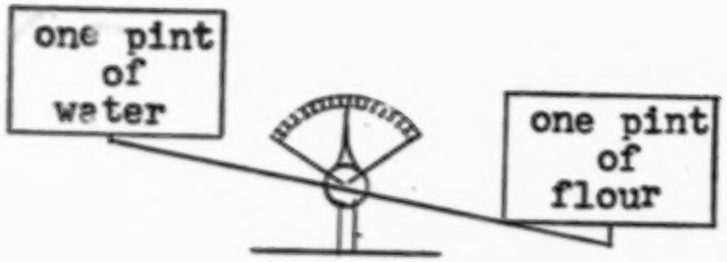


The perimeter of an object can be found as in the above examples, on any object that is not concave.

Teaching Hint: The symbol Pi (π) names the relationship between the diameter of a circle and its circumference.

To test the value of π compare the diameter of a circle with its circumference as illustrated above.

Circumference is estimated as being approximately three times the diameter.

C. Area	See unit on Geometry.
D. Volume	See unit on Geometry.
E. Dry and liquid measure	Two kinds of capacity units are used, one for liquid and one for dry measure. Ounces, pints and quarts do not mean the weight in these two measures. Teaching Hint: Demonstrate that an equal volume (one pint) of flour, does not weigh the same as an equal volume (one pint) of water. 
F. Mass and weight	Intuitively the mass of an object is the measure of the amount of material in it. Weight of an object is the pull of gravity on it. When motionless on the earth, at the same place, mass and weight can be considered the same. Teaching Hint: Discuss what happens in an automobile when traveling fast over a rise or dip in the road. Also discuss "G's" (the force of gravity) in launching rockets.
G. The measure of time	Time is related to motion. This can be indicated by the sandclock, waterclock, the rotation of the earth on its axis and the revolution of the earth around the sun.

H. Temperature

Develop the difference between Fahrenheit and Centigrade thermometers.

Practice comparing the scales.

There are 180° between boiling and freezing points on the Fahrenheit but only 100° on the Centigrade scales. Discuss why the centigrade scale has been adopted for the metric system.

I. Changing units of measure

Changing units by the proportion method is quite simple and accurate. This method is also used throughout other areas of mathematical computation.

Example: Changing a larger unit to a smaller unit.

3 ft. to ____ in.

$$\frac{3\text{ ft.} \cdot 12\text{ in.}}{1\text{ ft.}} - \frac{(3\text{ ft.})}{1} \cdot \frac{(12\text{ in.})}{1\text{ ft.}} =$$

$$\frac{(3 \cdot 12) \cdot (\text{ft.in.})}{1\text{ ft.}} = \frac{36}{1} \cdot \frac{\text{ft.}}{\text{ft.}} \cdot \text{in.} = 36\text{ in}$$

A conversion table is used on ordinary problems.

1 ft. = 12 in. 3 ft. = ____in. 12 is the conversion factor so multiply 3 x 12 = 36 in. Another method that is used frequently:

3 ft. = △ in.

$$\frac{\text{ft.}}{\text{ft.}} = \frac{\text{in.}}{\text{in.}} \qquad \frac{3\text{ ft.}}{1\text{ ft.}} = \frac{\triangle\text{in.}}{12\text{ in.}}$$

$$\frac{3\text{ ft.} \cdot 12\text{ in.}}{1\text{ ft.}} = \triangle\text{in.}$$

$$36\text{ in.} = \triangle\text{ in.}$$

Substitution method:

$3 \text{ ft.} = 3 \cdot 1 \text{ ft.} \qquad 1 \text{ ft.} = 12 \text{ in.}$

$= 3 \cdot 12 \text{ in.} = 36 \text{ in.}$

IX. Metric System	Explain to the class that the use of the metric system is worldwide in all scientific areas and is in general use except in the English speaking countries.
A. Scope	A smaller unit of measure is obtained by dividing by a power of 10.
B. Use of a centimeter ruler	Supply each student with a sheet of paper on which are drawn line segments of various lengths, have them measure to the nearest centimeter and again to the nearest millimeter with a metric ruler.
C. Comparison of Metric and English units	The student may use the meter as a convenient unit for conversion of measurement to the English system. 1 m = 39.37 inches or (approximately) 1 inch = 2.54 cm. Students should compare units of measure in both systems and be able to convert measurements expressed in one system into measurements expressed in the other. Students also learn to use the common metric units for area and volume. The area and volume concepts as learned in English units can be expressed equally well with units of the metric system. Explain this to students. Students learn the concept and patterns established for changing from one unit to another unit within the metric system. (4.25 m = 425 cm.) Move the decimal two places to the right, etc.

X. Percent

A. Meaning of percent

"Percent" and "%" both mean "hundredths" so any numeral with 100 as a denominator may be converted to percent.

B. Pattern for changing the decimal numeral to a percent

The short way to convert a decimal number to the percent form. Example: $\frac{25}{100} = .25 = 25\%$

Multiplying a number expressed as a decimal by its identity, in the form of 100/100, results in a fraction which may be expressed as a percent.

Example:

$$.15 = .15 \times 1 \qquad .005 = .005 \times 1$$

$$= .15 \times \frac{100}{100} \qquad = .005 \times \frac{100}{100}$$

$$= \frac{15}{100} \qquad = \frac{.5}{100}$$

$$= 15\% \qquad = .5\%$$

Teaching Hint: Never annex the percent sign without moving the decimal point two places to the right.

C. Pattern for changing percents to decimal form

Students will use the fact that % means hundredths and will write the percent as a fraction with a denominator of 100.

Example: $31.5\% = \frac{31.5}{100}$

$$31.5 \div 100 = .315$$

Practice of a similar nature will show the pattern: To change a percent to a decimal form drop the % sign and move the decimal 2 places to the left. The opposite of the pattern will also be true.

Teaching Hint: This is an instance of another name for a number. i.e. When students receive change for a dollar bill, they have received an amount of equivalent value, not the bill itself.

Teaching Hint: Practice will develop facility in changing forms of fractions, decimals and percents. The following form may be used:

FRACTIONAL FORM	DECIMAL FORM	PERCENT FORM
$\frac{1}{4}$	?	?
?	.50	?
?	?	7.5%

D. Percentage

If students have difficulty understanding just what is meant by the percent of a number, explain that it means a part of, or a fraction of. Percentage problems may be solved either by the fractional form or the decimal form.

E. Basic Pattern

There is one basic pattern for solving the percentage problem. From this basic pattern two variations of the pattern, are derived.

The basic pattern, formula or equation is:

$P = br$, where P = percentage (amount)
b = base (principal)
r = rate (%)

Example: What is twenty percent of 200?

$P = br$

$P = 20\% \text{ of } 200$

$P = .20 \times 200 \text{ or } \frac{20}{100} \times 200$

$P = 40$

Teaching Hint: The formula P = br as usually encountered in written problems is reversed and becomes rb = P.

Teaching Hunt: Sometimes rate (%) may be confused with percentage (the amount gained).

F. Variations

Using the properties of equations the two variations of the basic pattern are derived:

$r = \frac{P}{b}$

Example: Ten is what percent of fifty?

$r = \frac{10}{50}$

$r = \frac{1}{5} \text{ or } \frac{20}{100} \text{ or } .20$

$r = 20\%$

$b = \frac{P}{r}$

Example: Twenty is ten percent of what number (base)?

$b = \frac{20}{.10}$

$b = 200$

G. Proportion

The idea of ratio has already been introduced as a means of comparison with the numbers of arithmetic (fractions). Proportion is the relationship which exists between two equivalent ratio's.

Example: $\frac{3}{4} = \frac{6}{8}$ $3:4 = 6:8$

Example:

$\frac{3}{4} = \frac{x}{100}$	$\frac{3}{4} = \frac{x}{100}$
$3 \times 100 = 4x$	$100 \cdot \frac{3}{4} = \frac{x}{100} \cdot 100$
$300 = 4x$	$\frac{300}{4} = x \, \frac{100}{100}$
$\frac{300}{4} = x$	$\frac{300}{4} = x \cdot 1$
$75 = x$	$75 = x$
(product of extremes equal product of means)	(properties of equations)

H. Percent of increase

The percent of increase is usually based on the original number which is the number of the set before the increase took place.

Example: If 32 is increased by 4 what is the percent of increase? The problem may be solved using the basic percentage formula or by finding a number having the same ratio to a hundred as 4 has to 32.

$\frac{4}{32} = \frac{n}{100}$

$\frac{4 \times 100}{32} = n$

$12.5 = n$ $\qquad$ $n = 12\frac{1}{2}\%$ (% of increase)

If the increase is from 32 to 40 then find the difference between the two numbers and proceed as above.

$\frac{40 - 32}{32} = \frac{n}{100}$

$\frac{8 \cdot 100}{32} = n$

$25 = n$

$n = 25\%$ (% of increase)

I. Percent of decrease

Percent of decrease is also based on the original number.

Example: 75 is decreased by 10. What is the percent of decrease?

$\frac{10}{75} = \frac{n}{100}$

$\frac{10 \cdot 100}{75} = n$

$13.33 = n$

$13\frac{1}{3}\% = n$ (% of decrease)

	The problems under these topics are solved by using the basic percentage method.
J. Discount, commissions, sales price, and taxes	If the rate of percent (25%) has a more convenient fraction equivalent ($\frac{1}{4}$), it may be simpler to use the fraction form.
K. Interest	Either the proportion method or the interest formula may be used in computing interest. Example: $\frac{i}{p\ t} = \frac{r}{100}$ $i = \frac{p\ r\ t}{100}$ (Interest expressed as if it were for one year.) Example: $i = p\ r\ t$ Computation may be done in fractions or decimals. Simple interest is that which is earned by the principal for the basic period (usually a year). When this interest is added to the principal, this amount becomes the principal for the next period. This is compounding interest.

XI. Geometry

A. Points, lines, planes and solids

Drawings of points, lines and planes represent abstractions. Present them in relation to the students physical environment, points in space lines and segments, planes and solids. Develop in sequence; points generate lines, lines generate planes, etc. From points of no dimension to lines with one dimension, to planes with two, etc.

Teaching Hint. Models of all geometric figures must be available for illustration. Clear plastic may be used as indicating no thickness.

B. Perimeter of triangles, quadrilaterals and circles

Develop perimeter by actually measuring triangles, quadrilaterals and circles with string or thread. Develop equations or formulas as a result of class discovery activities.

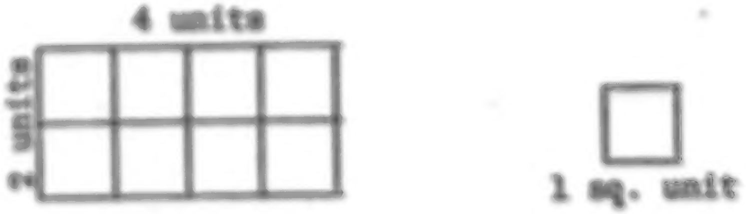
Teaching Hint: Peri-meter (Greek) around-measure. Develop the concept of π (3.14...) as the ratio of the circulference (perimeter) to the diameter. Use a number of circles of different sizes to develop this concept.

The derived measurements are an approximation of π.

$$C = \pi d,\ C = 2\pi r,\ \frac{C}{d} = \pi$$

C. Area, beginning with the rectangle

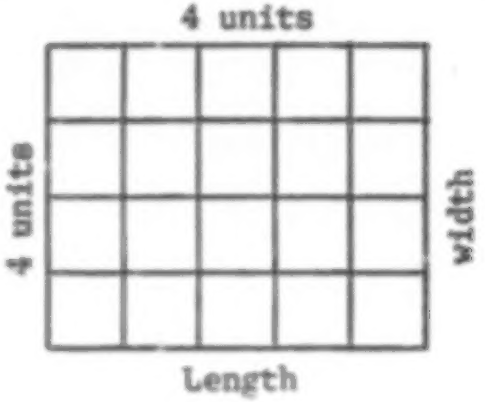
The formula for the area of a rectangle may be developed as follows, using a basic square unit.



Area is a number of square units. Two units of width, multiplied by four units of length shows the complete area. A = lw

Teaching Hint: By having the units cut out and moved into different positions the student can see that changing the shape of this figure does not change its area.

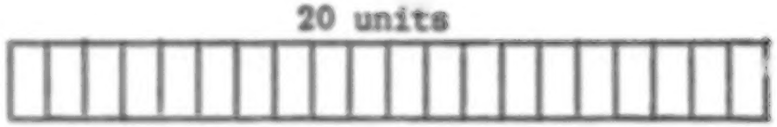
Example:



A = lw

A = (4 units) (5 units)

A = 20 square units



A = S · S

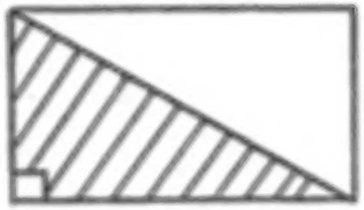
A = S^2

D. Other areas

The formulas for the areas of the triangle, parallelogram, trapezoid and circle are derived from the formula for the area of the rectangle. Once the square and rectangle are understood, the other formulas are derived.

Examples:

Right Triangle



$\square$ $A = lw$

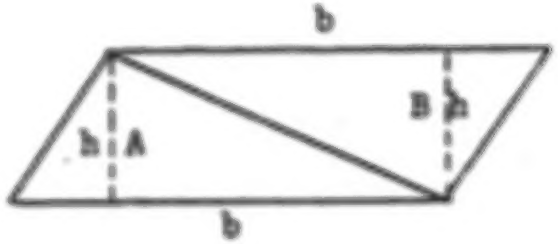
$\triangle$ $A = \frac{lw}{2}$

$\triangle$ $A = \frac{1}{2}bh$ (rename)

Acute or obtuse triangles use rectangle construction producing two right triangles. Area of triangle is the sum of one half the areas of the respective rectangles; therefore, the above formula applies.

Teaching Hint: Altitude may be inside or outside a figure, but must be perpendicular to the base.

Parallelogram



area $\triangle$ $A = \frac{1}{2}bh$

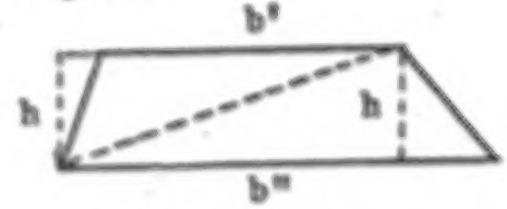
area $\triangle$ $B = \frac{1}{2}bh$

▱ Area $= \frac{bh}{2} + \frac{bh}{2}$

▱ $A = 2\,\frac{bh}{2}$

▱ $A = bh$

Trapezoid



"h" is the same for both triangles.

△ $A = \frac{1}{2} b'h$

△ $A = \frac{1}{2} b''h$

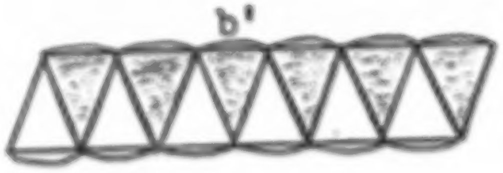
⏢ $A = \frac{1}{2} b'h + \frac{1}{2} b''h$

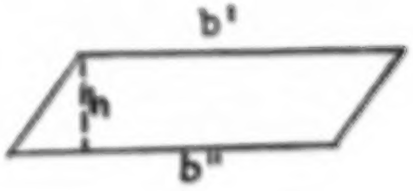
⏢ $A = \frac{b'h + b''h}{2}$

⏢ $A = \frac{1}{2} (b' + b'')h$

Circle

Arrange segments of a circle as a parallelogram.





Circumference = $2\pi r$

$b' + b''$

Radius = $r \approx h$

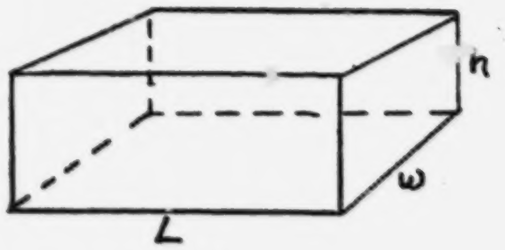
$A = bh$

$A = \frac{1}{2}(2\pi r) \cdot r$ (by substitution)

$A = r^2\pi$

Teaching Hint: It must be observed that there are alternate methods for deriving the above formulas.

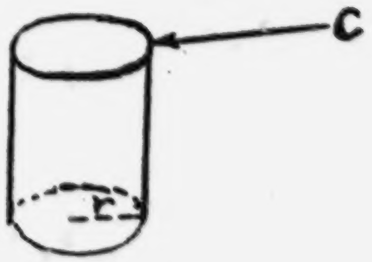
Right Rectangular Prism



Area is equivalent to the sum of the areas of all faces and bases.

$A = 2lw + 2wh + 2lh$

Right Circular Cylinder



Area is equivalent to the sum of the areas of the bases and the curves surfaces.

$A = \pi r^2 + \pi r^2 + Ch$ (where "C" is circumference)

$A = 2\pi r^2 + Ch$

$A = 2\pi r^2 + 2\pi rh$

$A = (2\pi r)r + (2\pi r)h$

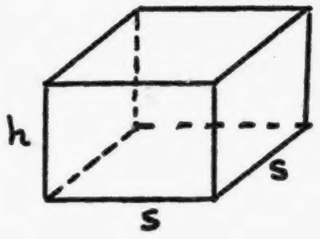
$A = 2\pi r(r + h)$

E. Volume, beginning with the cube

Volume is defined as the interior of a geometric solid. It is measured in cubic units.

Examples:

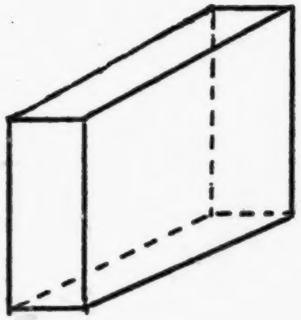
Cube



$V = s^2 \cdot h$

$V = Bh$ (where B represents the area of the base)

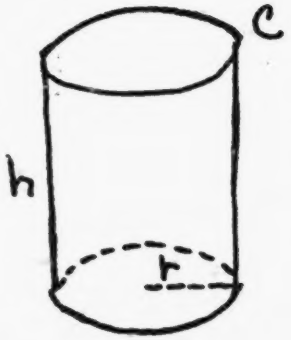
Right rectangular prism



$V = L \times W \times H$

$V = Bh$

Right circular cylinder



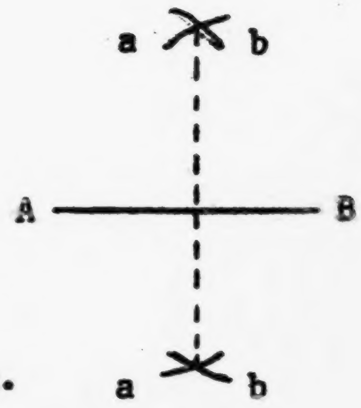
$V = \pi r^2 \cdot h$

$V = Bh$

F. Construction

Examples:

Bisecting a line segment

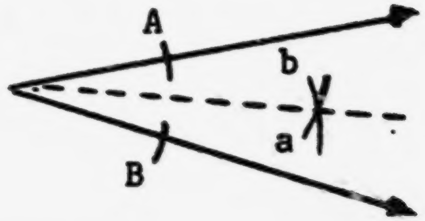


From point A strike arc a - a.

From point B strike arc b - b, with same compass setting.

The line drawn between the intersections of the arcs will bisect segment AB.

Bisecting an angle

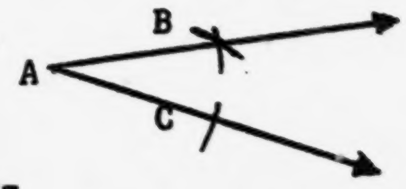


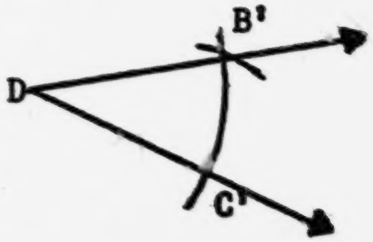
From the vertex, strike arc AB.

From points A and B strike arcs a and b respectively, using the same compass setting.

A ray drawn from the vertex through the intersection of the arcs will bisect the angle.

Congruent angles





From vertex A, draw arc BC.

Using same compass setting, draw arc B'C' using vertex D.

From point C strike an arc through B.

Using the same compass setting strike the same arc using C'.

Draw a ray from vertex D through the intersecting arcs to complete the congruent angle.

XII. Graphs

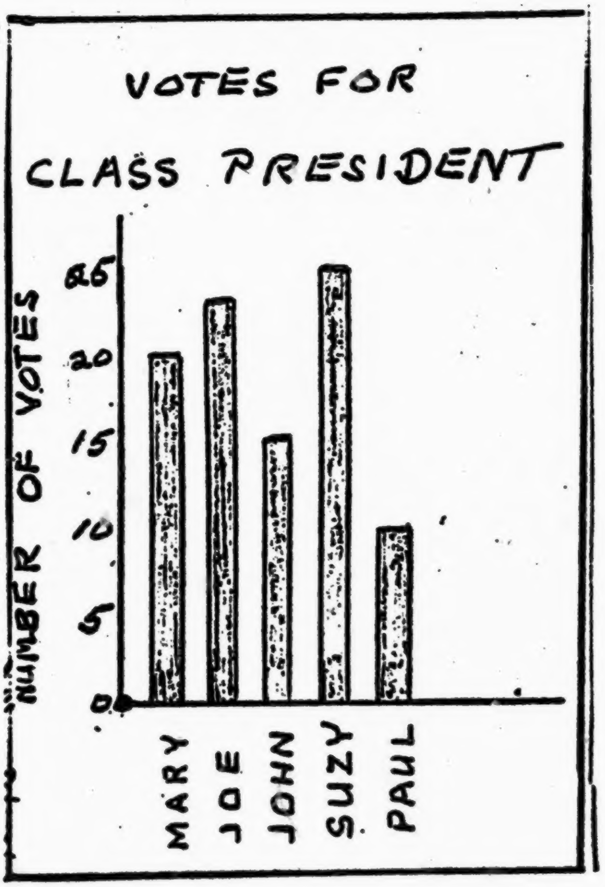
Use graphs from newspapers and magazines. Discuss various types of graphs and the use and reading of each type.

If a graph is easy to read and understand, has a suitable scale, has a title and clearly labeled information it is usually a good graph. Discuss the various kinds of graphs - bar, line, circle and picture graphs. A graph should show relationships in a clear and concise way.

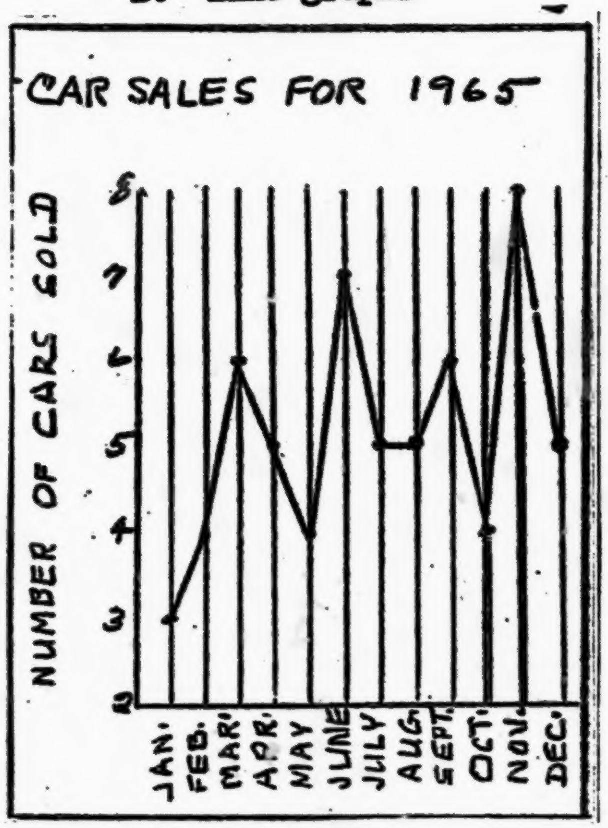
A. Bar graphs

Use graph paper to make neat graphs. Bar graphs may be either vertical or horizontal. The width of the bars on a certain bar graph should be the same and the spaces between the bars (if any) should be equal. The height or length of each bar indicates the quantity. The bar graph scale usually starts at zero.

The scale should be large enough to show relationship of data, but must be small enough to place on the page. Draw the vertical and horizontal axes, on which the number scales are placed and from which the bars are drawn. Leave enough margin for data. Draw and label the bars.



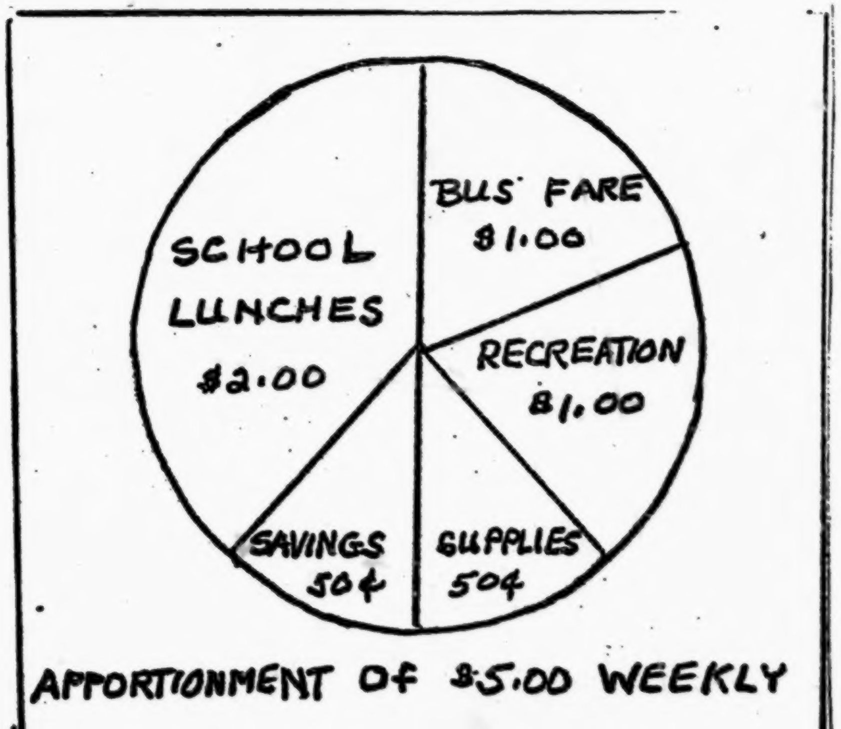
B. Line graphs



Line graphs are generally used to show that both axes represent sets of numbers. Such graphs, made of line segments, often are used to show changes that occur from time to time. The scale does not need to start at zero. In general, in graphs involving time, the number scale is on the vertical axis and time scale on the horizontal axis. Locate points in order of time and connect the points in order.

Teaching Hints: Make graphs of daily grades in a subject.

C. Circle graphs



A circle graph makes use of geometrical terms radius, sector and arc and degrees. The circle graph is used in representing the parts of a whole.

Circle graphs express percents or fractions of the whole as degrees of the central angles.

Teaching Hint: Graphs showing the apportionment of student allowances or the time spent in various activities during the day may be made.

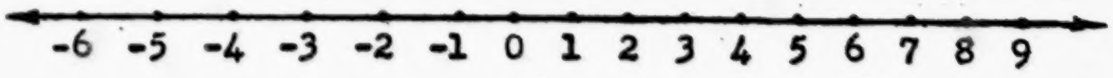
The making of graphs gives much actual practice in the use of compasses, and use and reading of rulers and protractors.

It is equally important that the student learn to read graphs.

XIII. Positive and negative integers

A. A number line

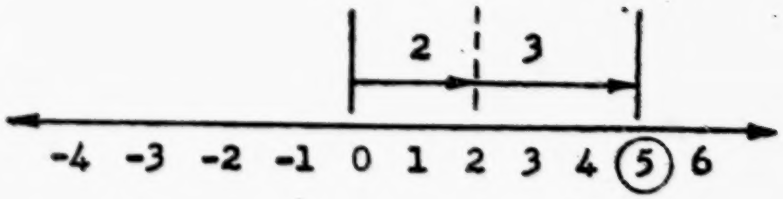
A number line is used with addition and subtraction of positive and negative integers. A number line extends indefinitely in both directions, and there is a one-to-one correspondence between each unit point on the number line and the members of the set of integers.



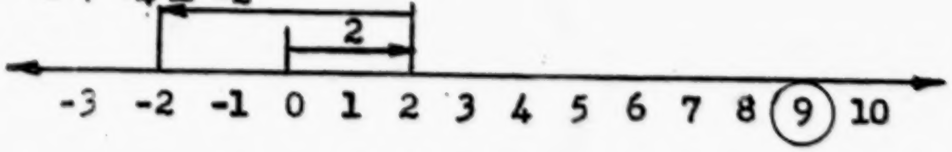
B. Addition (definition)

Addition may be represented on a number line as moving; to the right if a positive number, and to the left if a negative number.

$2 + 3 = 5$



${}^{+}2 + -4 = -2$



C. Subtraction (definition)

Subtraction may be graphed as moving in the opposite direction on the number line, from addition; to the left if a positive number, and to the right if a negative number.

Teaching Hint: Positive and negative signs may be written as follows:

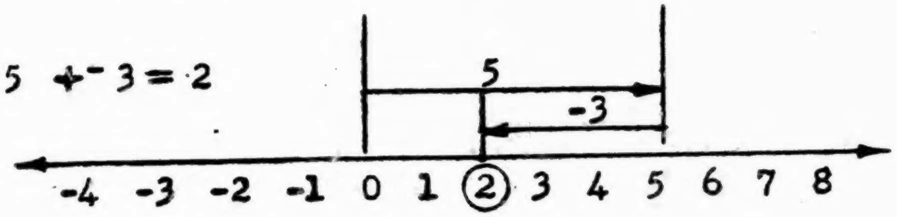
$3 + 2$ $\qquad$ $-3 + {}^{+}2$

${}^{+}3 + {}^{+}2$ $\qquad$ ${}^{+}2 + -3$

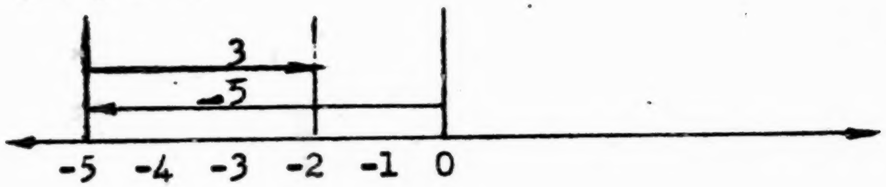
$-3 + -2$ $(-2) + (-3)$

$3 - (-2)$ $2 - (-3)$

$5 + {}^{-}3 = 2$



$-5 + 3 = -2$



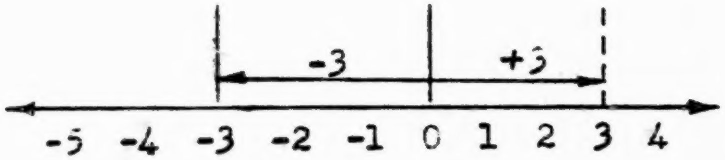
For every positive number there is an opposite (inverse) or negative number.

$(a) + (-a) = 0$ (additive inverse) $(-a) + (a) = 0$

Zero (0) is its own opposite and is expressed as 0.

${}^{+}0 = -0 = 0$

The opposite of +3 is -3 and is shown as follows on the number line:



Subtracting an integer is the same as adding its opposite.

(a) $\begin{array}{r} (+3) \\ -\underline{(-2)} \\ 5 \end{array}$ (b) $\begin{array}{r} (-3) \\ -\underline{(-2)} \\ -1 \end{array}$ (c) $\begin{array}{r} (+3) \\ -\underline{(+2)} \\ 1 \end{array}$ (d) $\begin{array}{r} (-3) \\ -\underline{(+2)} \\ -5 \end{array}$

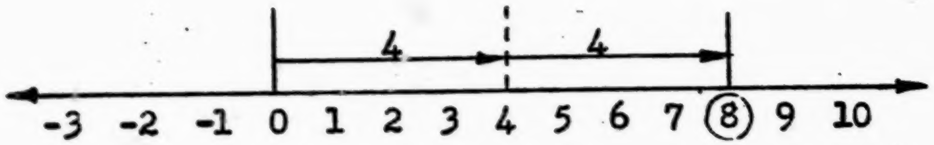
When subtracting change the sign of the subtrahend and proceed as in addition.

D. Multiplication of integers

1. Positive times a positive

Since multiplication is repeated addition, positive integers are expressed in multiplication as follows:

2 x 4 = 8 (two groups of fours)

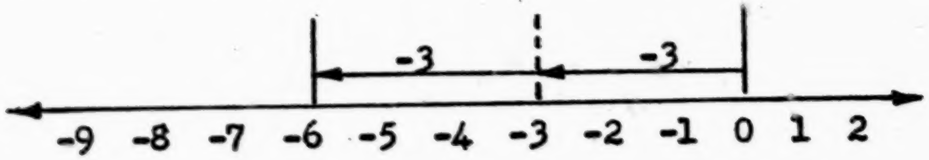


The product of two positive integers is a positive integer.

2. Positive times a negative

The principle for multiplication of a positive and negative number is as follows:

(+2) · (-3) = -6 (two groups of the inverse of three)



3. Negative times a negative

Observe the pattern

$(-3) \cdot 2 = -6$

$(-3) \cdot 1 = -3$

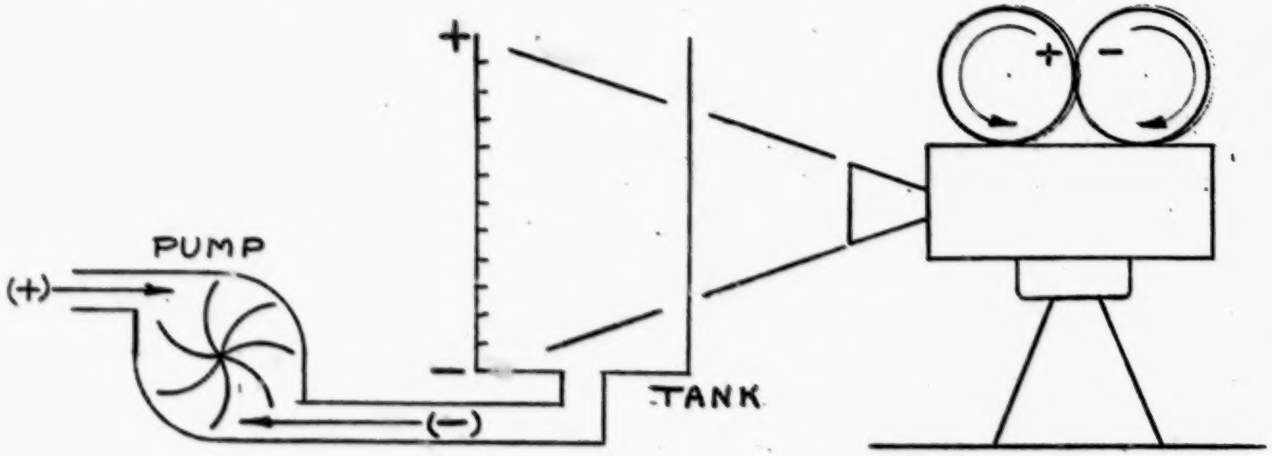
$(-3) \cdot 0 = 0$

$(-3) \cdot (-1) = 3$

$(-3) \cdot (-2) = 6$

Teaching Hint: Each of the above should be developed on the number line.

Teaching Hint: Illustrate by means of old time camera and water tank.



+camera and +water = +picture

+camera and -water = -picture

-camera and +water = -picture

-camera and -water = +picture

E. Division of integers

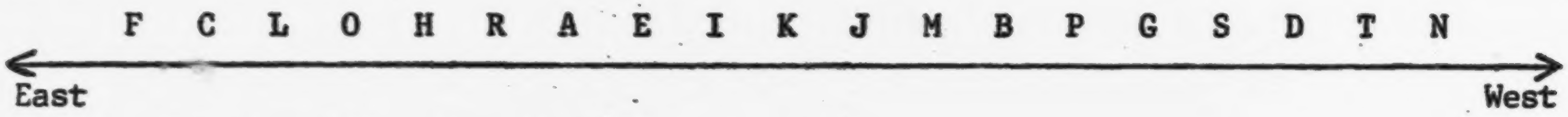
Division is the inverse form of multiplication.

$N = 6 \div (-3)$ so $(N)\,(-3) = 6$

since $(-2)\,(-3) = 6$ $\qquad$ $N = -2$

The above multiplication patterns hold true for division.

F. Problem Material for use with this unit.



1. How many miles from M to L? 9
2. How many miles from L to M? 9
3. If you are at P, which direction would you go to R? E How far? 8
4. If you start at H, go west 9 miles, then east 10 miles, where are you? O
5. If you start at I, go east 7 miles, then west 8 miles, where are you? K
6. How many miles did you travel in problem 5? 15
7. If you start at E, go 10 miles west, then 10 miles east, where are you? E
8. If you start at H, go 35 miles west, then 35 miles east, where are you? H
9. If you start at K, go 17 miles east, then 16 miles west, where are you? I
10. If you start at M, go 15 miles west, then 18 miles east, where are you? I
11. From J, go $\overrightarrow{7}$, then $\overleftarrow{10}$. Where do you stop? E
12. From B, go $\overleftarrow{11}$, then $\overrightarrow{12}$. Where do you stop? P
13. If you first go $\overrightarrow{7}$, then $\overleftarrow{8}$, and stop at R, where did you start? A
14. What would be a short cut for the trip in problem 11? $\overleftarrow{3}$
15. From K, go $\overrightarrow{8}$, then $\overleftarrow{11}$, then $\overrightarrow{3}$. Where do you stop? K
16. From K, go $\overrightarrow{7}$, then $\overrightarrow{2}$, then $\overleftarrow{12}$. Where do you stop? A
17. What shorter trip could you have taken in problem 16 and stop the same place? $\overleftarrow{3}$

18. From K, go $\overrightarrow{15}$, then $\overleftarrow{14}$. Where do you stop? J

19. From L, go $\overleftarrow{100}$, then $\overrightarrow{102}$. Where do you stop? H

20. If you first go $\overrightarrow{50}$, then $\overleftarrow{45}$, and stop at B, where did you start? E

21. If you start at A, go $\overrightarrow{55}$, then $\overrightarrow{45}$, then $\overleftarrow{100}$, where do you stop? A

22. From L go $\overrightarrow{1000}$, then $\overrightarrow{25}$, then $\overleftarrow{1023}$. Where do you stop? H

23. What would be a short cut for the trip in problem 22? $\overline{2}$

24. From M, go $\overrightarrow{25}$, then $\overleftarrow{17}$, then $\overrightarrow{17}$, then $\overleftarrow{25}$. Where do you stop? M

25. From K, go $\overrightarrow{35}$, then $\overrightarrow{30}$, then $\overrightarrow{15}$, then 2, then $\overleftarrow{75}$. Where do you stop? D

26. Using $^{+}$ for right and $^{-}$ for left, start at K, go $^{+}12$ then $^{-}10$. Where do you stop? M

27. Start at I, go $^{+}10$, $^{-}11$, $^{+}12$, $^{-}12$. Where do you stop? E

28. Start at G, go $^{+}98$, then $^{-}98$. How far from G do you stop? 0

29. Start at R, go $^{+}100$, then $^{-}80$. How far from R did you stop? +20

30. What shorter trip could you have taken in Problem 29 and stop the same? +20

31. What shorter trip could you take and stop at the same place you would if you made a trip of $^{-}58$ followed by $^{+}35$? -23

32. $^{+}15 + {}^{+}14 = +29$

33. $^{+}598 + {}^{+}2 = +60$

34. $^{+}795 + {}^{-}699 = +96$

35. $^{-}478 + {}^{+}352 = -126$

36. $^{-}32 + {}^{-}36 \; {}^{+}72 = {}^{+}4$

37. $^{-}16 + {}^{+}34 + {}^{+}2 + {}^{-}5 = {}^{+}15$

38. $^{-}728 + {}^{-}53 + {}^{+}400 = -381$

39. $^{-}395 + {}^{+}417 + {}^{-}23 = -1$

40. $^{-}38 + 0 = -38$

YOU SAVE MONEY—*If You Know What Credit Costs*

When you buy on credit, you pay an extra charge. Whether that extra charge is called "interest," "service fee," "carrying charge," or anything else, it raises the price of what you buy. It can keep you from buying other things you need. For example:

★ A store sells a refrigerator for $329.95. On a 24-month contract with a $10 down payment, you can pay the store $66 extra for credit—enough to buy 285 quarts of milk.

★ One of the new "compact" cars costs $2660.52, with deluxe accessories, sales tax, and license. With a $460.52 down payment, credit charges on the $2200 balance on a 36-month contract can cost you over $400—enough to pay cash for a washing machine and a drier.

How much credit costs you depends on the *rate of interest* and the *length of the loan*. Both affect the total cost of credit. The smart shopper gets credit at the lowest possible rate of interest for the shortest possible period of time. Credit costs vary widely, so it is worth your while to shop for credit carefully. Some commonly-quoted credit charges are expressed in true annual interest rates for you in the chart on the right.

Remember, it's smart to *shop* for credit. . . . It's even smarter—and cheaper, too!—to pay cash.

Δ GPO

HERE'S WHAT YOU PAY FOR CREDIT

If added to purchase price and total repaid in 12 equal monthly payments:

When they say	*you pay**
4% per year:	**7.3%**
6% per year:	**10.9%**
8% per year:	**14.5%**
10% per year:	**18.0%**
1% per mo. :	**21.5%**

*** In true annual interest**

If charged only on unpaid balance		**True annual interest**
3/4 of 1%	per month on unpaid balance	= 9%
5/6 of 1%	per month on unpaid balance	= 10%
1%	per month on unpaid balance	= 12%
1¼%	per month on unpaid balance	= 15%
1½%	per month on unpaid balance	= 18%
2½%	per month on unpaid balance	= 30%

(This brochure is a service to the people of California provided under an Act of the State Legislature, requested by Governor Brown. It was compiled by State Consumer Counsel Helen Nelson, Governor's Office, Sacramento.)

Grade 7

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APPENDIX

GRADE SEVEN

BASIC VOCABULARY IN ARITHMETIC

1. Abacus - an instrument used in counting.
2. Abstract number - a number that expresses a specific amount without reference to any objects or quantity of things.
3. Add - a mathematical and mental process used to group two or more numbers and to express the total amount with one number.
4. Addend - a number to be added to another number.
5. Addition - a mathematical and mental process used to find, without counting, the total value or amount of two or more addends.
6. Addition facts - the 81 primary combinations in addition that show the sum of 2 one-place numbers.
7. Altitude - a perpendicular line drawn from the vertex of the angle that is opposite the base of a triangle to the base.
8. Amount - the total of two or more objects.
9. Angle - a geometric figure formed by two lines that meet at a point.
10. Arabic numerals - the nine figures or digits used in the Hindu-Arabic number system: 1, 2, 3, 4, 5, 6, 7, 8, 9. The symbol, 0, or zero is used to express the absence of a digit in a place-value position.
11. Area - the amount of a surface when measured by a selected square unit of measure.
12. Cardinal number - a number that answers the question, "How many in all?"
13. Circle - a closed curved line, all points equidistant from a center point.
14. Common denominator - a denominator that is the same for two or more fractions.
15. Common fraction - a mathematical symbol or number to express one or more equal fractional units of a quantity, or one or more units of a given group of like units.

16. Computation - a mathematical and mental process used to find the answer to a given example or question.

17. Cost - the amount paid for an object or a service a person desires to obtain.

18. Cube - a rectangular solid bound by six equal squares or square surfaces.

19. Decimal fraction - a fraction with a denominator of 10 or a power of 10.

20. Decimal number system - a number system with place-value positions based on 10 and powers of 10.

21. Decimal point - a symbol used to separate the place value one from the place value one-tenth.

22. Degree - a unit for measuring an angle or temperature.

23. Denominator - a number that is written below the numerator of a common fraction to show the number of equal divisions made of one or any object or quantity.

24. Digit - any one of the numerals used in the Hindu-Arabic number systems.

25. Dividend - a number that is to be regrouped into numbers equal to the number that is the divisor.

26. Division - a mathematical and mental process used to change the dividend into numbers like the the divisor.

27. Division combination or facts - a division combination is an example where the dividend is either a one- or two-place-value number and the divisor is a one-place-value number.

28. Divisor - a number that shows the size of the numbers desired when the dividend is regrouped.

29. Foot - a unit of measure that is equivalent in length to 12 inches.

30. Fraction - a mathematical symbol to show the number of fractional units and their size in relation to any given base (object, group, or number).

31. Improper fraction - a number with a numerical value equal to or greater than one.

32. Inch - a unit of linear measure that is equal in length to $\frac{1}{12}$ foot.

33. Interest - the money paid on money borrowed or the money earned on money invested.

34. Least common denominator - a number that is the same or common for two or more fractions and expresses the smallest size common to each fraction.

35. Like fractions - fractions that have a like or the same denominator.

36. Measurement - a system of measures - also the measured size, capacity, or amount of a given quantity.

37. Mixed number - a number that contains an integer and a fraction.

38. Multiplicand - one of a given number of like numbers to be grouped and the product notated by one number.

39. Multiplication - a mathematical and mental process of rapid addition of a given number of like numbers.

40. Multiplier - a number that shows how many like numbers are to be grouped or added and the product notated by one number.

41. Multiply - a mathematical and mental process to find the product of a given number of like numbers and to notate the sum with one number.

42. Number - a mathematical symbol or symbols to show the idea of total amount of a quantity or total units in a group.

43. Ordinal number - a number used to show the position of an object or a number in an established series.

44. Ounce - a unit of measure equal to $\frac{1}{16}$ pound in common weight.

45. Percent - a word and a sign, %, to express hundredths.

46. Perimeter - the sum of the length of the sides of a surface plane.

47. Pint - a unit of measure of capacity equal to $\frac{1}{8}$ gallon.

48. Place value - an ordered position in the number system with a numerical value determined by the radix of the system.

49. Pound - a unit of measure in the United States system equal to 16 ounces.

50. Product - the total amount of a given number of equal addends.

51. Proper fraction - a fraction when the denominator is a number larger than the numerator.

52. Quart - a unit of measure in the United States system of measure equal in capacity to $\frac{1}{4}$ gallon or 2 pints.

53. Ratio - a mathematical expression of the relationship between quantities or numbers.

54. Rectangle - a surface plane bounded by four straight lines, the opposite sides of equal length and parallel, and each is a right angle.

55. Remainder - a quantity or number when a given part has been removed. Also the number left in uneven division.

56. Rounded off number - a large number expressed by two or three significant digits.

57. Square - a flat surface that has four equal sides and four right angles.

58. Square foot - the area of a square that has four sides, each equal in length to 1 foot.

59. Square inch - the area of a square with each of the four sides equal to 1 inch.

60. Square yard - the area of a square that has four sides, each equal in length to 1 yard.

61. Subtraction - a mathematical and mental process used to find the difference between two numbers or to find the remaining number when a component number is known.

62. Subtrahend - a number in subtraction that identifies one of the given component numbers of a given number.

63. Ton - a unit of weight in the United States that is equal to 2000 pounds.

64. Total - the whole or entire amount of quantity or quantities.

65. Trial divisor - the digit in the divisor that is in the place-value position of greatest value.

66. Triangle - a surface figure or plane that has three sides and three angles.

67. Unlike fractions - fractions that do not have a common or like denominator; fractions of unequal fractional units.

68. Weight - the amount of heaviness of an object or quantity.

69. Whole number - a counting or natural number.

70. Yard - a unit of measure in the English and United States systems used to measure length - equal to 3 feet or 36 inches.

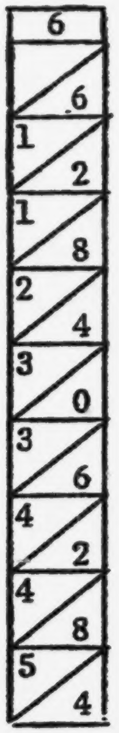
71. Zero - a mathematical symbol, 0, used in notation to show the absence of a digit in any place-value positions.

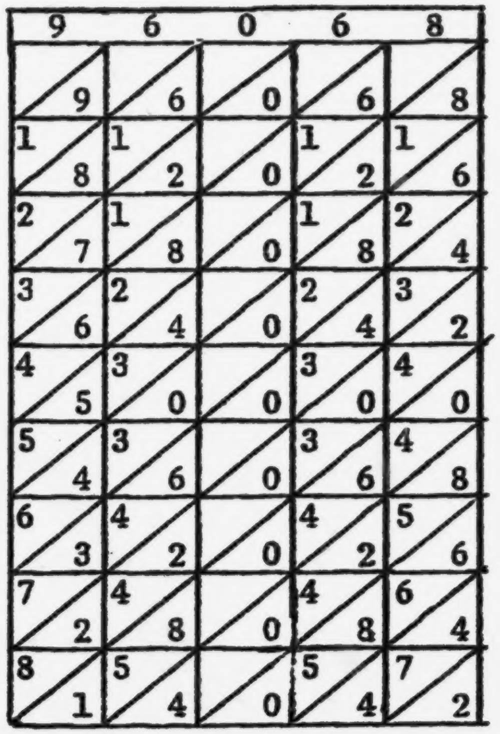
NAPIER'S RODS

The difficulty that was so widely experienced at one time in the multiplication of large numbers led to mechanical ways of carrying out the process. Very celebrated in its time was John Napier's (1550 - 1617) invention knows as Napier's rods or Napier's bones, and described by the inventor in one of his works published in 1617.

In principle the invention is the same as the Arabian Lattice, or grating method, only in this invention the process is carried out with the aid of rectangular strips of wood, bone, metal, or cardboard, and prepared beforehand. For each of the ten digits one should have some strips like the one shown to the left below for 6, and bearing the various multiples of that digit.

To illustrate the use of these strips in multiplication let us consider the problem of multiplying 96068 by 379. Place the strips headed 9, 6, 0, 6, 8, side by side as shown below.





3 x 96068 = 288204

7 x 96068 = 672476

9 x 96068 = 864612

864612
672476
288204
36409772

The results of multiplying 96068 by the 3, the 7, and the 9 of 379 are read off as shown above. Some simple diagonal additions of two digits being needed to obtain these results. The final product is found by the addition as is shown at the upper right of the bones.

THE MATHEMATICAL SENTENCE

The primary function of mathematical sentences is the solving of problems. Sentences are used to answer many questions, such as the simple inquiry of, "How many do you have?" and the more comples question, "What will happen if?" A mathematical sentence is the translation of a problem from its verbal form to a mathematical form.

To understand the nature of a mathematical sentence, one must first be familiar with the vocabulary of mathematics; its nouns, its verbs, and the other symbols that are used in writing the mathematical sentence. The following symbols stand for verbs in the writing of mathematical sentences:

$=$	Equal to;	$\neq$	Not equal to
$<$	Less than;	$\nless$	Not less than
$>$	Greater than;	$\ngtr$	Not greater than
$\leq$	Less than or equal to;	$\nleq$	Not less than or equal to
$\geq$	Greater than or equal to;	$\ngeq$	Not greater than or equal to
$\sim$	Similar to; is equivalent to;		
$\cong$	Congruent;		
$\in$	Is an element of;	$\notin$	Is not an element of
$\subset$	Is a subset of;	$\not\subset$	Is not a subset of
/	Note that the slash negates the symbol upon which it is imposed.		

Another group of symbols, serving as conjunctions, indicate the manner in which pairs of numbers are to be joined, or separated. Examples of these are:

+	Addition	$\frac{1}{2}$ or $\overline{)\ }$ or ÷	Division	$x^{\frac{1}{2}}$	Fractional exponent is the same as a radical
-	Subtraction	x^2	Exponent, or power (square, cube, fourth, etc...)	$\cup$	Union
x	Multiplication	$\sqrt{\ }$	Radical, (square root, cube root, fourth root...)	$\cap$	Intersection

Many symbols are used as nouns, especially in geometry. Examples of symbols are:

1, 2, 3...	**The numerals**
x, y, □ ⏢	**Variables (any letter or geometric figure will serve)**
<	**Angle**
△	**Triangle**

Symbols are also used to indicate feet, inches, and degrees when these terms are used as nouns in sentences.

Symbols that describe the relationship between nouns, but do not indicate an operation to be performed, act in the same manner as adjectives. Examples are:

//	Parallel (lines)
⊥	Perpendicular (lines)
⤫	Intersecting (lines)

These symbols are usually used with geometric figures. Example:

A || B Means line A is parallel to line B.

The last group of symbols are the punctuation symbols; parentheses, brackets, and braces. These are used to group numbers and to establish the order of operation.

The language for operating with signed numbers involves double meanings for some symbols. For example, "plus" and "minus" symbols are also positive and negative symbols that show direction. Use of signed numbers requires a clear and accurate understanding of such fundamental concepts as the commutative, associative, and distributive properties, and those involving multiplication by one and by zero, addition of zero, negative numbers as the opposite of positive numbers, and subtraction as T opposite of addition.

THE RULES OF ORDER

The structure of mathematical sentences follows definite patterns which are widely accepted. As with English, the order of a sentence makes a difference. Note the following:

I met a man with a good position in our bank.
In our bank, I met a man with a good position.
I, with a good position, met a man in our bank.

The rule of order in writing a sentence may be developed by the following sequence of steps beginning very early in the elementary school:

1. Addition and subtraction in order of appearance

   $2 + 3 - 1 =$ $\qquad\qquad$ $4 - 6 + 5 =$

   (Until the concept of negative numbers is understood, addition is performed before subtraction.)

2. Multiplication is sometimes repeated additions, so multiplication precedes addition.

   $4 + 3 \times 2$ becomes $4 + 6$

   Therefore, multiply, add - subtract

3. $3 + 8 \div 2$ may be written $3 + \frac{8}{2}$, so division precedes addition.

   $6 - 8 \div 2$ may be written $6 - \frac{8}{2}$, so division precedes subtraction.

4. Always do the process inside parentheses, then follow the order of appearance for multiplication and division, and then addition and subtraction. The fraction line in complex fractions operates the same as parentheses. Numerators and denominators must be cleared before dividing.

5. Powers and roots precede division and multiplication and/or addition and subtraction.

The rules of order may be summarized as follows:

Powers within parentheses (and fraction line), powers of parentheses, division and multiplication in order of appearance, addition and subtraction in order of appearance.

USE OF SENTENCES IN PROBLEM SOLVING

The translation of problems into mathematical sentences requires a knowledge of structure which in itself is not complex but which demands an adequate knowledge of the nature of the problem. For example, consider the simple problem:

> How much more money does Jim need to buy a ball that costs 65¢? He now has 35¢.
>
> The translation should read, $35 + X = 65$, to be in the best form. However, the form $65 - 35 = X$ is not misunderstood. It will produce the same result.

The issue raised is not which form is correct, but which is better. In teaching young children to write, rigid insistence upon a specific form may detract from the quality of thought. In mathematics, one must allow for flexibility in the problem solving approach. The maturity of the pupil will determine to some extent the degree to which the more precise form should be used.

The following are literal translations of selected one-step problem designs:

1.	The sum of two numbers	$7 + 5 = X$
2.	Subtraction, when a remainder is to be found	$12 - 7 = X$
	Subtraction, when "How many more are needed" is to be found	$7 + X = 12$
	Subtraction, when the original addend must be derived, "Jim found 7¢. He now has 12¢. How much did he have before he found the 7¢?	$X + 7 = 12$
3.	Multiplication, when the set of objects is 8 and the number of the sets is 5	$8 \times 5 = \square$
4.	Division, when the product of two numbers (dividend) is known, and the number of sets is known	$k \times 4 = 48$, or $48 \div k = 4$

5. Every problem of percent can readily be translated into an equation. The "three types" of percent often taught in the past, easily become but one type of equation, excepting for the different locations of the variable. For example:

What is 5% of $200?	$\frac{X}{200} = \frac{5}{100}$
$600 is what percent of $12,000?	$\frac{600}{12000} = \frac{X}{100}$
How much must I invest at 4% to earn $800 annually?	$\frac{800}{X} = \frac{4}{100}$

TYPES OF SENTENCES: VARIABLE

Sentences may be either closed or open. A closed sentence is a completed statement. It may be either true or false. An open sentence is not complete. It may lack a noun, a verb, or a conjunction. Each may be indicated by one of many symbols. When the value of a noun has not been definitely established, it is called a variable. Sometimes the terms "pronumeral" and "placeholder" are used as synonyms for the term "variable." Once the value(s) of a variable has been established, the open sentence becomes a closed sentence.

Examples of closed sentences:

$4 + 1 = 5$	True
$3 + 4 = 9$	False
$3 \quad 5$	True
$6 + 4 \neq 9$	True
$6 + 4 = 10$	False

Examples of open sentences and symbols of variables:

$4 + X = 12$ $\qquad \cdot 3 = 18$

$+ 3 = 8$ $\qquad \frac{X}{6} = 5$

$+ \quad =$ $\qquad 1 \quad K \quad 5$

$4 + 9 = X$

EQUALITIES AND INEQUALITIES

The verb of a mathematical sentence may indicate that two quantities are equal or unequal; the sentence, in turn, may be true or false.

The sequence of teaching may approximate the following steps:

1. Sentences of equality that are true
2. Sentences of equality that are false
3. Sentences of inequality that are true
4. Sentences of inequality that are false

$\{\ \}$ **SET** - any well-defined collection of discrete objects. If may contain no elements, only a few elements, or many elements.

$\in$ **ELEMENT or MEMBER** - the individual objects, things in a set.

$\emptyset$ **NULL SET** - the empty set; the set containing no elements.

FINITE SET - A set is finite if it is empty or if it can be counted by a natural number, has an end to numbers.

INFINITE SET - A set is infinite if it is not finite, or a set of numbers that can continue indefinately.

SPECIFYING A SET - by tabulation - simply list the names of all its members
by description or rule - give a name that denotes characteristics of the set

$A \subset B$ SUBSET - Set A is a subset of Set B if and only if Set A is a subset of Set B and at least one element of Set B is not an element of Set A.

DISJOINT SETS - sets that have no elements in common.

UNIVERSAL SET - when a particular set has one or more subsets, the over-all set is generally called the universal set or universe.

$\bar{A};\ A'$ COMPLEMENT OF A SET - If A is a set, then the set of just those things in the universe that do not belong to A is known as the complement of A.

$\sim;\ \leftrightarrow$ EQUIVALENT SETS - sets that can be put into a one-to-one correspondence.

$=$ EQUAL SETS - sets containing the same elements (and no other elements) that can be put into a 1-1 correspondence.

SOLUTION SET - the set whose members make true statements from a condition upon replacement for the variable.

VENN DIAGRAMS - drawings used to show graphically the relationships between sets and elements of sets.

$\cap$ INTERSECTION - the intersection of A and B is the set of elements which belong to both A and B.

$\cup$ UNION - if A and B are two sets, we can form a new set consisting of just those elements that belong either to A or to B or to both A and B.

| - such that, or satisfying the condition that

Set builder notation: $J = \{m \mid m$ is a man and m is more than 21 years of age$\}$

$>$ - is greater than

$<$ - is less than

$>$ - is greater than or equal to

$<$ - is less than or equal to

/ - negates (not) $\neq$

CAN YOU FOLLOW DIRECTIONS?

TIME LIMITS: 15 Minutes

DIRECTIONS: ANSWER THE FOLLOWING QUESTIONS

1. Read everything before doing anything.
2. Put your name in the upper right hand corner of your answer page.
3. Write the word "Name" under you name on your paper.
4. Draw five small squares in the upper left hand corner of your page.
5. Put an "X" in each square.
6. Draw a circle around each square.
7. Sign your name at the bottom of your paper.
8. Under the squares write "Yes, Yes, Yes."
9. Put a circle around each word on your paper.
10. Put an "X" in the lower left hand corner of your paper.
11. Put a triangle around the "X" you have just drawn.
12. On the reverse side of your paper, multiply 703 by 9805. __________
13. Call out your first name when you get to this point in the test.
14. Draw a rectangle around the answer for question 12.
15. If you think you have followed directions up to this point, call out, "I have!"
16. On the reverse side of your paper, add 8950 and 9805. __________

17. Put a circle around your answer.

18. Draw a square around the circle.

19. Count aloud in your normal speaking voice, backwards, from ten to 1.

20. Now that you have finished reading carefully, do only sentence 1 and 2.

TEST FOR JUNIOR THINKERS

Here are some problems that should make you think. Read each problem carefully, then try to solve it. Time is limited to 30 minutes.

1. A clock strokes only the hours. How many strokes will it make in 1 day?

2. In Roman numerals the letter M stands for 1,000. Can you make it mean 1,000,000?

3. How can you measure the thickness of a sheet of paper with an ordinary ruler?

4. After walking four miles in a straight line, Robert discovered he'd gone two miles in the opposite direction. Where was he when he stopped?

5. Mr. Patrick, a farmer, was counting his poultry. He figured that all but one hundred of his birds were chickens, all but one hundred were ducks, and all but one hundred were geese. How many barnyard fowl did Mr. Patrick have?

6. Ninety-five out of every hundred of the earth's inhabitants live in one place; the other five out of a hundred live somewhere else. Can you name the two places?

7. The fencing that encloses a square lot is fixed on upright posts placed ten (10) feet apart. If twenty (20) posts are necessary to fence in the lot, what is its total area?

8. Mike is thinking of a number. If he increases it by 2 and divides it by 2, the result will be the original number. What is the number that Mike is thinking of?

9. When Susan was asked how large her family was, she replied, "I have as many brothers as I have sisters." Her brother, Richard, added. "I have twice as many sisters as brothers." Can you figure how many children are in that family?

10. "Hey, Dad, how old are you?" asked Joe. "Well, son," replied his father, "when I was as old as you are now, I had to wait ten years for you to be born. But when you're two-thirds as old as I am, I'll be twice as old as you'll be." How old is Dad?

Do not write on test. Make answer column. Check.

1. $4[58 - (98 \div 7)] =$

2. $198 - [(4 \times 7) + (81 \div 3)] =$

3. $36 \div 9 + 5 \times 9 - 5 \times 8 =$

4. $45 + [36 - (51 \div 3)] =$

5. $7(\frac{7}{8} - \frac{3}{8}) - 2 =$

6. $(\frac{5}{6} - \frac{1}{3})(\frac{2}{3} - \frac{1}{6}) =$

7. $2.7 + (5.8 + 6.7) =$

8. $22.2 + [2.2 \div (20 + 2)] =$

9. $9\frac{1}{5} - [\frac{3}{10} + (2\frac{1}{2} \times \frac{4}{5})] =$

10. $47 + 4 \times 14 =$

11. $54 - 72 \div 6 + 2 =$

12. $(27 \div 3)(7 + 9) - 65 =$

13. $[21(3 + 4)] \div 7 =$

14. $107 - (\frac{2 + 3}{7 - 2} + 39 \div 13) =$

15. $.5[\frac{1}{4}(7 - 5.4)] \div 1 =$

16. $4[(48 \div 6) - 6] \div 2 =$

17. $\frac{7 + 4}{13 + 9} - \frac{6 - 3}{5 + 1} =$

18. $7(6 + 5) - 12 \times 6 =$

19. $\frac{5 + 9}{7} + \frac{12 \div 4}{3 \div 3} =$

20. $5.3(.7 + .6) =$

21. $6(5 + 9) \div 2 =$

22. $7(4 + 12 \div 3) + 4 =$

23. $\{32 - [40 - (3 \times 8)]\} \div 4 =$

24. $\frac{1}{4}\{([\frac{1}{4} \div (\frac{1}{2} - \frac{1}{4})] + \frac{1}{4}\} =$

25. $6\{.64 + [6.4 - (6 - 4.6)]\} =$

Do not write on test. Make answer column. Check.

1. $4\left[68 - (91 \div 7)\right] =$
2. $189 - \left[(81 \div 3) + (7 \times 4)\right] =$
3. $42 \div 6 + 6 \times 9 - 6 \times 8 =$
4. $54 + \left[31 - (51 \div 3)\right] =$
5. $9(\frac{7}{8} - \frac{3}{8}) - 2 =$
6. $8.5 + (7.2 + 6.7) =$
7. $(\frac{2}{3} - \frac{1}{6})\ (\frac{5}{6}\quad \frac{1}{3}) =$
8. $33.3 + \left[3.3 \div (30 + 3)\right] =$
9. $8\frac{1}{5} - \left[(2\frac{1}{2} \div \frac{4}{5}) + \frac{3}{10}\right] =$
10. $74 + 4 \times 16 =$
11. $56 - 72 \div 6 \times 4 =$
12. $(6 \times 9)\ (27 \div 3) - 56 =$
13. $\left[21(3 + 5)\right] \div 7 =$
14. $98 - \left(42 \div 13 + \frac{3 + 4}{10 - 3}\right) =$
15. $.5\left[\frac{1}{4}(8 - 6.4)\right] + 2 =$
16. $6\left[(48 \div 6) - 6\right] \div 2 =$
17. $\frac{7 + 5}{17 + 7} - \frac{7 - 3}{3 + 5} =$
18. $6(5 + 6) - 12 \times 5 =$
19. $\frac{6 + 8}{7} + \frac{3 \div 3}{12 \div 4} =$
20. $3.7(.7 + .6) =$
21. $8(6 + 9) \div 4 =$
22. $6(5 + 12 \div 3) + 4 =$
23. $\left\{32 - \left[48 - (4 \times 8)\right]\right\} \div 4 =$
24. $\frac{1}{2}\left\{\left[\frac{1}{4} \div (\frac{1}{2} - \frac{1}{4})\right] + \frac{1}{4}\right\} =$
25. $8\left\{.46 + \left[4.6 - (6 - 4.6)\right]\right\} =$

Solve each problem for N, in simplest form. Watch signs. Please make answer column.

1. $\frac{5}{7} + \frac{9}{14} = N$

2. $\frac{11}{12} - \frac{5}{8} = N$

3. $\frac{3}{4} \times \frac{5}{6} = N$

4. $\frac{2}{3} \times 1\frac{1}{4} = N$

5. $6\frac{2}{3} \div 1 \quad = N$

6. $\frac{8}{15} \div 6 = N$

7. $4 - 1\frac{3}{4} = N$

8. $8\frac{1}{5} - 3\frac{1}{3} = N$

9. $6 \div 2\frac{1}{4} = N$

10. $5\frac{1}{15} - 4\frac{2}{5} = N$

11. $2\frac{3}{8} + 4\frac{1}{2} = N$

12. $3\frac{1}{3} \div 2\frac{1}{2} = N$

13. $1\frac{7}{8} \times 4 = N$

14. $18\frac{3}{4} + 4\frac{3}{8} = N$

15. $8\frac{1}{6} - 3\frac{1}{3} = N$

16.
$$\begin{array}{r} 307\frac{3}{5} \\ 546\frac{5}{6} \\ +\ 291\frac{3}{8} \\ \hline \end{array}$$

17. $\frac{7}{16} \times 9\frac{3}{5} = N$

18. $9\frac{3}{16} - 2\frac{1}{4} = N$

19. $5\frac{5}{12} \times 11\frac{1}{9} = N$

20. $\left[5 + (72 \div 8) - 11\right] = N$

21. $\left(6\frac{2}{7} \times \frac{0}{11}\right) \div \left(7\frac{2}{3} + 11\frac{5}{8}\right) = N$

22. $8(9 + 4) = 9(11 - 3) + 8(N)$

23. $N\left[5(2 \times 3) + 10\right] = 120$

24. $\frac{N + 4}{7} = \frac{24}{15 + 6}$

25. $\frac{31 + 9}{5} - 6 = N - 7$

Solve each problem for N, in simplest form. Watch signs.

1. $1\frac{5}{14} - \frac{5}{7} = N$

2. $\frac{5}{8} + \frac{7}{24} = N$

3. $\frac{5}{8} \div \frac{5}{6} = N$

4. $\frac{5}{6} \div \frac{2}{3} = N$

5. $4 \times 1\frac{2}{3} = N$

6. $\frac{4}{45} \times 6 = N$

7. $9 - 2\frac{3}{8} =$

8. $8\frac{1}{5} - 4\frac{1}{3} = N$

9. $2\frac{2}{3} \times 2\frac{1}{4} = N$

10. $4\frac{2}{5} + \frac{2}{3} = N$

11. $6\frac{7}{8} - 2\frac{1}{2} = N$

12. $1\frac{1}{3} \times 2\frac{1}{2} = N$

13. $7\frac{1}{2} \div 1\frac{7}{8} = N$

14. $28\frac{5}{12} - 18\frac{3}{4} = N$

15. $3\frac{1}{3} + 4\frac{5}{6} = N$

16.
$$\begin{array}{r} 703\frac{5}{6} \\ 645\frac{3}{5} \\ 192\frac{3}{8} \\ \hline \end{array}$$

17. $4\frac{1}{5} \div \frac{7}{16} = N$

18. $7\frac{3}{16} - 2\frac{1}{4} = N$

19. $11\frac{1}{9} \times 5\frac{5}{12} = N$

20. $\left[6 + (63 \div 7) - 11\right] = N$

21. $\left(7\frac{3}{8} \times \frac{0}{13}\right) \div \left(7\frac{2}{3} + 11\frac{5}{8}\right) = N$

22. $7(9 + 4) = 9(11 - 4) + 4(N)$

23. $N\left[5(2 \times 3) - 10\right] = 120$

24. $\frac{8}{3 + 4} = \frac{N}{15 + 6}$

25. $\frac{33 + 7}{5} - 6 = N = 5$

Round to the nearest 10

1. 3454.5 2. 7845.6

Round to the nearest 100

3. 3454 4. 6446

Round to the nearest 1000

5. 87654 6. 12345

Round to the nearest $\frac{1}{10}$

7. 29.987 8. 6.543

Round to the nearest $\frac{1}{100}$

9. .008 10. 7.654

Answers

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.

True or False

These are natural numbers?

1. 15%
2. $\overline{IV}$
3. $\frac{14}{2}$
4. $\frac{13}{5}$
5. 4.02

These all name zero?

6. $\frac{2-2}{3}$
7. $\frac{3}{2-2}$
8. $\frac{40-2}{(y+2)(x-x)}$
9. $\frac{18-(4^2+2)}{4(3+2)}$

True or False

10. $(5 \times 3) - 8 = 32 - (9 + 16)$
11. $6 + (7 \times 2) = (6 + 7) \times 2$
12. $15.8 - 8.9 < 7$
13. $17 + \left[8 - (16 \div 4)\right] > 5^2$
14. $98 \div \left[(36 \div 9) + 45\right] > 1$

True or False

15. $\frac{27 \div 9}{4} + \frac{13 - 11}{8 \div 2} < 2$
16. $7 + \left[(2 \times 8) - 9\right] = 32 - (4 + 9)$
17. $(13 + 4)\,(3 - 3) = (9 \times 7) - 63$

Solve

18. $35 + 4N = 107$
19. $6N + 5 = 29$
20. $81 = 9(N + 2)$
21. $\frac{N}{13} = 22$
22. $7N = 98$
23. $\frac{N}{5} - 17 = 13$
24. $\frac{2N}{7} + 4 = 16$
25. $\frac{5N}{8} = 15$

Answers

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.
13.
14.
15.
16.
17.
18.
19.
20.
21.
22.
23.
24.
25.

1. List 12 multiples of 4 ____________________

2. List 12 multiples of 3 ____________________

3. What is the G. C. F. (Greatest Common Factor) of two relatively prime numbers? ____________________

4. What is the L. C. M. (Least Common Multiple) of two relatively prime numbers? ____________________

5. How do we tell whether a number is divisible by 2? ____________________

6. How do we tell whether a number is divisible by 3? ____________________

7. How do we tell whether a number is divisible by 5? ____________________

8. CIRCLE the following pairs of numbers that are RELATIVELY PRIME

12 and 18   6 and 10   15 and 14   60 and 13   60 and 61
2 and 4   9 and 10   24 and 25   21 and 24   15 and 25

9. What number is neither prime nor composite?

Write exponential notation for each of the following

10. 6 · 6 · 6 = ____________   11. 7 · 3 · 3 · 7 · 5 ____________

FACTOR COMPLETELY

12. 175 = ____________________

13. 108 = ____________________

14. 221 = ____________________

FIND THE G.C.F. (greatest common factor) OF EACH GROUP OF NUMBERS. GIVE ANSWER IN FACTORS & SIMPLIFIED

15. 63, 27, 45 ______________________________

16. 700, 420, 1540 ______________________________

17. 17, 39, 20 ______________________________

FIND THE L. C. M. (least common multiple) OF THESE NUMBERS. GIVE ANSWER IN SIMPLEST FORM AND FACTORS.

18. 12, 18, 40 ______________________________

19. 35, 45 ______________________________

20. 6, 12, 18, 24 ______________________________

Make answer column. Replace variables to make true sentences.

1. $8N = 5\frac{1}{3}$

N = ____

2. $7\frac{5}{6} - N = 1\frac{1}{4}$

N = ____

3. $4\frac{2}{3} + N = 10\frac{5}{12}$

N = ____

4. $\frac{x}{1.5} = 5.2$

N = ____

5. 7.7Q = 4928

Q = ____

6. $\frac{N}{5} = 8.3$

N = ____

7. $N - 7\frac{2}{3} = 2\frac{7}{9}$

N = ____

8. N - 76.6 = 8.9

N = ____

9. 5.7A = 5415

A = ____

10. $\frac{N}{4} = \frac{2}{15}$

N = ____

11. $10R = 8\frac{1}{8}$

12. .82T = 64.78

T = ____

13. $\frac{1}{3}A = 35$

A = ____

14. $\frac{1}{6}K = 16$

K = ____

15. X + 32 = 80

X = ____

16. 8R = 58

17. B - 17.2 = 43

B = ____

18. .08A = 2.752

A = ____

18. 08A = 2.752

A = ____

19. $\frac{1}{3}H = 2.8$

H = ____

20. $\frac{A}{13} = 30$

21. $\frac{X}{18} = 54$

22. .06W = 16.5

W = ____

23. N - .9 = 4.8

N = ____

24. 3x = 225

x = ____

25. $K + 12\frac{3}{4} = 39$

K = ____

1. $2000   607.43 =

2. $305.2 \times 4.05 =$

3. $14.543 + 162.98 =$

4. $7.3 \times 1.375 =$

5. $80.87 - 3.9 =$

6. $9.6\overline{)64}$

7. $.003\overline{)6.51}$

8. $\dfrac{2\frac{1}{4}}{4\frac{1}{4}} =$

9. $12\frac{1}{2} + 4 =$

10. $3\frac{1}{2} + 5\frac{1}{4} =$

11. $2\frac{1}{4} \times 4\frac{2}{3} =$

12. $8\frac{1}{3} - 3\frac{3}{4}$

13. $10\frac{1}{2} \times 5\frac{1}{3} =$

14. $\frac{9}{10} \times 8\frac{1}{3} =$

15. $4\frac{1}{2}\overline{)1\frac{1}{3}}$

Change to a common fraction.
Reduce to lowest terms.

16. 0.875 =

17. $\frac{1}{4}\%$ =

18. 350% =

19. 0.0035 =

20. 3.5% =

Change to a decimal fraction.

21. 3.5% =

22. 125% =

23. $\frac{1}{2}\%$ =

24. 4% =

25. $37\frac{1}{2}\%$ =

Change to a percent.

26. 0.875 =

27. $\frac{5}{8} =$

28. .38 =

29. $\frac{3}{5} =$

30. $\frac{4}{7} =$

31. Find the area of a
14 x 16 foot room.

32. What is the perimeter of the room?

33. $3^3 =$

34. 4^5

35. $2^6 =$

36. What is $33\frac{1}{3}\%$ of 39?

37. 15% of what = 20?

38. 25% of 40 = what?

39. What % of 75 = 50?

40. 40% of \$1.75 =

41. 60% of 25 =

42. $\dfrac{\frac{4}{5}}{\frac{2}{3}} =$

43. $4\frac{1}{5}\quad 1\frac{2}{3}$

44. $4\frac{1}{3} + 2\frac{1}{2}$

45. 60% of 25 =

46. $1\frac{1}{5} \times 1\frac{2}{3} \times 3\frac{1}{2} \times \frac{3}{7} =$

47. 5% of \$1.75 =

Follow directions!

1. $\begin{array}{rl} 34 & \text{five} \\ +\ 43 & " \\ \hline & \text{five} \end{array}$ 2. $\begin{array}{rl} 424 & \text{five} \\ +\ 133 & " \\ \hline & \text{five} \end{array}$ 3. $\begin{array}{rl} 424 & \text{five} \\ -\ 133 & = \\ \hline & \text{five} \end{array}$ 4. $\begin{array}{rl} 330 & \text{five} \\ -\ 231 & " \\ \hline & \text{five} \end{array}$ 5. $\begin{array}{rl} 34 & \text{seven} \\ +\ 43 & " \\ \hline & \text{seven} \end{array}$

6. $\begin{array}{rl} 43 & \text{seven} \\ -\ 34 & " \\ \hline & \text{seven} \end{array}$ 7. $\begin{array}{rl} 102 & \text{three} \\ 221 & " \\ +\ 101 & " \\ \hline & \text{three} \end{array}$ 8. $\begin{array}{rl} 202 & \text{three} \\ -\ 121 & " \\ \hline & \text{three} \end{array}$ 9. $\begin{array}{rl} 222 & \text{three} \\ 111 & " \\ +\ 201 & \\ \hline & \text{three} \end{array}$ 10. $\begin{array}{rl} 46 & \text{twelve} \\ +\ 28 & \\ \hline & \text{twelve} \end{array}$

11. $\begin{array}{rl} \text{T1} & \\ +\ \text{1T} & \\ \hline & \text{twelve} \end{array}$ 12. $\begin{array}{rl} 46 & \text{twelve} \\ -\ 28 & " \\ \hline & \text{twelve} \end{array}$ 1

Change to base ten:

13. 2112 three = ____

14. 2112 five = ____

15. 342 five = ____

16. 342 seven = ____

17. 10 twenty = ____

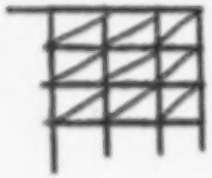
18. 12 twelve = ____

Solve: (they're base ten!)

19. $\begin{array}{r} 2112 \\ 4897 \\ +\ 3204 \\ \hline \end{array}$ 20. $\begin{array}{r} 80012 \\ -\ 34567 \\ \hline \end{array}$ 21. $\begin{array}{r} 346 \\ -\ 897 \\ \hline \end{array}$

22. $\begin{array}{r} 2014 \\ -\ 897 \\ \hline \end{array}$ 23. $\begin{array}{r} 487 \\ \times\ 203 \\ \hline \end{array}$ 24. $\begin{array}{r} 326 \\ \times\ 824 \\ \hline \end{array}$

25. $28 \overline{)7784}$



Why not?

1	five
2	five
3.	five
4.	five
5.	seven
6.	seven
7.	three
8.	three
9.	three
10.	twelve
11.	twelve
12.	twelve
13.	
14.	
15.	
16.	
17.	
18.	
19.	
20.	
21.	
22.	
23.	
24.	
25.	

Part I True and False

1. The numbers which man invented for counting are called whole numbers.
2. The number of pupils in your class is a natural number.
3. Zero is a natural number.
4. $1 + (3 + 4) = (1 + 3) + 4$
5. Letters may be used as symbols to represent unknown numbers.
6. In the statement $(2 + 3) \cdot 6 = (2 \cdot 6) + (3 \cdot 6)$ we may interchange the "+" and the "·" and write $(2 \cdot 3) + 6 = (2 + 6) \cdot (3 + 6)$.
7. $(4 \cdot 2) \cdot 3 = 4 \cdot (2 \cdot 3)$.
8. The set of counting numbers is closed with respect to all arithmetic operations.
9. $(5 - 3) - 2 = 5 - (2 - 3)$.
10. The statement $73 \cdot 25$ may be written as $(75 + 5) \cdot (20 + 3)$.
11. The sum of $\underline{a}$ plus $\underline{b}$ plus $\underline{c}$ is equal to the sum of $\underline{b}$ plus $\underline{c}$ plus $\underline{a}$.
12. $(20 \cdot 5) + 4 = 20 \cdot (5 + 4)$
13. If the sum of $\underline{a}$ and $\underline{n}$ is equal to $\underline{n}$ then $\underline{a}$ must be equal to $\underline{n}$.
14. The set of numbers which are multiples of 3 is closed with respect to multiplication.
15. When we say the set of counting numbers is closed with respect to addition, we mean the sum of two counting numbers is always a counting number.
16. If $c + 2d + 3 = 13$ and $d = 5$, then a must equal zero.
17. Putting shoes and putting on stockings is an example of the commutative property.
18. Parentheses do not change a problem when the operation is associative.

Part II Multiple Choice

1. The sum of any two natural numbers...
   - A. is not a natural number.
   - B. is sometimes and natural number.
   - C. is always a natural number.
   - D. is a natural number equal to one of the numbers being added,
   - E. none of these.
2. If a and b are whole numbers then a + b = b + a is an example of
   - A. commutative property
   - B. associative property
   - C. distributive property
   - D. closure
   - E. none of these
3. Chose the number from the following which is not a counting number.

   A. $\frac{2}{1}$ B. 56 C. 7 D. 1 E. none of these
4. "If a ticket to the movies cost 50 cents, how many could you buy for a quarter?
   - A. 50 cents divided by 25 cents = 2.
   - B. 25 cents divided by 50 cents = one half.
   - C. 50 cents divided by zero = not possible.
   - D. Zero divided by 50 cents = infinity.
   - E. Zero tickets because half a ticket would have no value.

5. In which of the following numbers is zero NOT a placeholder?

A. 110 B. 101 C. 1.01 D. 0.11 E. 10.1

6. Solve the following without doing the computation. "If the number named by (987 x 654) + (987 x 473) is divided by 987 the remainder would be..."

A. zero

B. 987

C. a number between 473 and 654

D. the sum of 654 and 473

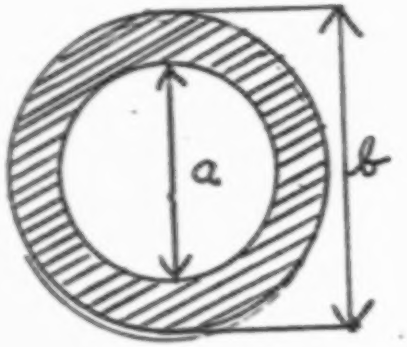
E. none of these

I. In each blank space at the right write the letters of all formulas from Column B which match the item in Column A

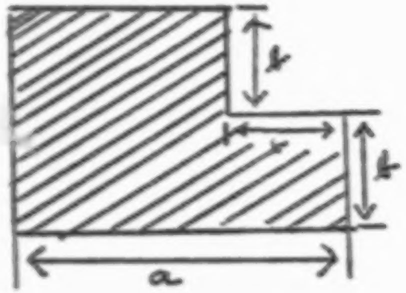
Column A	Column B	Answers
1. Area of a triangle	a. $a = \pi \cdot d$	1. ____________
2. Area of a rectangle	b. $A = \pi \cdot r^2$	2. ____________
3. Area of a trapezoid	c. $A = 1 \cdot w$	3. ____________
4. Area of a circle	d. $A = \frac{1}{2} \cdot a \cdot b$	4. ____________
5. Area of a parallelogram	e. $A = \frac{1}{2} \cdot a \cdot (b_1 + b_2)$	5. ____________
	f. $A = a_2 \cdot b$	
	g. $A = s^2$	

II. In each blank space at the right write the letter of the formula from Column B which may be used to determine the area of each shaded region in the following figures.

1.



2.



a. $A = a \cdot b = b^2$

b. $A = a^2 - b^2$

c. $A = 2 \cdot a \cdot b - b^2$

d. $A = a \cdot b - 2 \cdot b^2$

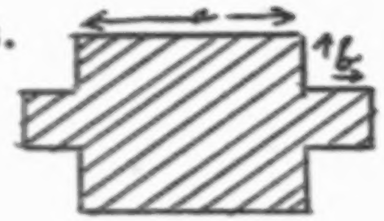
e. $A = \frac{\pi \cdot a^2}{2} - \frac{\pi \cdot b^2}{2}$

f. $A = 3 \cdot a \cdot b - 4\ b^2$

g. $A = 4 \cdot a \cdot b - 4 \cdot b^2$

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________

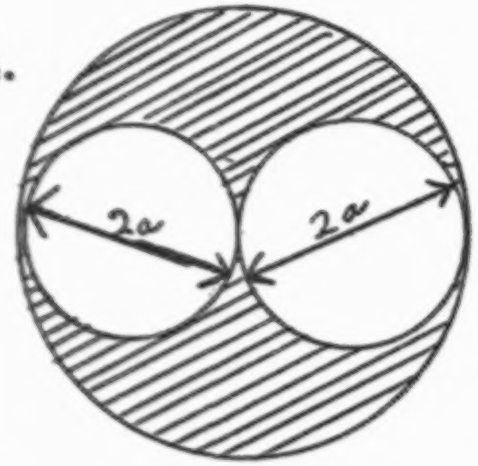
3.



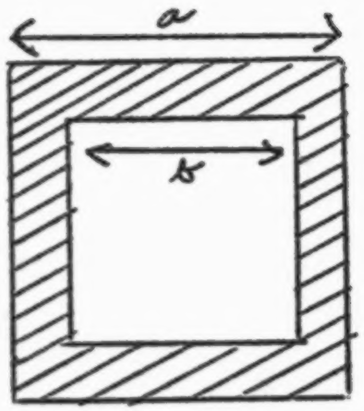
h. $A = \frac{\pi \cdot a^2}{2} - \frac{\pi \cdot b^2}{4}$

i. $A = 2 \times a^2$

4.



5.



III. In the blank space at the right write a response that will make a true statement.

1. The area of all one-dimensional figures is ____ 1. ____________

2. In a right triangle the side opposite the right angle is called the ____ 2. ____________

3. The Pythagorean property for any right triangle is ____ where c is the length of the hypotenuse and the legs have lengths a and b. 3. ____________

4. If a polygon has all sides with the same measure and all angles with the same measure, it is a (an) ________ polygon. 4. ____________

5. The angle determined by two radii of a circle is called a (an) ___ angle. 5. ____________

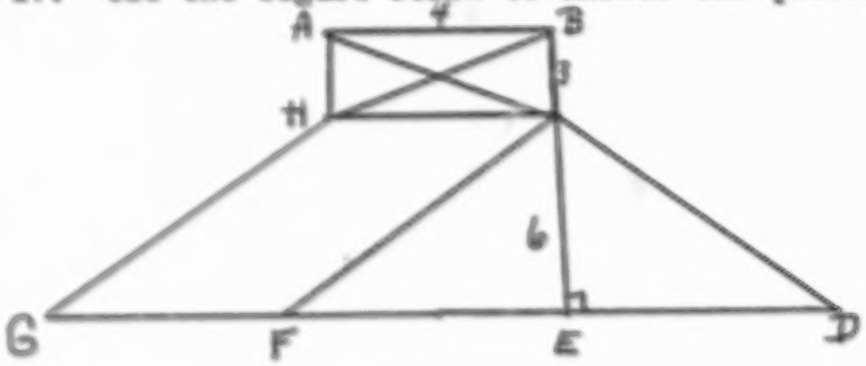
6. The measure of a two dimensional figure is called its.______. 6. ____________

7. In a triangle, the segment perpendicular to a base from the opposite vertex is called a (an) __________ 7. ____________________

8. For any polygonal region the polygon itself is called the ______________ of the region. 8. ____________________

9. If a base of a triangle has the same measure as the longer side of a parallelogram and if the altitude has the same measure as the altitude of the parallelogram, the area of the triangle is ___________ that of the parallelogram. 9. ____________________

IV. Use the figure below to answer the questions which follow.



1. What is the area of the rectangular region ABCH? 1. ____________________

2. What is the area of the interior of parallelogram HCFG? 2. ____________________

3. What is the area of the interior of trapezoid HCDG? 3. ____________________

4. What is the area of the triangle region CDF? 4. ____________________

5. What is the measure of $\overline{HB}$? 5. ____________________

V. Do the following problems and write the answers in the spaces provided at the right.

1. What is the length of a diagonal of a rectangle whose sides measure 5 feet and 12 feet respectively? 1. ____________________

2. A 10 foot ladder is placed against the side of a building. The top of the ladder is 8 feet above the ground. How far from the side of the building is the foot of the ladder is the ground is level?

2. ____________________

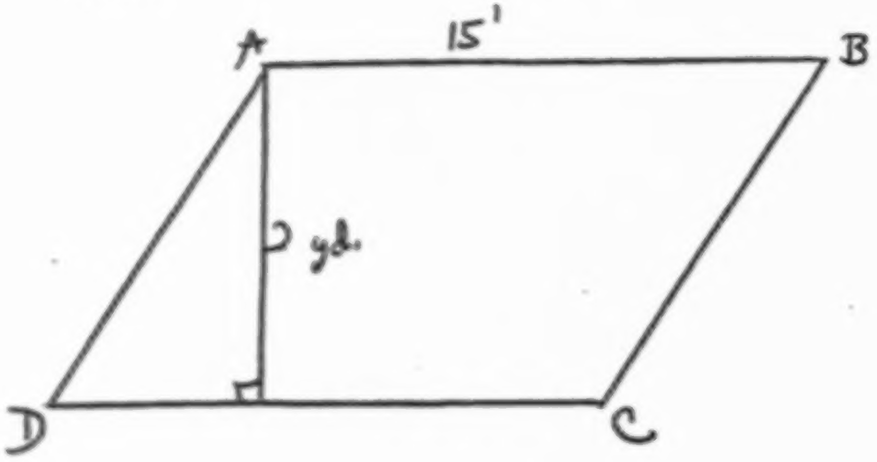
3. Mr. Smith has a portable swimming pool on his lawn. It has a circular base with a diameter measuring 15 feet. Approximately how many square feet of lawn does Mr. Smith lose? ($\pi = \frac{22}{7}$)

3. ____________________

4. A punch press punches out circular discs with 7 foot diameters. How much would one disc cost at 40¢ per square foot? ($\pi = \frac{22}{7}$)

4. ____________________

VI. Find the area of the interior of the following parallelogram in each of the units shown at the right.



1. ________________ sq. ft.
2. ________________ sq. yd.
3. ________________ yd. ft.

** In the blank space at the right write a word of symbol that would make a true statement.

ANSWERS

1. "The second power of the third power of the third power of 4" is the way we read the symbol ________. 1. ________

2. The simplest name for $(a^{-1})^{-1}$ is ______ 2. ________

3. To simplify $(x^a)^b$ we ______ the exponents. 3. ________

4. The simplest exponential notation for $x^2 \; y^3$ is ____. 4. ________

** In the blank space at the right write the letter of the item from column B which matches each item in Column A.

Column A	Column B	
5. $x^a \; x^b$	a. $x^{-a \; b}$	5. ________
6. $x^{-a} + y^{-b}$	b. 1	6. ________
7. $(x^{-a})^b$	c. x^{a+b}	7. ________
8. $x^a \; x^{-a}$	d. $x^{a \; b}$	8. ________
9. $x^a \; x^{-a}$	e. $\frac{y^b}{x^a}$	9. ________
	f. $x^{(-a) \; (-b)}$	
	g. x^{a-b}	

** Write the simplest exponential notation for each of the following

10. $\frac{8^6}{8^2}$ 10. ________

11. $\underline{6^5 \times 6^5} \times \underline{6^3}$ 11. ________

12. $\frac{4^0 \times 3^2 \times 5^4 \times 3 \times 4}{4 \times 5^3 \times 5^5}$ 12. ________

13. $(3^{-4})^{-2} x (4^{-3})^2 x 3^2 x 4^8$

14. $\frac{(a^3)^4 \ x \ (b^3)^5 \ x \ a^3 \ x \ b^5}{(a^5)^4 \ x \ (b^{-2})^3}$ $a \neq 0, b \neq 0$

15. $(m^5)^2 \ x \ (n^{-3})^2 \ x \ (m^2)^{-3}$ $m \neq 0, n \neq 0$

13.________________

14.________________

15.________________

**Perform the following operations. Write the simplest name for each result.

16. ${}^-13 + {}^-19 + 17 - 21 - {}^{-31}$

17 $|{}^-22| + |300\%| - 700\%$

18. ${}^-12 + {}^{-15} - {}^{-55} + {}^{-20} - {}^{-4} + 15 + {}^{-13} + 16$

16.________________

17.________________

18.________________

Write the formulas for finding the areas of the following.

19. Triangle____________ 20. Circle____________ 21. square____________

22. Parallelogram______________ 23. trapezoid______________

24. rectangle________________

5 pts. 1. The Y-coordinate of the point $(3, {}^{-}7)$ is ____________________.

5 pts. 2. $({}^{-}4, {}^{-}3)$ names a point located in the __________________ quadrant.

5 pts. 3. Each point in the coordinate plane has __________________ numbers associated with it.

5 pts. 4. a) Find c^2 when $c^2 = a^2 + b^2$ and $a = 6$ and $b = 9$. ________________________

5 pts. b) How can you represent the value of c in part (a)? _________________

5 pts. 5. The graph of $y = 2x$ is a ________________.

10 pts. 6. The point of intersection of $y = x$, $y = 2x$, $y = 3x$ is __________________.

5 pts. 7. The distance between $(3, {}^{-}4)$ and $({}^{-}7, {}^{-}4)$ is _________________.

5 pts. 8. The hypotenuse of a right triangle is 25 units in length, and one side is 24 units in length. The third side is __________________ units in length.

15 pts. 9. State the Pythaogrean Property in words (do not use "a", "b", or "c".)

____

____

5 pts. 10. The graph of y x is ____

5 pts. 11. Find the distance between the points whose coordinates are (1.3) and (6.15). __________

5 pts. 12. The coordinates of a point which lies in Quadrant IV are _________________________

20 pts. 13. Graph the two equations below and give their point of intersection.________________

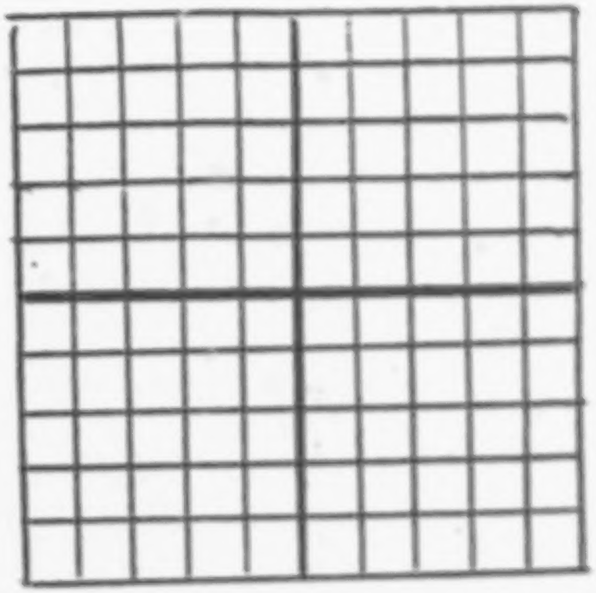
$2X = Y + 1$ $2Y = X$

$2X = Y + 1$

X						
Y						

$2Y = X$

X						
Y						



1. What is the name given to a set of points made up of 2 points and all the points between them? Draw an example.

2. How many points determine a ray? Draw one.

3. How many points determine a segment? Draw one.

4. What is the difference between a closed segment and an open segment?

5. What is a half line? Draw one.

6. What is the endpoint of ray $\overrightarrow{EF}$?

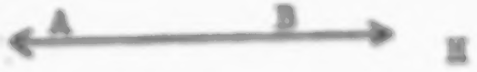
7. What must be added to a half line to make a line?

What is the intersection of the following sets:

8. The set of natural numbers and the set of whole numbers?
9. The set of odd whole numbers and the set of even natural numbers?

Use this diagram and give the intersection of each of the following:

10. Line H and segment $\overline{AB}$



11. Ray $\overrightarrow{AB}$ and ray $\overrightarrow{BA}$

12. The half line to the right of A and the half line to the left of A.

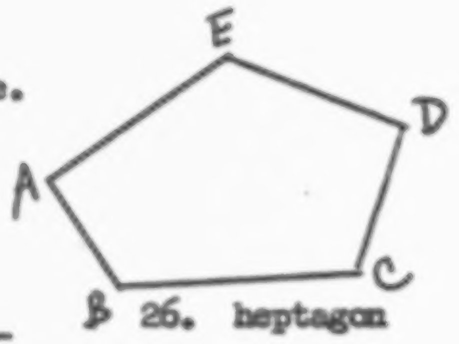
13. Point A and segment $\overline{AB}$.

14. A point separates a line into how many sets? Name them.

15. A line separates a plane into how many sets? Name them.

16. A plane separates space into how many sets? Name them.

18. A non-simple closed curve looks like ??????? Draw one.



19. Name 2 adjacent sides of this polygon.
Tell how many sides and diagonals each polygon has:

20. triangle __________

21. quadrilateral ______

22. pentagon _____, ______

23. nonagon _____, _____

24. decagon _____, _____

25. hexagon _____, _____

26. heptagon

27. dodecagon

28. octagon

29. What is a concave polygon (draw one) and a convex polygon (draw one).

30. Two rays with a common endpoint make up what kind of geometric figure?

Name ____________________
Period____

Part 1. (matching)

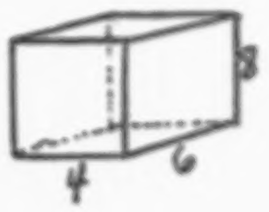
________1. Cylinder. area
________2. Triangle, area
________3. Cone, volume
________4. Cylinder, volume
________5. Parallelogram, Perimeter
________6. Parallelogram, area
________7. Pyramid, volume
________8. Trapezoid, area
________9. Circle, circumference
________10. Circle, area

a. $A = 2\pi r(r+h)$
b. $V = Bh$
c. $V = \frac{1}{3} Bh$
d. $A = \frac{1}{2}bh$
e. $V = \frac{1}{3} Bh$
f. $V = \frac{4}{3}\pi r^2$
g. $P = 2(l + w)$
h. $A = bh$
i. $S = 4\pi r^2$
j. $A = r^2\pi$ or πr^2
k. $C = \pi 2r$
l. $A = \frac{1}{2}h(B + b)$

Pi = 3.14

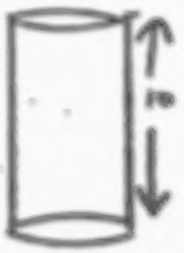
Part 2. Find the volume of each figure.

1. ________ cu. in.

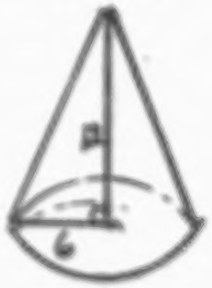


2. ________ cu. in.

Radius = 3

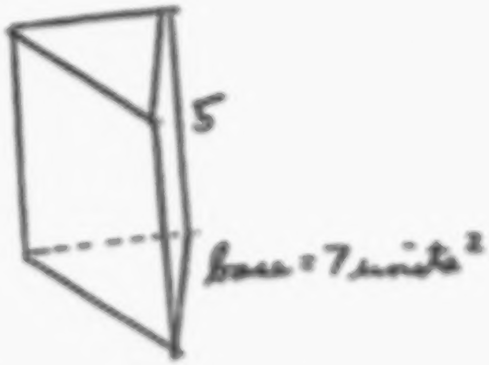


3. ________ cu. ft.

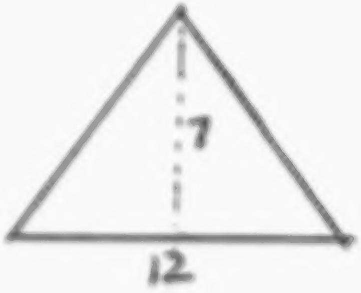


Find area of each figure:

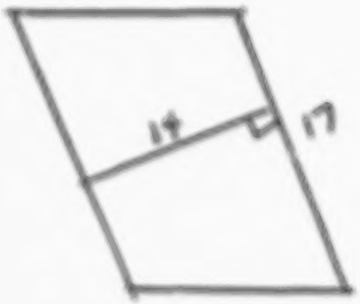
4. ________ cu. ft.



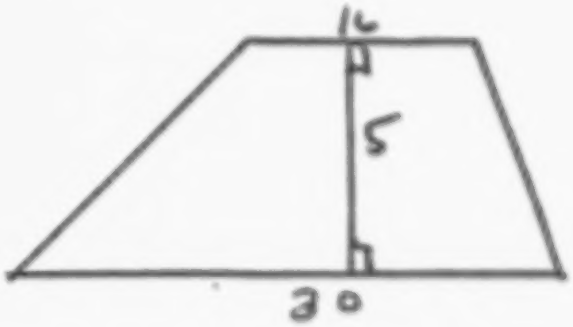
5. ________ sq. in.



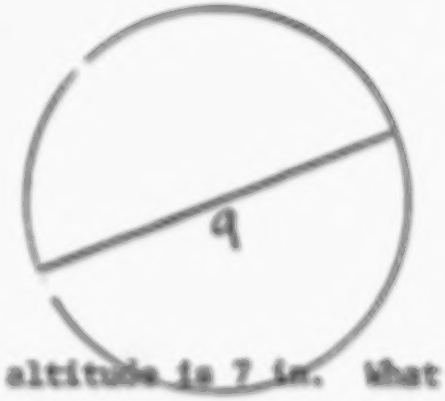
6. ________ sq. in



7. ________ sq. ft.



8. ________ sq. ft.



9. The bases of a trapezoid of 6 in. and 12 in. and its altitude is 7 in. What is its area? ____

10. One side of a triangle is 16 in. long and the altitude to that side is 15 in. long. Find the area of the triangle. ______

11. A circle has a radius of 6 in. Find the circumference. ________

12. Find its area.________

13. The radius of a cylinder is 3 in. and its altitude is 10 in. What is its volume??________

14. Its area? ________

15. By how much is the volume of a cylinder changed if the radius and the altitude are both tripled?? ________

DO NOT WRITE ON TEST.   Make Answer column.   Watch signs.

1) $2' + 4^3 =$ ____   2) $5^2 \times 5^4 \times 5^0 =$ ____   3) $10^7 - 10^5 =$ ____   4) $(4.8)^2 \times (.5)^2 =$ ____

5) 4 = ___% of 76   6) 15 = 125% of ____   7) 16 = __% of 10   8) 69 = __% of 76

9) 75% of ___ = 15   10) 48 = 25% of ___   11) 6% of $49.50 = ___   12) 120% of $5.95 = ____

13) 105% of 212 = ____

14. Find the volume of cube; each side is 5 in.
15. Find perimeter of square; each side is 7 in.
16. Find volume of box for biscuits; measures 3" by 5" by 10".
17. Find perimeter of equilateral triangle; sides are 8 in.   Area of = r^2; Circumference = d;
    So: $3\frac{1}{7}$
18. Find area of   if r = 7 in.
19. Find area of   if d = 14 ft.
20. Find circumference; if r = $3\frac{1}{2}$ ft.
21. Find area of square; 8 in. per side.
22. Find area of rectangle;   7"   3"

23. x   40   30   18   32   24   Find x

24. $\frac{24}{16} = \frac{72}{x}$ x = ?

25. $\frac{x}{3} = \frac{16}{9}$ x = ?

26. $\frac{x}{4} - 32 + 56 = 25$ x = ?

27. $\frac{x}{6} + 24 = 84$ x = ?

28. $4x + \frac{1}{2}x = 27$ x = ?

29. If $\frac{1}{16}$" = 1 ft., how much does $1\frac{5}{8}$ represent on a map?

30. If Joe earns $25.00 for working 20 hours after school, how much will he earn for 60 hours work.

31. What is the cost of a 9 x 12 carpet at $7.95 per square yard?

32. If 4 pounds of meat can serve 9 people, how much could serve 15?

33. Mary is 13 years old. Her age is 3 years less than 1/3 of her father's age. How old is her father.

1. If 275 persons came to see a play and used just ½ of the seats in the theater, how many persons could have been seated?__________

2. Mr. Adams earns $7500 a year. If 5/6 of his salary is used for family expenses, what amount is used for these expenses?__________

3. John saved $44 and spent 3/4 of it for Christmas gifts. How much money did he spend for the gifts? __________

4. At $2.25 a week, how many weeks will be required for a family to pay the $94.50 balance due on a television set? __________

5. Mrs. Martin told Joan that they had driven about 3/4 of the way to her grandmother's house. They had driven 348 miles. About how far is it from Joan's to her grandmother's?__________

6. How much will 24 boxes of candy at $1.20 a box cost?__________

7. Mr. Alfred's average driving speed is 45 miles an hour. How long should it take him to drive 855 miles?__________

8. At $1785.00 each, what will be the cost of a fleet of 25 new cars?__________

9. Last year Mr. Dorn worked 239 days and earned $4780. What were his average daily earnings?__________

10. Mr. Alt bought 4 new tires for his car; the total cost was $115.88. What was the price per tire? __________

11. Mr. Hart earned $120 at a wage of $3 per hour. How many hours did he work that week?__________

12. At a price of $35 each, how many electric motors can a dealer buy for $525?__________

13. Jack has a part time job that pays him $18 a week. How much could he earn in 52 weeks?__________

14. In 1919 a plane flew across the Atlantic Ocean in about 16 hours. If the distance is about 1960 miles, what was its average speed per hour? __________

15. If a stock car completes a 500 mile race in 3½ hours, what will be its average speed in miles per hour?__________

16. Find the cost of 12 women's coats at $59.95 each.______________

17. At 14½¢ per gallon, what is the cost of 3775 gallons of fuel oil?______________

18. Mr. Bell drove 50 miles in 1¼ hours. What was his average speed per hour?______________

19. Mrs. Cearhs bought a 3½ pound piece of beef for 6 persons. What fraction of a pound per person will it provide?______________

20. Mr. Tee bought a $275 TV set at a sale. The set was marked down 1/5 of the price. How much did he pay?______________

NAME

WORK PROBLEMS HERE

NAME ______________________ PERIOD________ SCORE________

Which number is named by the following expressions? ANSWERS PUT BELOW PROBLEMS

(1) 10^4

(2) $2^3 \cdot 5$

(3) 7^3

(4) $1^5 \cdot 3^2 \cdot 10^4$

(5) $2^3 \cdot 3 \cdot 5$

Write the prime factors of each of the following numbers. If the number is prime, write PRIME for your answer. Use exponential notation where possible.

(6) 120

(7) 253

(8) 252

(9) 400

(10) 89

Find the greatest common factor of each set of numbers below:

(11) 15, 21
(12) 5, 15, 25
(13) 24, 32, 16
(14) 5, 11
(15) 10, 12, 15
(16) 27, 36, 81
(17) 42, 63, 91
(18) 36, 60, 210
(19) 60, 140, 180, 22
(20) 20, 32, 48, 56

ANSWERS

1.________
2.________
3.________
4.________
5.________
6.________
7.________
8.________
9.________
10.________
11.________
12.________
13.________
14.________
15.________
16.________
17.________
18.________
19.________
20.________

DO NOT WRITE ON TEST.

Simplify. Be Careful

1. $\frac{7}{8} + \frac{3}{4}$

2. $4\frac{1}{2} + 9\frac{5}{16} + \frac{3}{8}$

3. $1\frac{1}{3} + 2\frac{1}{2} + 3\frac{5}{9}$

4. $(\frac{7}{20} + \frac{0}{9}) + \frac{7}{15}$

5. $\frac{1}{5} + \frac{2}{3} + \frac{7}{10}$

6. $3\frac{5}{9} - 1\frac{1}{3}$

7. $\frac{17}{36} - \frac{7}{12}$

8. $3\frac{7}{36} - \frac{7}{9}$

9. $\frac{5}{18} - \frac{5}{12}$

10. $57\frac{11}{21} - 29\frac{5}{28}$

11. $\frac{3}{4} \times \frac{4}{9} \times \frac{3}{5}$

12. $8\frac{1}{4} \times 3\frac{1}{3} \times 3\frac{1}{5}$

13. $2\frac{2}{5} \times 1\frac{1}{4} \times 2\frac{1}{3}$

14. $15 \times 3\frac{1}{3} \times 4\frac{1}{5} \times 10\frac{3}{10}$

15. $2\frac{2}{9} \times 3\frac{1}{3} \times 2\frac{1}{2} \times 1\frac{2}{5} \times 2\frac{7}{10} \times \frac{9}{10}$

16. $\frac{7}{8}\,\big)\overline{\frac{5}{8}}$

17. $\frac{2}{3} \div 8$

18. $3\frac{1}{3} \div 19\frac{1}{2}$

19. $\dfrac{\frac{0}{11}}{4\frac{1}{5}}$

20. $1\frac{2}{7} \div \frac{2}{3}$

21. $(\frac{5}{8} \div 7\frac{1}{2}) \times \frac{2}{3}$

22. $(\frac{6}{7} - \frac{5}{14}) \times \frac{0}{7} =$

23. $(\frac{1}{8} + \frac{3}{4}) \div 5\frac{1}{3}$

24. $-\frac{2}{3} \times (\frac{1}{4} + \frac{5}{6})$

25. $(5 \times 3\frac{1}{3}) \div \frac{5}{21}$

NAME________________________________ SCORE_________ PERIOD______

(1) Add:
82,162
48,279
14,086
53,328
13,497
62,173

(2) Subtract:
415,037
316,287

(3) Multiply:
786
598

(4) Divide:
85)71,995

(5) Add:
6 9/10
4 3/5

(6) Subtract:
7 3/8
3 5/6

(7) Multiply:
6 3/4 x 4 1/6

(8) Divide:
84 ÷ 2 5/8

(9) Reduce 48/64 to lowest terms

(10) What part of 90 is 75?

(11) Which is larger: 2/3 or 7/12?

(12) Add: .96 + .096 + 9.6

(13) Subtract: 8.3 - .49

(14) Multiply: 2.08 x .15

(15) Divide:
.14) .006

(16) Find 4% of $5.00

(17) Find 150% of $264

(18) Find 133 1/3 % of $600

(19) Solve for N: $\frac{N}{.02} = \frac{N}{10}$

(20) $\frac{.005}{N} = \frac{1.4}{.28}$

(21) $\frac{N}{.01} = \frac{.2}{.16}$

(22) $\frac{9}{1\frac{1}{4}} = \frac{N}{10}$

(23) $\frac{21}{N} = \frac{7/10}{3/5}$

(24) $\frac{108}{243} = \frac{N}{27}$

(25) $\frac{56}{105} = \frac{N}{15}$

ANSWERS

1.__________
2.__________
3.__________
4.__________
5.__________
6.__________
7.__________
8.__________
9.__________
10.__________
11.__________
12.__________
13.__________
14.__________
15.__________
16.__________
17.__________
18.__________
19.__________
20.__________
21.__________
22.__________
23.__________
24.__________
25.__________

REVIEW TEST

DIRECTIONS: Choose carefully the best answer and circle it on the answer sheet. Do not write on test.

Choose the way you would write each number.

1. Seven hundred fifty: (a) 507 (b) 705 (c) 750 (d) 7,050 (e) 7,005

2. Thirteen thousand two hundred seventy-one. (a) 130,271 (b) 13,721 (c) 103,271 (d) 13,271 (e) none of these

Round off the numbers as indicated:

3. 36 (to the nearest ten). (a) 35 (b) 300 (c) 30 (d) 40 (e) 360

4. 999 (to the nearest ten). (a) 99 (b) 990 (c) 900 (d) 9,000 (e) 1,000 (f) none of these

5. 1,989 (nearest hundred). (a) 2000 (b) 1980 (c) 1990 (d) 1890 (e) 1900 (f) none of these

6. 4124,527 (nearest hundredth). (a) 4124,528 (b) 4100 (c) 4124.52 (d) 4124,53 (e) none of these

From the examples given, choose the number that is the example of the word.

$$\begin{array}{r}11\\+11\\\hline 22\end{array}\qquad\begin{array}{r}19\\-13\\\hline 6\end{array}\qquad\begin{array}{r}20\\\times 44\\\hline 80\\80\\\hline 880\end{array}$$

$$16\overline{)32}^{\,2}$$

$$7/9\qquad 10\tfrac{1}{3}\qquad 8/5$$

7. subtrahend: (a) 13 (b) 20 (c) 11 (d) 19 (e) 44

8. quotient: (a) 32 (b) 16 (c) 880 (d) 2 (e) 6
9. partial product: (a) 80 (b) 880 (c) 22 (d) 6 (e) 16
10. addend: (a) 19 (b) 13 (c) 11 (d) 20 (e) 44
11. difference: (a) 22 (b) 880 (c) 2 (d) 6 (e) 32
12. numerator: (a) 9 (b) 8 (c) 5 (d) 4 (e) none
13. divisor: (a) 16 (b) 32 (c) 2 (d) 44 (e) 13
14. multiplicand: (a) 19 (b) 13 (c) 20 (d) 11 (e) 44
15. mixed number: (a) 7/9 (b) 8/5 (c) 880 (d) 22 (e) none

Choose the answer.

16. 6 + 9 + 5 + 4 = (a) 25 (b) 23 (c) 22 (d) 26 (e) none

3

17. 34 ÷ 87 + 45 + 71 = (a) 273 (b) 227 (c) 237 (d) 291 (e) none

18. 285 - 132 = (a) 417 (b) 153 (c) 143 (d) 53 (e) none

19. 1004 - 627 = (a) 377 (b) 487 (c) 1631 (d) 477 (e) none

20. 42 x 23 = (a) 210 (b) 96 (c) 126 (d) 966 (e) none

21. 804 x 465 = (a) 339 (b) 1269 (c) 373,860 (d) 40,274 (e) none

22. 678 x 809 = (a) 60,342 (b) 548,502 (c) 869 (d) 1,457 (e) none

23. 80 - 40.1 = (a) 49.9 (b) 9.9 (c) 120.1 (d) 39.9 (e) none

24. 36 x 24¢ = (a) 86.4 (b) $8.64 (c) $864 (d) 86¢ (e) none

Second Semester Examination

Part I - Questions 1 - 30 are True-False. Use Column A for True and Column B for False. If a statement is not always true, then it is false.

1. The complement of an acute angle is an obtuse angle. (F)
2. The supplement of an acute angle is an obtuse angle. (T)
3. The complement of a right angle is a 0° angle. (T)
4. The supplement of an obtuse angle is an obtuse angle. (F)
5. Triangles are classified by sides and angles. (T)
6. All triangles are polygons. (T)
7. All polygons are quadrilaterals. (F)
8. A quadrilateral with four right angles is a square. (F)
9. A right triangle has two right angles. (F)
10. $9 + 8 - 7 = 1 + 4 + 5 + 1$ (F)
11. $8 + 4 - 3 = 3 + 6$ (T)
12. $6 + 4 \cdot (0) = 5 + 1$ (T)
13. $\frac{8 - 8}{3 + 5} = 8$ (F)
14. $13 \cdot \frac{(13)}{(13)} = 13$ (T)
15. $7 + 2 < 5 + \frac{8}{2}$ (F)
16. $2\frac{3}{2} > 8 - 2$ (F)
17. $15 = \sqrt{200 + 20 + 5}$ (T)
18. The basic metric unit for measuring weight is the gram. (T)
19. 33 is $33\frac{1}{3}\%$ of 88. (F)
20. 15 is 75% of 20. (T)
21. The L.C.M. of 25 and 15 is 375. (F)
22. The members of the set of all geometric figures are points. (T)
23. The G.C.F. of 14 and 35 is 7. (T)
24. $-x \cdot y = x \cdot -y$ (T)
25. $-(x + y) = -x + y$ (F)
26. 4500 cwt = 225T (T)
27. A convex polygon is a polygon with at least one of its diagonals on the outside. (F)
28. The product of two prime numbers is a composite number. (T)

29. $\sqrt{38 - 2}$ is another name for a whole number. (T)
30. $1006 = 1 \cdot 10^3 + 6 \cdot 10^1$ (F)

Part II - Questions 31 - 50 are multiple choice. Select the best answer from those given, except that if no correct answers are given you may mark answer space "E".

31. $\frac{11}{12} = \frac{n}{144}$ Solve for n

A. 86
B. 140 (C)
C. 132
D. 48

32. If the least count of a ruler is $\frac{1}{16}$", then the greatest possible is:

A. $\frac{1}{32}$"
B. $\frac{1}{16}$" (A)
C. $\frac{1}{8}$"
D. $\frac{1}{2}$"
E. None of these

33. Which of the following states that A is a subset of B.

A. $A \cap B$
B. $A \cup B$ (C)
C. $A \subset B$
D. $A \rightarrow B$
E. None of these

34. For any non-zero number of arithmetic $\frac{a}{b}$, we know that $\frac{a}{b} \div \frac{2a}{2b}$ is:

A. 1
B. $\frac{a}{b}$ (E)
C. 2
D. 4
E. $\frac{1}{2}$

35. If $m = y \cdot y$, then y is one of the following:

A. The inverse of itself
B. A square root of m
C. The identity element (B)
D. An operation
E. None of these

36. Which of the following is a prime number?

A. 39
B. 51 (E)
C. 57
D. 87
E. None of these

37. One of the following is NOT a name for the number zero:

A. $0 \div 5$
B. $0 \cdot N$
C. $\frac{m - m}{10}$ (E)
D. $2(3 - 3)$
E. $\frac{6}{0}$

38. One of the following is NOT a name for the number one:

A. $\frac{m + 1}{m + 1}, m \neq -1$
B. $\frac{n - n}{3 - 2}$
C. $\frac{1}{6 - 5}$ (b)
D. $\frac{400\%}{3 + 1}$
E. $\frac{3 - 2}{1 + 0}$

39. If a sentence is an equation, then it has one of the following for its verb:

A. >
B. < (C)
C. =
D. Any of the first three
E. Any variable

40. The set of whole numbers is closed under:

A. Division
B. Multiplication
C. Taking square root (B)
D. Subtraction
E. None of these

41. A man is saving $1500 at 5% per annum. If the interest is compounded semi annually at the end of the year he will have:

A. $1575
B. $1537.50 (C)
C. $1575.93
D. $75
E. None of these

42. The sum of the measures of the angles of a triangle is:

A. 90 °
B. 360° (E)
C. 45°
D. 540 °
E. None of these

43. A geometric figure consisting of two rays with a common end point is called:

A. An angle
B. A line segment (A)
C. Another ray
D. A line
E. None of these

44. If a number has a multiplicative inverse, then the inverse can be named by:

A. ^{-}n
B. $\frac{1}{n}$ (B)
C. $\frac{1}{\frac{1}{}}$
D. $^{-}\left(\frac{1}{n}\right)$
E. None of these

45. One of the following is NOT an example of a variable:

A. =
B. X
C. □ (A)
D. △
E. None of these

46. One of the following is NOT an example of using a cardinal number:

A. There were three boys.
B. The score was 2 to 1.
C. He was present three times. (D)
D. They are in seventh place.
E. There are nine weeks before vacation.

47. If an instrument has a smaller count than another, we know:

A. It is more accurate.
B. It has more tolerance.
C. It has smaller relative error. (D)
D. It is more precise.
E. None of these.

48. The G.C.F. of 9 and 37 is:

A. 153 C. 17 E. None of these (D)
B. 9 D. 1

49. If a and b are whole numbers and $a < b$, then the number of whole numbers between a and b is:

A. $b - a$
B. $b - (a + 1)$
C. $(b + 1) - (a + 1)$ (A)
D. $(b + 1) - a$
E. None of these

50. One of the following is NOT a name for a natural number.

A. VIII
B. 11_3
C. $1 . 1_5$
D. $10 \div 5$
E. 100%

Make answer column. Replace variables to make true sentences.

1. $8\ N = 5\frac{1}{3}$
   N = ________

2. $7\frac{5}{6} - N = 1\frac{1}{4}$
   N = ____

3. $4\frac{2}{3} + N = 10\frac{5}{12}$
   N =

4. $\frac{x}{1.5} = 5.2$
   x = ______

5. $7.7\ Q = 4928$
   Q = ________

6. $\frac{}{5} = 8.3$
   N = ________

7. $N - 7\ 2/3 = 2\ 7/9$
   N = ________

8. $N - 76.6 = 8.9$
   N = ________

9. $5.7\ A = 5415$
   A = ______

10. $\frac{N}{4} = 2/15$
    N = ________

11. $10\ R = 8\frac{1}{3}$
    R = ____

12. $.82\ T = 64.78$
    T = ______

13. $\frac{1}{3}A = 35$
    A = ______

14. $\frac{1}{6}K = 16$
    K = ______

15. $X + 32 = 80$
    A = ______

16. $8\ R = 58$
    R = ______

17. $B - 17.2 = 43$
    B = ________

18. $.08\ A = 2.752$
    A = ________

19. $\frac{1}{3}H = 2.8$
    H = ________

20. $\frac{A}{13} = 30$
    A = ______

21. $\frac{x}{18} = 54$

22. $.06\ W = 16.5$
    W = ______

23. $N - .9 = 4.8$
    N = ______

24. $3x = 225$
    x = ________

25. $K + 12\frac{3}{4} = 39$
    K = ________

Do you remember addition?

1. $\begin{array}{r} 23 \\ 14 \\ \hline \end{array}$

2. $\begin{array}{r} \frac{1}{3} \\ \frac{1}{3} \\ \hline \end{array}$

3. $\begin{array}{r} \frac{1}{4} \\ \frac{1}{4} \\ \hline \end{array}$

4. $\begin{array}{r} 4.1 \\ 3.4 \\ 1.2 \\ \hline \end{array}$

5. $\begin{array}{r} 805 \\ 332 \\ \hline \end{array}$

6. $\begin{array}{r} \frac{3}{5} \\ \frac{4}{5} \\ \hline \end{array}$

7. $\begin{array}{r} \frac{7}{8} \\ \frac{3}{8} \\ \hline \end{array}$

8. $\begin{array}{r} \$42.14 \\ 6.20 \\ .56 \\ 91.87 \\ \hline \end{array}$

9. $\begin{array}{r} 277 \\ 199 \\ \hline \end{array}$

10. $\begin{array}{r} 5\frac{2}{3} \\ 3\frac{2}{5} \\ \hline \end{array}$

11. $\begin{array}{r} 8 \\ 7\frac{1}{3} \\ \hline \end{array}$

12. $\begin{array}{r} 4.104 \\ 3.714 \\ 3.289 \\ 1.347 \\ \hline \end{array}$

13. $\begin{array}{r} 525 \\ 559 \\ 76 \\ 269 \\ \hline \end{array}$

14. $\begin{array}{r} 19\frac{7}{10} \\ 73\frac{7}{10} \\ \hline \end{array}$

15. $\begin{array}{r} 10\frac{2}{5} \\ 6\frac{1}{2} \\ 25\frac{3}{10} \\ \hline \end{array}$

16. $\begin{array}{r} \$\ 2.08 \\ 15.97 \\ 3.86 \\ .26 \\ \hline \end{array}$

17. $\begin{array}{r} \$\ \ .99 \\ 74.59 \\ .95 \\ 27.87 \\ 3.68 \\ 35.24 \\ \hline \end{array}$

18. $\begin{array}{r} \frac{1}{6} \\ 5\frac{5}{8} \\ 9\frac{1}{3} \\ \hline \end{array}$

19. $\begin{array}{r} 9\frac{2}{3} \\ \frac{1}{4} \\ 8\frac{1}{10} \\ \hline \end{array}$

20. 7.4 + .109 + 15.08 =

Multiplication

1. $\begin{array}{r} 32 \\ \underline{\times\ 3} \end{array}$

2. $8 \times \frac{1}{4} =$

3. $\frac{3}{10} \times 6 =$

4. $\begin{array}{r} .6 \\ \underline{\times\ 9} \end{array}$

5. $\begin{array}{r} 34 \\ \underline{\times\ 12} \end{array}$

6. $3/4 \times \frac{5}{6} =$

7. $\begin{array}{r} 3.6 \\ \underline{\times .4\ 5} \end{array}$

8. $1 \frac{1}{6} \times 8 =$

9. $\begin{array}{r} 756 \\ \underline{9} \end{array}$

10. $1\frac{1}{4} \times 2/3 =$

11. $1\frac{1}{2} \times 4 \frac{4}{5} =$

12. $\begin{array}{r} .56 \\ \underline{.07} \end{array}$

13. $\begin{array}{r} 368 \\ \underline{917} \end{array}$

14. $\frac{1}{2} \times 3/4 \times \frac{5}{6} =$

15. $5\ 2/3 \times 3\ 3/4 \times \frac{3}{10} =$

16. $\begin{array}{r} 7.98 \\ \underline{\times\ .68} \end{array}$

17. $\begin{array}{r} 6070 \\ \underline{5040} \end{array}$

18. $\frac{5}{6} \times 12 =$

19. $7 \frac{1}{2} \times 2 \frac{1}{10}$

20. $\begin{array}{r} .708 \\ \underline{.075} \end{array}$

Division

1. $4\overline{)48}$ 2. $\frac{1}{2} \div 2 =$ 3. $3\frac{1}{4} \div 2 =$ 4. $7\overline{)2.8}$ 5. $4\overline{)68}$

6. $10 \div 2/5 =$ 7. $2 \div 2\frac{1}{3} =$ 8. $\$.80\overline{)\$4.80}$ 9. $4\overline{)828}$ 10. $\frac{2}{3} \div \frac{3}{5} =$

11. $\frac{3}{8} \div 1\frac{3}{4} =$ 12. $.8\overline{)276}$ 13. $392\overline{)246176}$ 14. $1\frac{1}{4} \div \frac{1}{4} =$ 15. $2\ 2/3 \div 1\frac{1}{3} =$

16. $.54\overline{).513}$ 17. $32\overline{)29056}$ 18. $1\ 2/5 \div 1\frac{1}{2} =$ 19. $8\frac{1}{4} \div 5\frac{1}{2} =$ 20. $2.1\overline{)\$63.42}$

Do you remember subtraction?

1. $\begin{array}{r}58\\ -\ \underline{43}\end{array}$
2. $\begin{array}{r}\frac{3}{5}\\ \underline{\frac{1}{5}}\end{array}$
3. $\begin{array}{r}2\frac{2}{3}\\ \underline{\frac{1}{3}}\end{array}$
4. $\begin{array}{r}8.3\\ \underline{2.1}\end{array}$
5. $\begin{array}{r}799\\ \underline{508}\end{array}$
6. $\begin{array}{r}6\frac{7}{8}\\ \underline{4\frac{1}{8}}\end{array}$
7. $\begin{array}{r}8\frac{1}{6}\\ \underline{3\frac{5}{6}}\end{array}$
8. $\begin{array}{r}\$25.69\\ \underline{14.31}\end{array}$

9. $\begin{array}{r}936\\ \underline{143}\end{array}$
10. $\begin{array}{r}6\\ \underline{1\frac{1}{4}}\end{array}$
11. $\begin{array}{r}\frac{5}{6}\\ \underline{\frac{1}{2}}\end{array}$
12. $\begin{array}{r}9.30\\ \underline{7.96}\end{array}$
13. $\begin{array}{r}852\\ \underline{55}\end{array}$
14. $\begin{array}{r}6\frac{3}{4}\\ \underline{\frac{1}{3}}\end{array}$
15. $\begin{array}{r}5\frac{1}{4}\\ \underline{4\frac{5}{6}}\end{array}$

16. $\begin{array}{r}\$12.14\\ \underline{4.49}\end{array}$
17. $\begin{array}{r}9000\\ \underline{8019}\end{array}$
18. $\begin{array}{r}4\frac{1}{6}\\ \underline{\frac{3}{10}}\end{array}$
19. $\begin{array}{r}9\frac{1}{6}\\ \underline{8\frac{2}{3}}\end{array}$
20. $\begin{array}{r}6.001\\ \underline{2.12}\end{array}$

ED0 26270

O.C.S.E.I.P. SYLLABUS

Grade 8

SE 006 070

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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INTRODUCTION

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

The first draft of this syllabus was written during an 8 week session at University of California, Irvine during the summer of 1966 by:

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We wish to thank all the participants in this program for their hard work and fine cooperation.

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PREFACE

The "new mathematics" involves the use of symbols and terms not previously used in the junior high program. It also introduces and integrates algebra, geometry and arithmetic.

The teacher new to this program should, however, be aware of a much broader academic emphasis. The "new mathematics" is designed to promote an attitude, on the part of students and teachers, of exploration and discovery. For those who subscribe to this approach, it may become necessary to change radically the previously used teaching techniques in favor of allowing the children to explore an area or concept of mathematics with the teacher as a guide, rather than be told certain specifics and then simply exercise themselves with the new knowledge. The previous approach suggests that the student must bring to his exploration all of his working knowledge, apply it, and draw conclusions prior to further understanding.

It is with this approach in mind that we feel teachers and students will both receive a new impetus in teaching and learning.

Understanding precedes skill in solving problems.

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I. Numeration Systems: Base Numbers and Place Values

A. Comments and philosophy

The student should already be familiar with our decimal numeration system which is structured from the first through the seventh grade. If he is not, it is suggested that a study of other numeration systems such as ancient Egyptian, Hindu-Arabic, and/or Babylonian be used as an introduction to this unit. (See).

The teacher should make a definite distinction between number and numeral, face and place value (digital and positional value). However, this distinction should not be stressed to the point where progress in developing skills is hindered.

The study of number bases $n \neq 10$ may be, and often is, introduced as an aid in helping the students better to understand base $n = 10$. At the eighth grade level, this is usually done as supplementary work beyond that of the regular textbooks. It is important to keep the use of this aid in perspective, however, so that it does not become an end to itself.

In addition to helping achieve better understanding of the workings in base 10, work in other bases can also contribute to the later work in algebra in polynomials.

B. Place value

1. Explanation and chart

It is suggested that the teacher place emphasis on place value and use a chart as a tool. This chart should include places for numbers greater and less than one in several bases including base ten. For ease in notation on the chart, the base 10 numeration for the base may be used.

Care should be taken never to use the term "decimal point" except when referring to base ten. Points in other systems are named according to their base (eg. base 5, quinary point; base 2, binary point).

decimal point

10^3 (10x10x10) (1000)	10^2 (10 x 10) (100)	10^1	units	10^{-1} .1 $\frac{1}{10}$ quinary point	10^{-2} .01 $\frac{1}{100}$	10^{-3} .001 $\frac{1}{1000}$
5^3 (5x5x5) (125_{ten})	5^2 (5x5) (25_{ten})	5	units	5^{-1} .1five $\frac{1}{5}$ binary point	5^{-2} .01five $\frac{1}{25}$	5^{-3} .00 $\frac{1}{125}$
2^3	2^2	2	units			

(caution should be taken not to stress fractional notation until students understand whole number systems)

2. Teaching aids

a. A counting game in different bases can be effective in helping the understanding of digit sequence structure in general. Students count in turn, in base five, and are eliminated as they miss. The winner chooses a new base for counting and must catch the mistakes of others.

Have students make place value charts. Small squares of colored paper may be used to represent units and strips of paper to represent 10, 10^2, 10^3 etc., or 5, 5^2, 5^3 etc., so that students may represent various numerals in different bases. This may help less able students.

b. Numeration bingo

Bingo in other numeration systems. -- Blank grids are given to students. The students then fill in the grids with numerals in a given base. The teacher then could orally give problems in addition or subtraction and the students would check their papers for the answers. BINGO is achieved as in the regular game.

3. Reinforcement

For the slower students, the teacher might like to make a grid on a ditto sheet and then have students complete the grid for an addition chart (which may also be used for its inverse, subtraction), and multiplication chart (which may be used for division, its inverse operation). These charts can be constructed for any numeration base system including base 10, and comparisons then drawn between base $n = 10$ and base $n \neq 10$.

Examples:

⊖

Base six

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

$4_{six} + 5_{six} = 13_{six}$

$12_{six} - 4_{six} = 4_{six}$

ⓧ

Base seven

+	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

$5_{six} \times 4_{six} = 26_{six}$

$42_{six} + 5_{six} = 6_{six}$

Note: The teacher may wish to help the class develop the observation that bases n 10 have more combinations to remember than base n = 10, while bases n 10 have fewer such combinations requiring memorization.

C. The General Case: Base K

Place value in any base

Base K

by agreement $K^0 = 1$ if $K \neq 0$

$K^0 = 1$

$K^1 = K$

$K^2 = K \cdot K$

$K^3 = K \cdot K \cdot K$

etc.

$K^5\ K^4\ K^3\ K^2\ K^1\ K^0$ base K

$\text{base}^5\ \text{base}^4\ \text{base}^3\ \text{base}^2\ \text{base}^1\ \text{base}^0$ base 5

$5^5\quad 5^4\quad 5^3\quad 5^2\quad 5^1\quad 5^0$

$\text{base}^5\ \text{base}^4\ \text{base}^3\ \text{base}^2\ \text{base}^1\ \text{base}^0$ base 2

$2^5\quad 2^4\quad 2^3\quad 2^2\quad 2^1\quad 2^0$

D. Miscellaneous Teaching Aids

1. Base puzzle

a	b		▨	c		d	e
f		▨	▨		▨	g	
	▨	h			▨		▨
▨	i		▨	▨	j		k
l			m	n		▨	o
p		▨	q		▨	r	
	▨	s		▨	t		
▨	u		▨	v			▨

Across - change to base ten

a. 1300_6
c. 14261_8
f. 210_4
g. 10001_2
h. 4102_5
i. 2211_3
j. 1101111101_2
l. 514836_9
p. 10100_2
q. 234_5
r. 131_7
s. 202_6
t. 10243_5
u. 1012_3
v. 3212_5

Down - Change to base ten

a. 2324_5
b. 222_3
c. 212221_4
d. 6160_7
e. 32_5
h. 20311_4
i. 10300_5
j. 102_9
k. $1T42_{12}$
l. 2241_5
m. 565_9
n. 1001_2
r. 11132_5
s. 2200_3
t. 333_4

2. Sample test

a. Write the following bases with base ten symbols to show the place values in the indicated positions. Use exponents above on upper line and a pure base ten value below. The first one is an example.

Base ten		10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	
Base ten		100,000	10,000	1,000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$	
Base five												
Base five	Do not write in here											Do not write in here
Base two												
Base two												
Base eight												
Base eight												

b. Use base ten symbols - write expanded notation using exponents

1. $3124_5 = 3\cdot5^3 + 1\cdot5^2 + 2\cdot5^1 + 4\cdot5^0$

2. $6305_7 =$

3. $10111_2 =$

4. $1TE4_{12} =$

5. $34152_8 =$

6. $42.332_5 = 4\cdot5^1 + 2\cdot5^0 + 3\cdot5^{-1} + 3\cdot5^{-2} + 2\cdot5^{-3}$

7. $1T.2E_{12} =$

c. Change to base ten by use of expanded notation.

1. $3012_5 = 3\cdot5^3 + 0\cdot5^2 + 1\cdot5^1 + 2\cdot5^0$

$$= 3(125) + 0(25) + 1(5) + 2(1)$$

$$= 375 + 0 + 5 + 2 = 382$$

2. $4176_8 =$

3. $1T0E_{12} =$

d. Add - give answer in base indicated. All addends are written in the base indicated.

1. base five

$$\begin{array}{r} 1324 \\ 301 \\ \underline{4132} \end{array}$$

2. base seven

$$\begin{array}{r} 6032 \\ 514 \\ \underline{3046} \end{array}$$

3. base two

$$\begin{array}{r} 10110 \\ 1010 \\ \underline{1101} \end{array}$$

4. base four

$$\begin{array}{r} 301 \\ 133 \\ \underline{444} \end{array}$$

4. base two

$$\begin{array}{r} 10110111 \\ 1011011 \\ \underline{11101111} \end{array}$$

5. base five

$$\begin{array}{r} 4312304 \\ 430121 \\ \underline{3230434} \end{array}$$

e. base ten base "k"

0 = X

1 = J

2 = L

3 = M

4 = R

+ = $\triangle$

(x) = $\boxed{\triangle}$

Write the following base "K" in expanded notation with base "K" symbols. Use exponents in base "K".

1. 3241_k = ____________________

2. 20343_k = ____________________

3. 4322_k = ____________________

The numerals are in base "K" using base ten symbols.

1. 3241

2. 20343

3. 4322

E. Vocabulary

The teacher and students should be cognizant of the following terms which will be used in the study of numeration systems in other bases. A speaking vocabulary by the teacher of these terms greatly help the students.

natural number
set
counting number
brace
grouping
element
empty set
whole number
exponent
subscript
place value
digit
power

II. Natural Numbers and Zeros

A. Set Notation

By this time the student should have learned that a set is an undefined term and the elements of a set may be enclosed between braces.

The teacher may explain that a capital letter denotes a set and that lower case letters are used to denote the elements of a set. For example $a \in A$ means that "a" is an element of set A.

If the set contains no elements then it is said to be the empty or null set $\emptyset$. The Greek letter Phi, $\emptyset$, is often used to designate the empty set.

The student should have a clear understanding of equal sets which means two sets containing exactly the same elements. For example: Set A = $\{1, 2, 3, 4\}$, Set B = $\{1, 2, 3, 4\}$, Set A is equal to set B. Two sets are equivalent means the two sets contain the same number of elements. If set A = $\{1, 2, 3, 4\}$, set B = $\{a, b, c, d\}$ then set A is equivalent to set B since there is a one-to-one correspondence between them.

The teacher should point out the cardinality of sets (how many) and the difference between a number used in the cardinal and ordinal sense.

Examples:

cardinal	ordinal
1. 34 boys	1. 34th boy
2. 7 streets	2. 7th street
3. 2 in line	3. 2nd in line

The teacher may note that: John lives on 51st Street, not 51 Streets

B. Matching sets

The teacher may have the student draw diagrams or work examples showing one-to-one correspondence, or two-to-one correspondence etc.

(a)	$\{1, 2, 3\}$	Set of heads	One to correspondence
	$\{a, b, c\}$	Set of tails	One
(b)	$\{(LR)_1, (LR)_2, (LR)_3\}$	Set of legs	Two to correspondence
	$\{1, 2, 3\}$	Set of bodies	One

The student may make up his own problems or diagrams.

C. Symbols for Grouping

The student may avoid misunderstanding by learning to use the parentheses ( ) symbols for grouping. The parentheses will enclose that part of a mathematical sentence which is thought of as a numeral naming a single number.

$8 \cdot 5 + 4 =$ not clear
$(8 \cdot 5) + 4 = 44$
$8 \cdot (5 + 4) = 72$

Textbooks contain many examples in the use of parentheses.

D. Rules of Order

The teacher may wish to review the conventional rules of order for the students. For convenience, in some sentences, we may omit the parentheses. In such cases we agree that when more than one operation is involved, multiplication and division are to be performed before addition and subtraction.

A key phrase such as: My Dear Aunt Sally (MDAS) may be very useful in keeping the order fixed in the student's mind. Have the students answer roll with a phrase they like, sometimes the more nonsensical the better. The teacher will find it beneficial to use his or her key phrase often and refer students back to when they forget.

Example: $5 + 9 \cdot 3 = 32$, $12 \div 3 + 5 = 9$

E. Brackets

In some sentences, students will learn, one set of parentheses are not enough to make the meaning clear. Here, we use both parentheses and brackets.

$n = 75 - 4 \cdot 5 + 12 - 8$

(a) $n = 75 - [(4 \cdot 5) + (12 - 8)]$

(b) $n = [75 - (4 \cdot 5)] + (12 - 8)$

(c) $n = [(75 - 4) \cdot 5] + (12 - 8)$

(d) $n = (75 - 4) \cdot [(5 + 12) - 8]$

After the concept is clear the student will need many problems to fix it in his mind. One approach that has been used is to get the student to think of "boxes inside of boxes" and that he is locked inside the innermost box - when he has a single numeral then he has a key which unlocks the box and permits him to get into the next larger one, etc.

When working problems on the board or prepared work papers be sure to leave the groupings off of every 3rd or 4th problem - this will aid the student in remembering not to use your knowledge as his crutch, and will reinforce the order properties much stronger in his mind.

F. Closure Property

Point out to the student that in order to have closure we must have a set of numbers and the appropriate operation.

Take any element in a set and any other element in the same set, or the same element again and put them together (however the operation tells you) and the result must be an element of the same set.

Example: Given set - {2,4,6,...} Set of even naturals
Operation - Addition
Checks for closure (1) 2 + 2 = 4
(2) 2 + 6 = 8
(3) 10 + 14 = 24
(4) 132 + 16 = 148

Conclusion: any even natural added to any even natural will always be an even natural, therefore the closure property holds

Given set - {1,3,5,7,...} Set of odd naturals
Operation - Addition
Checks for closure (1) 5 + 7 = 12

Conclusion: two odd naturals added together will not always give an odd natural, therefore closure does not hold. Two odd naturals added together will always give an even natural.

G. Associative, Commutative and Distributive Properties

The student should have had the associative, commutative, and distributive properties of addition and multiplication. A brief review with examples should be enough for this topic.

1. Property

Take at least 3 elements of a set and you may group any two together first and then combine with the third as long as you do not change the order in which they were first written.

Example: 1. $(A + B) + C = A + (B + C)$
2. $(A.B).C = A \, . \, (B.C)$
3. $(3 + 4) + 5 = 3 + (4 + 5)$
$7 + 5 = 3 + 9$
$12 = 12$
4. $(5.6) \, . \, 2 = 5 \, .(6.2)$
$30.2 = 5.12$
$60 = 60$

a. Are the following associative? (Yes or No)

1. $(3 + 7) + 8 = 3 + (8 + 7)$ Answer No
Notice the order has changed
2. $(3.2) \, .5 = 3 \, .(5.2)$ Answer No
Notice the order has changed
3. $5 + (7 + 3) = (5 + 7) + 3$ Answer Yes
Notice the order did not change
4. $[5\,(2.3)]\,7 = [(5.2)3]\,7$ Answer Yes
Notice the order did not change

b. The associative property only permits the changing of grouping -- not the order in which they are written.

2. Commutative property - the order of the elements may be changed without changing the results

Examples: 1. $a + b = b + a$
2. $ab = ba$
3. $5 + 7 = 7 + 5$
4. $8.5 = 5.8$
5. $(3.7) + (2.5) = (2.5) + (3.7)$
6. $[(5 + 8) + 4]\,6 = 6\,[(5 + 8) + 4]$

3. Distributive property - multiplication is distributive over addition

Examples: 1. $a(b + c) = ab + ac$
2. $3(5 + 6) = (3.5) + (3.6)$
$3(11) = 15 + 18$
$33 = 33$

3. $4\,[(2 + 3) + 5] = [(4.2) + (4.3) + (4.5)]$

$4\,[(2 + 3) + 5]$, is a product, one of its factors is '4', the other factor is a sum consisting of three addends - notice that the factor '4' is distributed to each addend when the distributive property is used.

H. Special Properties of 'One' and 'Zero'

1. 'One'

a. Smallest natural number

b. One added to any natural produces the next natural number.

$1 + 1 = 2,\ 2 + 1 = 3,\ 3 + 1 = 4,\ 4 + 1 = 5,\ 5 + 1 = 6$, etc.

c. One to any whole number power is '1'

$1 = 1,\ 1^1 = 1,\ 1^2 = 1,\ 1^3 = 1,\ 1^4 = 1,\ 1^5 = 1$, etc.

d. '1' times any number is that number

$1.3 = 3,\ 1.5 = 5,\ 1.36 = 36,\ 1.a = a$, etc. because of the commutative property of multiplication any number times one is that number.

e. Due to the special property described in 'D' above we call 'one' the identity element of multiplication.

f. Any number divided by '1' is that number

$\frac{k}{1} = k;\ \frac{8}{1} = 8;\ \frac{3/4}{1} = 3/4$

2. 'Zero'

a. 'Zero' times any number or any number times 'zero' is 'zero'

b. 'Zero' added to any number or any number added to 'zero' is that number

$3 + 0 = 3,\ 0 + 5 = 5$, etc. (zero is the identity element for addition and subtraction)

c. Due to the special property described in 'B' above 'Zero' is called the identity element of addition.

d. When a product is 'zero', one or more of its factors is zero.

e. Division by 'zero' is meaningless

$\frac{6}{2} = 3$ if and only if $3 \cdot 2 = 6$ because division is the inverse of multiplication,

therefore $\frac{6}{0} = \triangle$ is meaningless because $\triangle \cdot 0 \neq 6$

* The teacher might wish to use the following approach in bringing out this point to the slower students

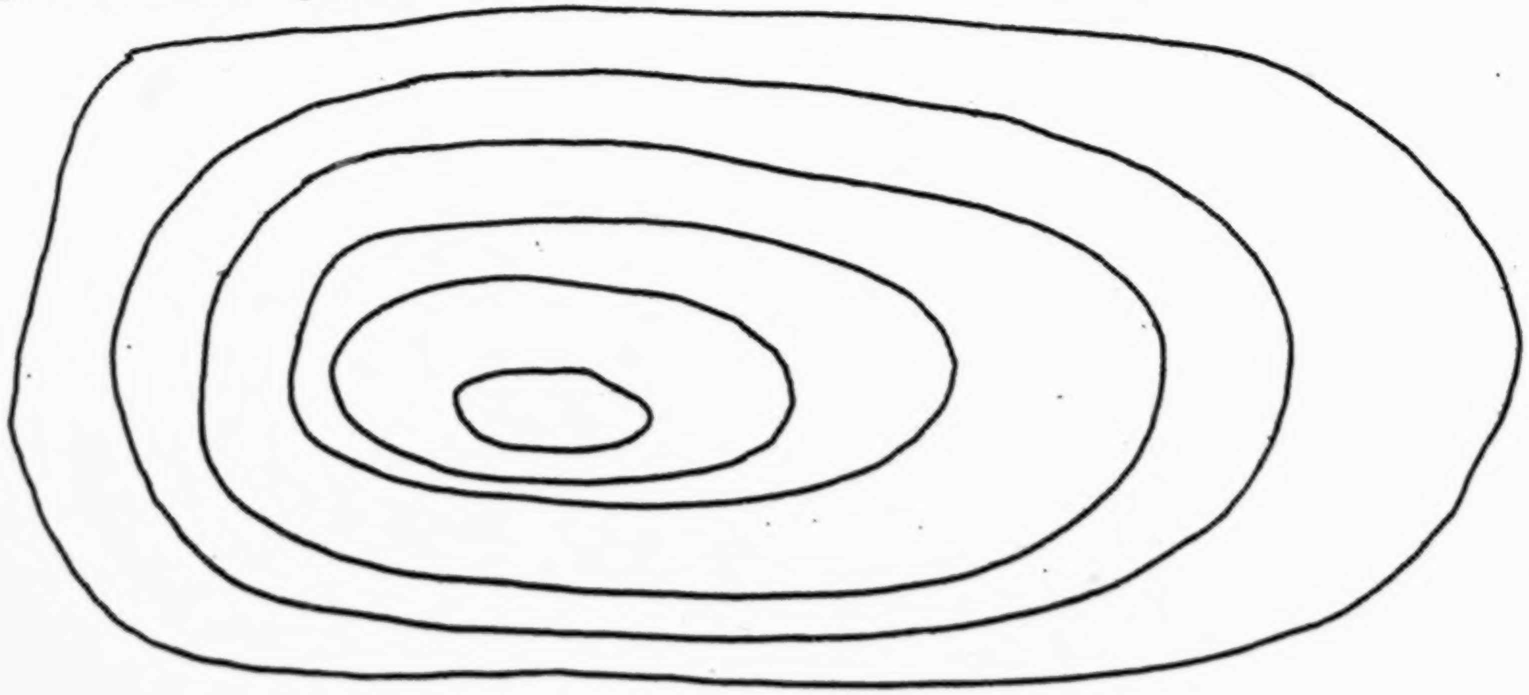
1. Give several of your students names of numbers
2. Name another set of students numbers with one of them 'zero' and another 'one'
   a. the second set of students are mirrors
3. Have the first student walk up and look into the face of the first mirror
   a. the person who is the mirror will then multiply by his number and say the results as a reflection
4. Looking is the same as the operation - multiply
5. The students will soon learn what mirror (element) will give them their identity.

** The same approach as * may be used for addition also.

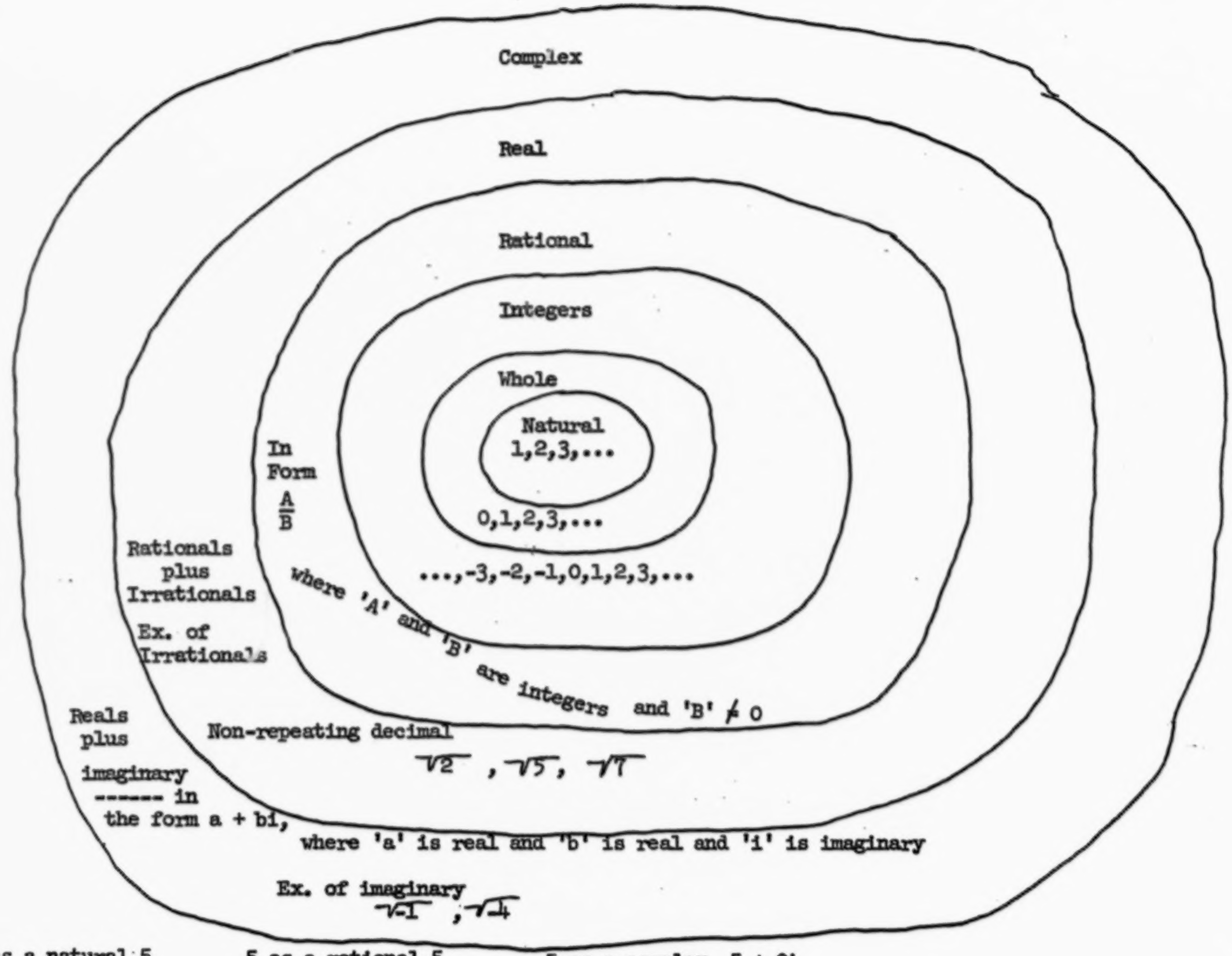
III. Integers

A. Number system

In beginning the unit on integers the following chart will help the student orient integers in the general number system.



NUMBER SETS


Complex
Real
Rational
Integers
Whole
Natural
1,2,3,...
0,1,2,3,...
...,-3,-2,-1,0,1,2,3,...
In Form $\frac{A}{B}$
where 'A' and 'B' are integers and 'B' $\neq$ 0
Rationals plus Irrationals
Ex. of Irrationals
Non-repeating decimal
$\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$
Reals plus imaginary ------ in the form a + bi,
where 'a' is real and 'b' is real and 'i' is imaginary
Ex. of imaginary
$\sqrt{-1}$, $\sqrt{-4}$

5 as a natural 5
5 as a whole 5
5 as an integer 5

5 as a rational $\frac{5}{1}$
5 as a real 5

5 as a complex 5 + 0i

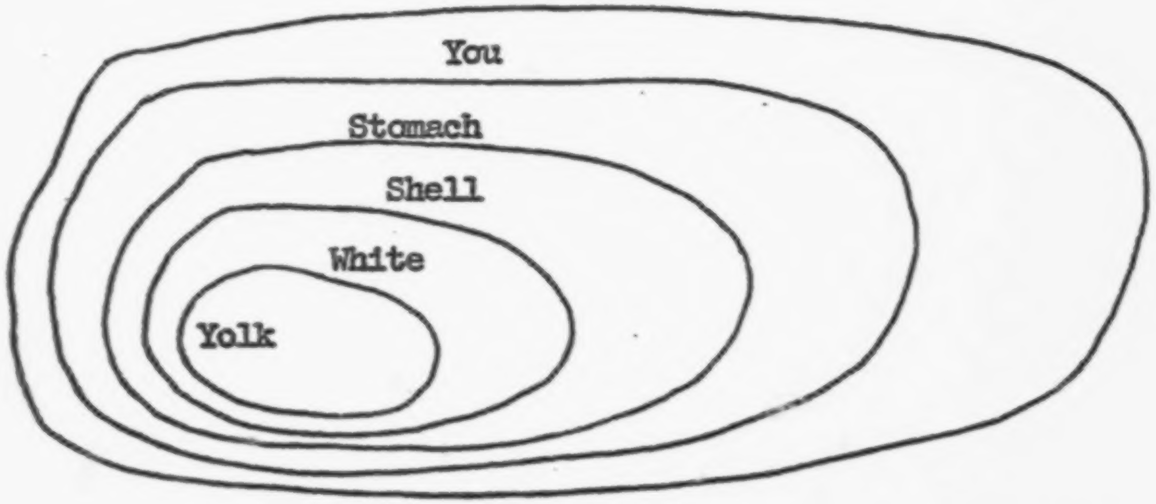
Although this chart is found in many texts, the teacher may find the one on the following page more useful, especially when talking about sets and sub sets.

Giving drill questions about sub sets and using the chart can be most helpful. Examples of such could be as follows:

Are all natural numbers whole numbers?	yes
Are all natural numbers integers?	yes
Are all integers natural numbers?	no
Are all whole numbers natural numbers?	no

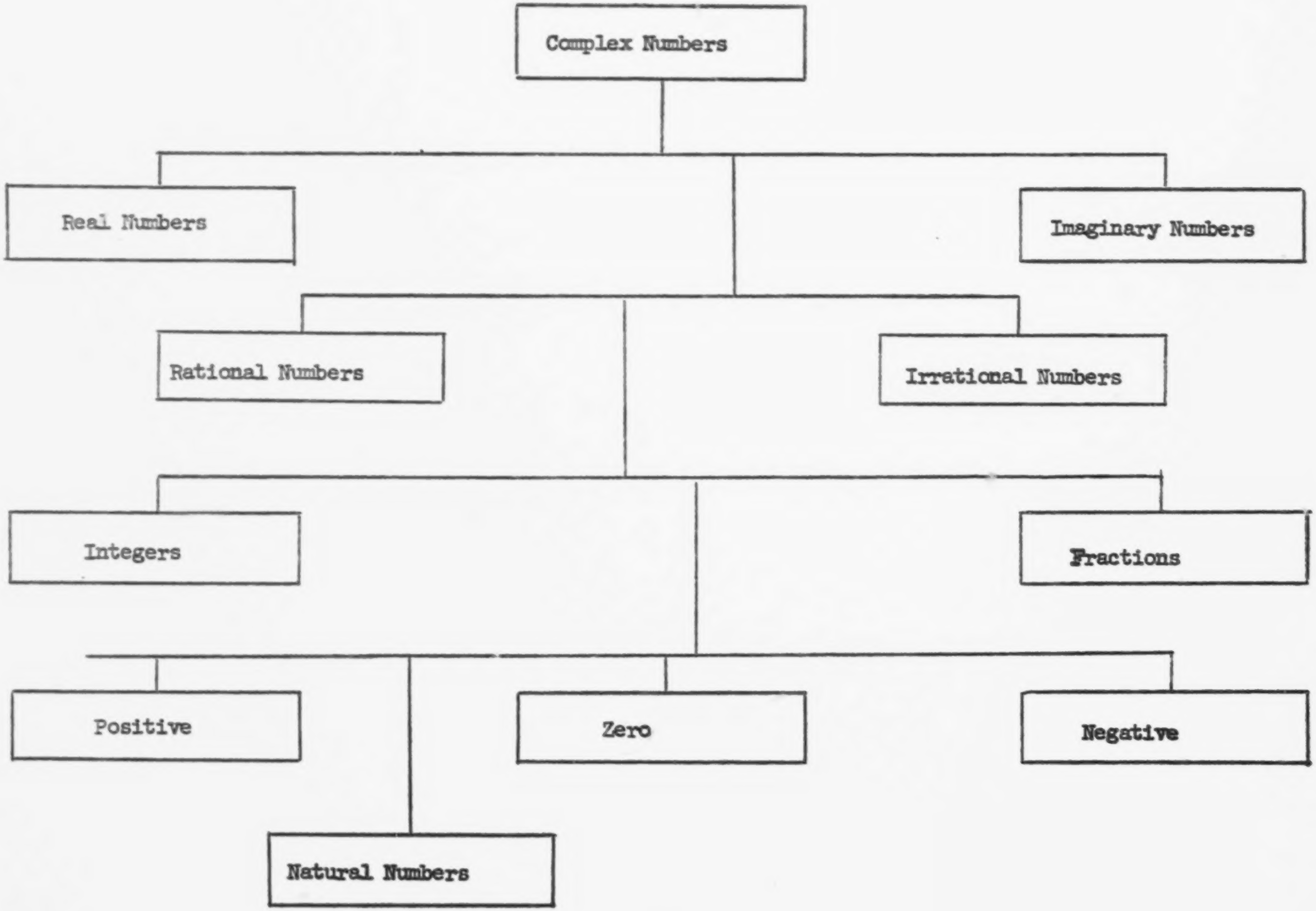
If the student has trouble grasping the idea of sets and sub sets the teacher may wish to use something such as the following.

1. Tell the students they are very hungry and that you hard boiled some eggs for them.
2. Each student eats the hard boiled egg without taking off the shell
3. You may now use the following drawing for sets and sub sets using questions such as:
    a. Is the egg in your stomach? yes
    b. Is the yolk in the shell? yes
    c. Is your stomach in the egg? no
    d. Etc. until the idea of a sub set is gained.

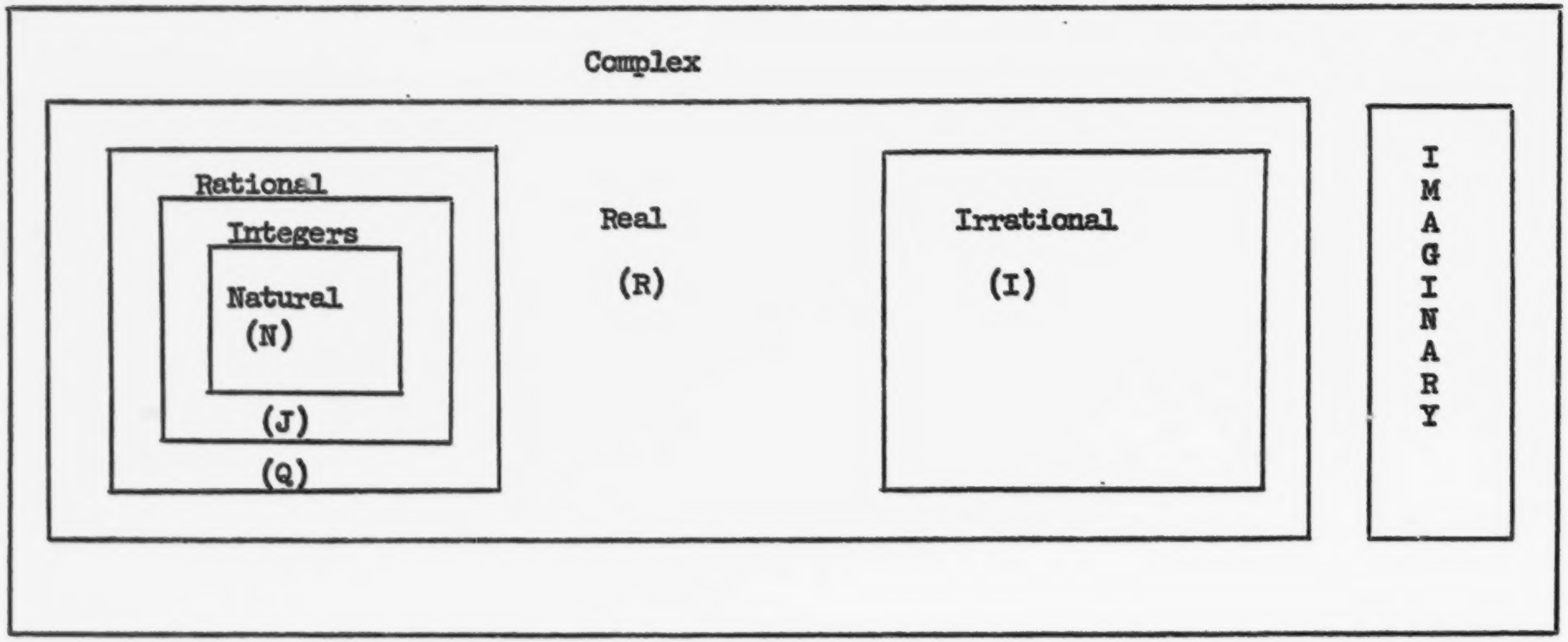


The teacher may wish to use this chart for students to fill in and ask questions about

NUMBER STRUCTURE



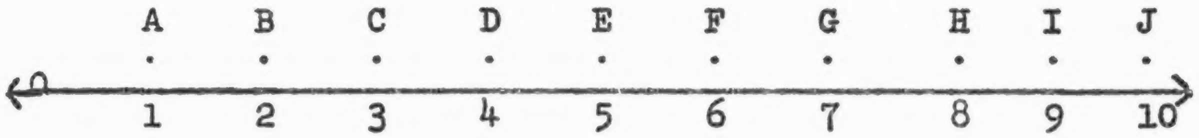
Another Look at Number Structure



B. Using a number line

1. Number line

Emphasize for students that we speak of "a" number line, not "the" number line, as there are several other number lines besides the one showing natural numbers and zero. Show, by arrows, that the line is endless. The teacher should make a number line on the blackboard with points marked on it.

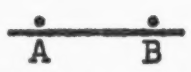


Then explain that on a number line the number associated with a point on the line is called that point's coordinate.

Example: Coordinate of D is 4, etc.

Intuitive definition of a line:

A line is a set of points in space and is usually understood to be a straight line. A line extends indefinitely in each of two directions. We designate this by means of naming any pair of points on the line and by placing a double arrow over this notation.

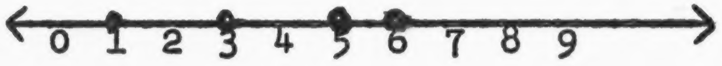


is read "line AB."

(The teacher might wish to look up collinear and non-collinear in the mathematics dictionary.)

2. Graph of a set of numbers

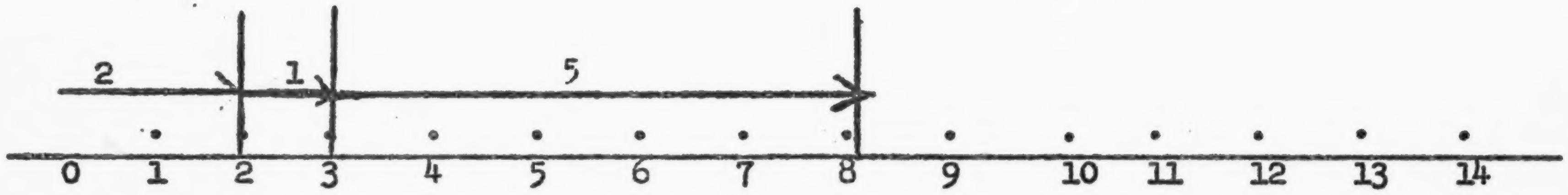
The following is an illustration of a graph of a set of numbers $\{1,3,5,6\}$



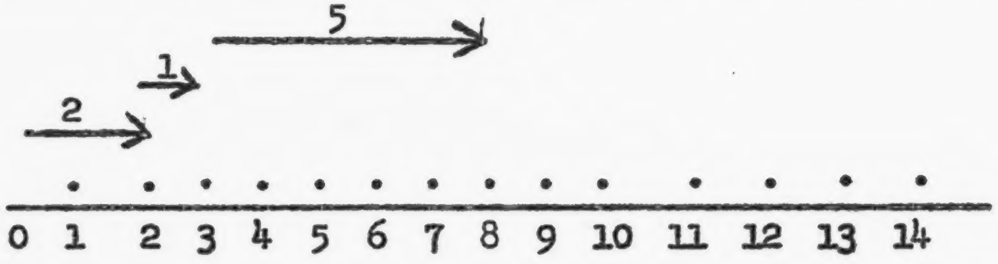
3. Addition, subtraction, and multiplication using a number line

On the board place diagrams similar to the ones below to explain addition, subtraction and multiplication on a number line.

Addition: Example - 2 + 1 + 5 = n

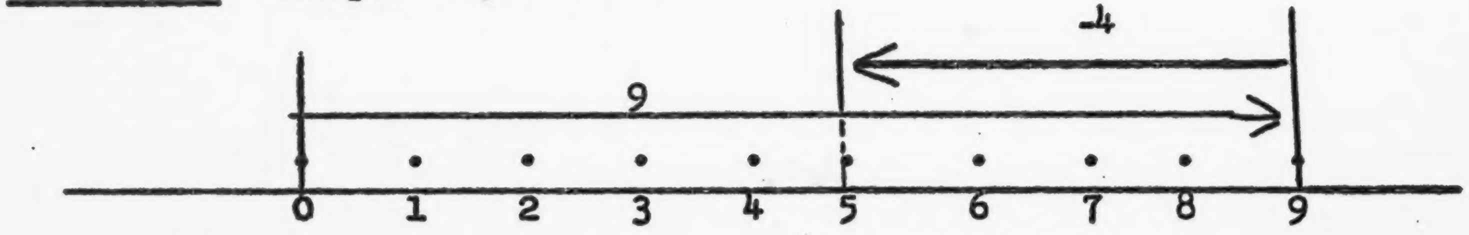


The use of the following number line may be more useful than the one above, in that, first, when the student adds negative intgers he will not try to come back over the line he has just made, and second, the first addend is closest to the number line and the second addend next, etc. This will aid in reading.



The student sees that "add" means go the same direction on a number line.

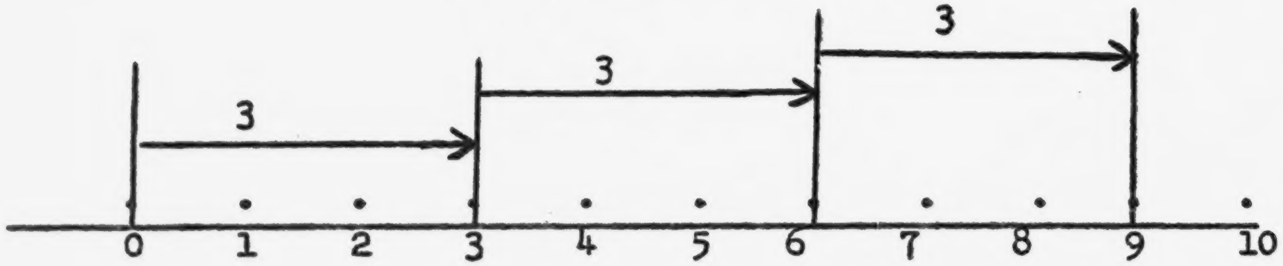
Subtraction: Example - 9 - 4 = n



"Subtract" means a change in direction on a number line.

Multiplication: By using a number line the teacher may demonstrate that multiplication is similar to addition. Example: $3 \times 3 = n$

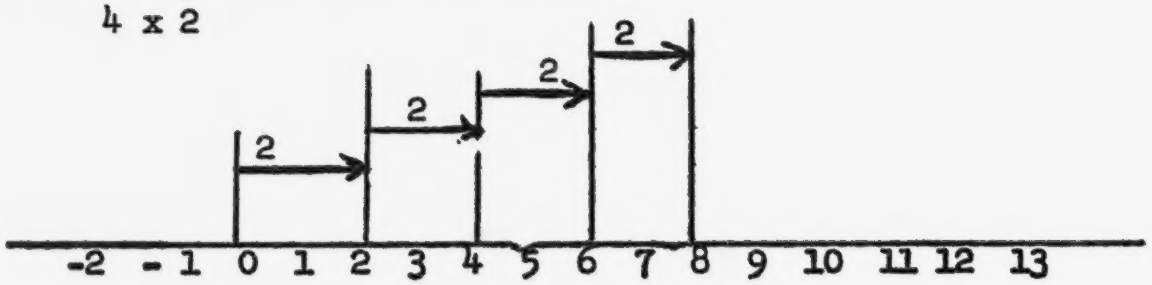
$3 \times 3 = n = 3 + 3 + 3 = n$



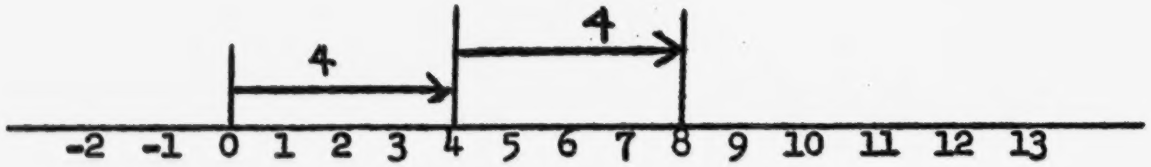
The above problem is an example of a poor illustration to use because the multiplicand and multiplier are the same. $4 \cdot 2$ and $2 \cdot 4$ would be much better. The teacher should take note that many students cannot actually read $4 \cdot 2$ properly - that is, that $4 \cdot 2$ is a notation for 4 sets, each set containing two and that $2 \cdot 4$ would be two sets each containing 4. The teacher will need to ask repeatedly which is the multiplier and which is the multiplicand.

The teacher may wish to use the following as an example: Because of the commutative property of multiplication $5 \cdot 25 = 25 \cdot 5$ five quarters and twenty-five nickels are the same amounts but certainly do not look the same.

4×2



2×4



C. Negative integers

Explain to students that for every positive integer there exists a negative integer, and conversely

$$(+K) + (-K) = 0$$

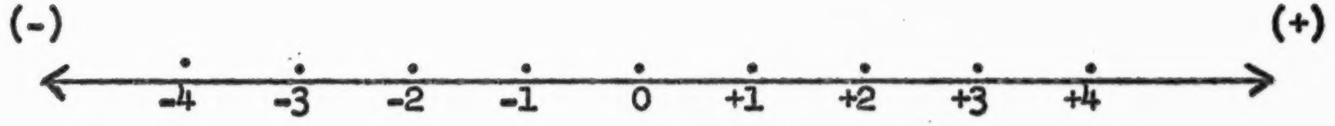
$$(-K) + (+K) = 0$$

The teacher should know that the set of integers may be expressed as the set of whole numbers and their additive inverses. The additive inverse of a whole number is a second number which when added to the first makes the result zero -- which is the identity element for the operation. The teacher will find it useful later on to use "add the opposite" and "add the additive inverse" jointly.

1. Notation and terminology

Terminology and notation should be stressed: Positive integer, negative integer opposite. The notation: +1, +2, +3, -3, -2, -4. Occasionally raised signs $(^+2)$ are used in textbooks but this is not necessarily preferred to (+2). Students have known about positive and negative integers because most are familiar with a thermometer. To further develop the concept of negative integers extend a number line to the left of zero.

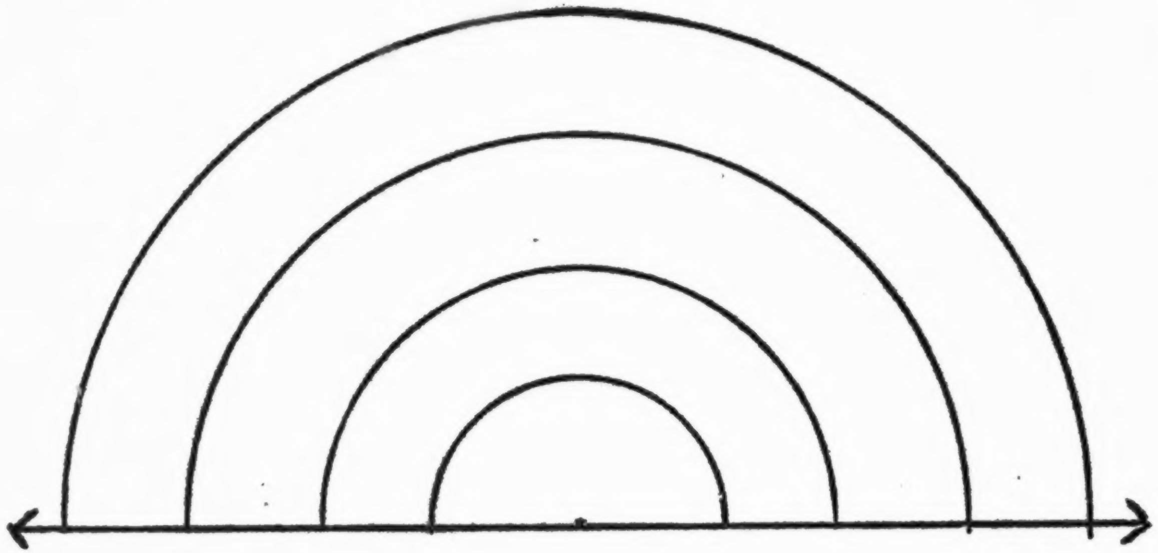
2. A number line



Note: It is suggested that a chart number line, at least 6 feet long, should be placed in the classroom for student reference during the work on positive and negative integers.

3. Opposites

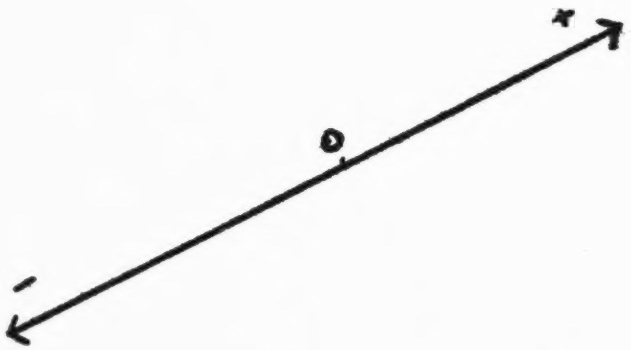
The following chart may be used on the chalk board to show that points the same distance from zero on a number line and on opposite sides of zero are called opposites.

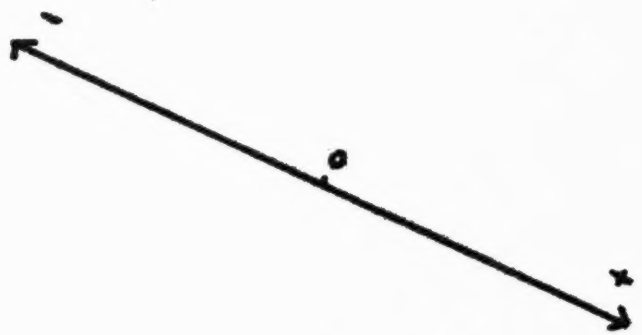


If the minus symbol (-) means "the opposite of," then $-3 = -3$ may be read, "negative 3 is the opposite of 3" and $2 = -(-2)$ means "2 is the opposite of negative 2."

D. Addition, subtraction and multiplication of integers

After using a number line for a short time students will see that counting to the right is in the positive direction and to the left is in a negative, or opposite direction of positive movement.





1. Addition of integers

Number lines need not be horizontal. Using a number line students will see that $3 + (-5) = -2$ means 3 units in the positive direction then 5 in the negative direction which gives (-2) for the sum. Likewise $(-3) + (-2)$ means (-5); and $(-2) + 4 = 2$.

The teacher should be on the lookout for students doing the problem who go back to zero to start their second addend. The teacher may wish to use a walking number line on the floor and show that you move to the first addend; from there you take your direction instructions and then move the number of required units.

Axioms for addition of integers (associativity, identity and additive inverse), and demonstrations on how to use the axioms and rules for addition of positive and negative integers, are not covered here as they are fully discussed in the textbooks.

In adding a pair of positive and negative integers on a number line it may be seen by the student that the integer farther from zero determines whether the sum is positive or negative. The teacher may choose to explain to students the traditional rule for addition of positive and negative integers, namely: find the difference of their absolute values and take the sign of the larger integer. It is still a valid rule!

By using the chart the student may see that the sum of any integer and its opposite is zero. (Remember zero is the identity element for addition and that the additive inverse of zero is zero.)

Here is a game which will help the students formulate a rule for adding integers which does not require the rote memorization of a rule which will be forgotten soon.

Las Vegas Game

"Positive" means "glad"
"Negative" means "sad"

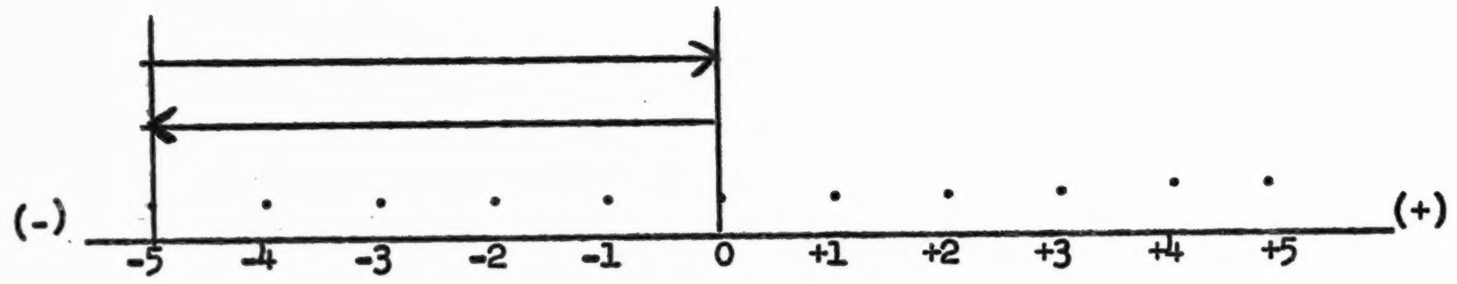
$-8 + 15$ means you play two games. The first game you are "sad" 8 dollars, the second game you are "glad" 15 dollars -- you don't play anymore games. First question: Are you "glad" or "sad"? The student will quickly see that because he was "sad" and "glad" that his answer must be the difference of his "gladness" and "sadness."

(-3) + (-6) means that the first game was a "sad" 3 and the second game "sad" 6. First question: Are you "sad" or "glad"? The students will quickly say that they are sad. Second question: How sad? The students will quickly see that the answer is the sum of their sadness.

2. Subtraction of integers

It is more difficult for students to subtract than to add by use of a number line. Since subtraction is the opposite of addition students will become aware that when they subtract a negative number they really add a positive number. Example: $-5-(-5)=-5+5=0$.

Example: $-5-(-5)=0$



a. By number line

Think: Count 5 to the left of zero, then to subtract the opposite of 5 count 5 places back to zero. The student will then get the concept that anything subtracted from itself gives zero.

Another example of subtraction the students will understand is the "counting back" that takes place in a store when change is made. If the customer purchases something worth 55¢ and gives the checker a dollar, the checker will usually say, "55¢, 65¢, 75¢, one dollar" as the customer receives two dimes and a quarter. This is known as the complementary method of subtraction, and is used in computors.

Several such situations should be discussed by the class. The students should then understand that when subtracting a negative number, change the sign and proceed as in addition.

The teacher may wish to show by many examples that all subtraction problems may be rewritten as addition by adding the "additive inverse" of the subtrahend.

$6 - 3 = 6 + (\text{the additive inverse or opposite of } 3)$
$= 6 + (-3)$
$= 3$

$6 - (-3) = 6 + (\text{the additive inverse or opposite of } 3)$
$= 6 + 3$
$= 9$

$-4 - (-5) = -4 + (\text{the additive inverse or opposite of } (-5))$
$= -4 + 5$
$= 1$

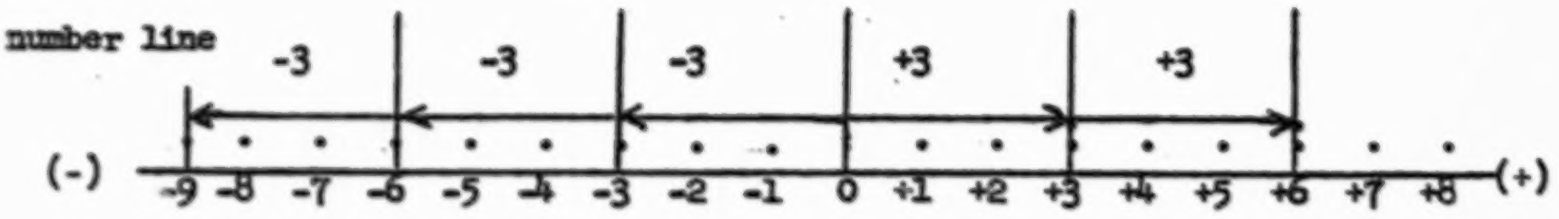
The students will have problems in locating just exactly what is the subtrahend in the problem. Once the teacher has "put across" what the subtrahend is, then adding the additive inverse or opposite will be easy.

3. Multiplication of integers

It is not hard for students to use a number line and understand that the product of a positive and a negative integer is negative: $3 \cdot (-3) = (-9)$ means "mark off (-3) three times on a number line." Also, it is easy to see that the product of 2 positive integers is positive: Ex: $2 \cdot (+3) = 6$ by marking off (+3) twice on a number line.

$$3 \cdot (-3) = (-9) \quad 2 \cdot (+3) = 6$$

a. By number line



But students will find that some problems may not be explained on a number line, for example $-3 \cdot 5 = n$. However, using the commutative law, $(-3) \times 5 = 5\ (-3) = -15$. The following little "chart" may be used as a teacher's aid to help students remember which signs to use in multiplication of integers if all else fails. (The teacher will find that for some students the excellent material found in textbooks simply doesn't explain the rules for positive and negative signs in multiplication, and that such students may be helped by a "game" chart.)

b. Rules for signs

in multiplication

+ Friend	+ win	+ good
- Foe	- lose	- bad

1. For a friend to win is good $(+ \cdot + = +)$ "The product of 2 positive integers is a positive integer."

2. For a foe to lose is good $(- \cdot - = +)$ "The product of 2 negative integers is a positive integer."

3. For a friend to lose is bad $(+ \cdot - = -)$ For a foe to win is bad $(- \cdot + = -)$ "The product of a negative and a positive integer is a negative integer."

For those students who don't understand or can't remember the above game there is the very good game of Rattle Snake. Multiplication is considered fighting and rattle snakes are the (-) symbols. The positive (+), shown or not shown, are non-fighters; only the rattle snakes are fighters. The student must consider himself in the fight. If a rattle snake (-) can be paired with another rattle snake (-) then they fight each other. The student is then asked does this make him "glad" or "sad" and he and she will quickly say "glad." If a rattle snake (-) cannot be paired with another (-) then the rattle snake will fight the student which will make him feel "sad."

The student will quickly see that if there is an even number of rattle snakes, he will be "glad" and if an odd number, then he will be "sad." (See previous Las Vegas" game.)

E. Properties of the operations with integers

The properties of integers (closure, commutative, associative, distributive) are adequately covered in the state textbooks so no other comments seem necessary.

F. Division of integers

1. Inverse of multiplication

By using the inverse relationship between multiplication and division and the rules for multiplying integers the students develop the rules for dividing integers.

2. Sign rules for division similar to multiplication

Since multiplication and division are inverse operations, $\frac{a}{b} = c$ means $a = bc$. For example, $\frac{20}{5} = 4$ means $20 = 4 \cdot 5$. Because of this we can derive the sign rules for division of integers from the multiplication rules.

3. Division by (-1)

It should be explained to students the effect of dividing by (-1). For any integer, a, $a \div -1 = (-a)$. So the result of dividing a non-zero integer by (-1) is the opposite of the integer. This concept is useful to students in solving equations which might otherwise be confusing at first.

Example: $\overline{-}\, y = 6$

$$\frac{\overline{-}\, y = 6}{-1 \quad -1} \qquad \frac{-y}{-1} = \frac{6}{-1}$$

$$y = \overline{-}6$$

4. Division by zero

Students should have the explanation that while zero may be used as a divident, it should not be used as a divisor because division by zero is undefined.

IV. Open Sentences and Problem Solving

A. Open Expression

Students should first learn that an open expression is one in which a variable is used to represent any one of a given set of numbers.

Students should be given some open expressions such as b, b + 4, 3 b + 9, etc. and asked to translate these. There will generally be a great variety of translations. This shows the students that a variable may represent any number in its replacement set.

1. Reading open expressions or phrases

In reading the expressions certain words may be subsituted for the signs (+, -, x, ÷) which appear or may designate operations in the expressions. For instance, (+) may be expressed as "more than," increased by" or "added to," etc. The expression b + 4 could be translated as "the number increased by 4" or "the sum of a number and 4." Students should become adept at translating phrases before beginning to translate verbal problems.

B. Translating English phrases and sentences into open phrases and open sentences

The student's ability to read and analyze will result in an understanding of phrase before the translation begins. In this phase of mathematics the poor reader is greatly handicapped.

1. Translating English phrases to numerical phrases

Students should have much oral practice in translating English phrases into numerical phrases. For example: If a boy is now 13 years old, give a numerical phrase representing his age 5 years ago (13-5) or 3 farmers have 2 horses each and one farmer has 4 horses (3 · 2 + 4).

2. Open sentences and English sentences

An open sentence implies that a certain place in the sentence has been left open which is meant to be filled by making the proper numerical choice. A variable of any kind represents a number.

a. Oral practice

Have students volunteer to give two expressions. Have another student connect them with an equality symbol (=) or one of the inequality symbols ($\neq$, <, >). A numerical sentence has now been stated: If the equal symbol (=) was used, the numerical sentence is called an equation.

C. Solving problems

Although no teacher can give students a set of rules for solving problems, hints to use as guideposts may be given:

1. Hints for problem solving

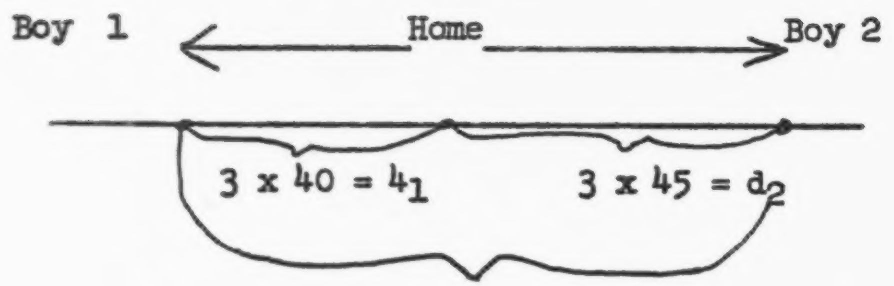
a. study and understand the problem
b. make a diagram to help you
c. make a reasonable estimate
d. translate into an open sentence
e. solve the open sentence to get the solution set, thus obtain an answer (or answers)
f. check the answer in the original problem (not in the sentence)

2. Diagrams

a. Value of

Diagrams ("thinking-aids") will often help a student translate an English sentence into an equation. These diagrams are seldom accurate scale drawings but help a student "think through" a problem. For instance, the rate-time-distance problems so difficult for 8th grade students are often clarified by a diagram. Ex: Two boys leave home at the same time and travel in opposite directions. One walks at an average speed of 45 m.p.h. and the other at an average speed of 40 m.p.h. How far apart are the 2 boys at the end of 3 hours?

b. Example of



$D = d_1 + d_2$ = distance apart at end of 3 hours

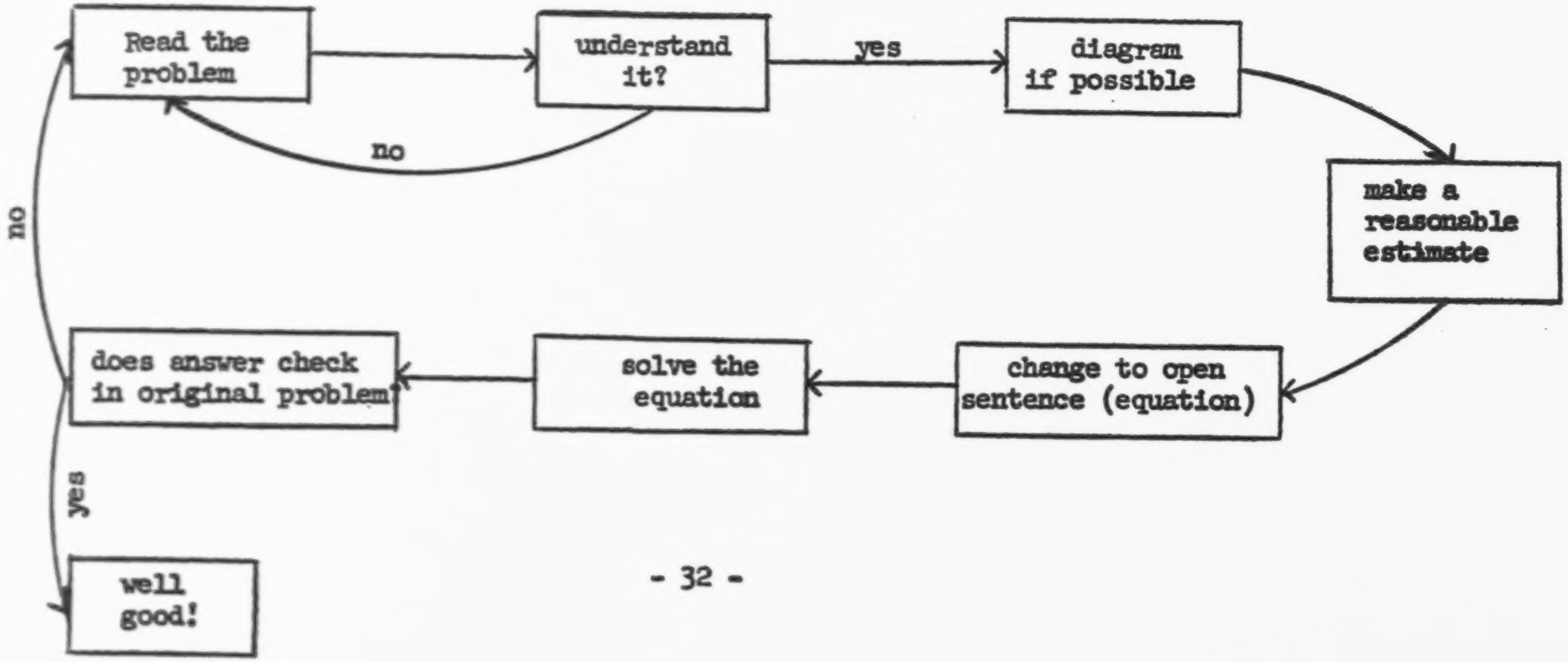
$D = d_1 + d_2$

$D = (3 \times 40) + (3 \times 4.5)$

$D = 12.0 + 13.5 =$

$D = 22.5$ miles

1. Chart of hints for solving problems


Read the problem
understand it?
yes
no
diagram if possible
make a reasonable estimate
change to open sentence (equation)
solve the equation
does answer check in original problem?
no
yes
well good!

D. Using formulas and estimating answers

1. Formulas

The students have developed various formulas in previous work. For example, the area of a rectangle ($A=l \cdot w$); Distance-rate-time ($d = r \cdot t$) Interest formula ($i = p \cdot r \cdot t$) and Relationship between Centigrade and Fahrenheit thermometer readings ($F = \frac{9}{5} C + 32$). If

students know only one statement of the formula they may use the properties of equations (see Sec. IV) to find or state the formula for any variable desired. These formulas may often be used to help translate an English sentence into an open sentence.

2. Estimating answers

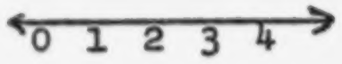
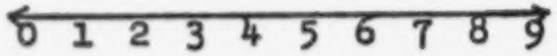
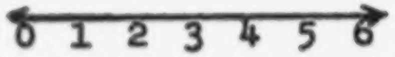
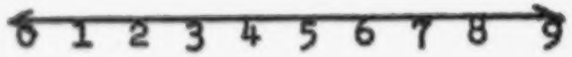
Students should be encouraged to mentally estimate an answer for every equation. This they may use as a check on their work. Advise students to round off numbers in an equation and compute mentally before beginning the solution.

V. Solving Equations

One type of mathematical sentence is the statement. Statements are those sentences which can be identified either as true or false. Statements of equality, whether true or false, are called equations.

A. Symbols and properties of equality

The teacher may construct charts showing symbols with their definitions and uses, such as the following:

Symbol	English (spoken)	Example
$=$	is equal to	$3 + 5 = 8$
$\neq$	is not equal to	$4 + 2 \neq 7$
$>$	is greater than	$3 > 2$
	is to the right of (number line)	
$\ngtr$	is not greater than	$6 > 3 \cdot 2$
	is not to the right of (number line)	
$<$	is less than	$2 < 3$
	is to the left of (number line)	
$\nless$	is not less than	$6 < 3 \cdot 2$
	is not to the left of (number line)	

Symbol	English (spoken)	Example
$\geq$	is greater than or equal to (same as $\nless$)	$7 \geq 3 \cdot 2$
$\leq$	is less than or equal to (same as $\ngtr$)	$6 \leq 3 \cdot 2$
$\ngeq$	is not greater than nor equal to (same as $<$)	$2 \ngeq 3$
$\nleq$	is not less than nor equal to (same as $>$)	$3 \nleq 2$

B. Properties of equations

Addition property: If $a = b$ then $a + c = b + c$

Multiplication property: If $a = b$ then $a \cdot c = c \cdot b$

Division property: If $a = b$, $c \neq o$ then $\frac{a}{c} = \frac{b}{c}$

(c = c is reflexive property)

Equality is reflexive; that is for any expression a, we may write $a = a$.

Equality is symmetric, that is for any expressions a and b, if $a = b$ then $b = a$.

Equality is transitive; that is, for any expressions a, b and c, if $a = b$ and $b = c$, then $a = c$.

C. Properties for operations with natural numbers

1. We postulate these properties for the operation addition:

   a. closure: For any two natural numbers a and b there exists one and only one natural number c such that $a + b = c$.

b. Commutative: For any two natural numbers a and b, $a + b = b + a$

c. Associative: For any natural number a, b and c, $a + (b + c) = (a + b) + c$

2. We postulate these properties for the operation multiplication:

a. Closure: For any two natural numbers a and b, there exists one and only one natural number c such that $a \cdot b = c$.

b. Commutative: For any two natural numbers a and b, $a \cdot b = b \cdot a$

c. Associative: For any natural numbers a, b, and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

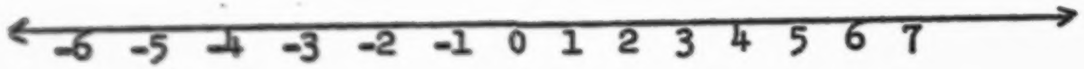
3. We postulate this property for combined use of multiplication and division.

Distributive: Multiplication of natural numbers is distributive with respect to addition, $a\,(b + c) = ab + ac$

It then follows that by the symmetric property of equality $ab + ac = a\,(b + c)$. Also since multiplication is commutative, $(b + c)\,a = a\,(b + c)$.

Note: The teacher familiar with "traditional" material can readily recall that "factoring" - - use of distributive property - - comprised well in excess of half the usual first year algebra course.

D. The number line



1. Greater than $(>)$ means the number whose numeral is farther to the right on the line.

Example: -4 is to the right of -6 therefore $-4 > -6$

0 is to the right of -3 therefore $0 > -3$

3 is to the right of -5 therefore $3 > -5$

It is then helpful to point out to the student that the point of the symbol is toward the smaller quantity and the symbol "opens" towards the larger quantity. $3 > 2$ and $2 < 3$

Note: The teacher will have some trouble with students seeing that "to the right of" means "to the right of" something and that the "something" must be "to the left" of what we are comparing in the first place. The teacher may wish to use some of his or her English background here in the use of subject and verbs.

2. The teacher may provide opportunities for discussion so that the meaning of symbols and axioms are quite clear. An ample amount of material is usually found in most textbooks. Have the students read these signs in many problems, picture the number line in their minds and compute as true or false statements. Remember the symbol $(\leq)$ less than or equal to has two parts to it, = and < eg: $-1 \leq -1, 0, 1, 2 \ldots$ (This is read negative - one is less than or equal to negative one, zero, one, two and so on.)

3. If for two natural numbers a and b, there is a natural number x such that $a + x = b$, we may say that $a < b$ and write "$a < b$," or that b is greater than a.

   a. This order is not reflexive; i.e. $a \nless a$

   b. This order is not symmetric since $a > b$ does not imply that $b > a$

   c. This order is transitive because $a < b$ means $a + d = b$ for some natural number c. $b < c$ means $b + e = c$ for some natural number e. Then substituting $a + d$ for b $(a + d) + e = c$.

   d. By the associative property of addition $a + (d + e) = c$. Since natural numbers are closed under addition, there is a natural number $d + e$ which when added to a gives c, thus by definition of order $a < c$.

E. Open sentences

Students may confuse open sentences with the variable unless emphasis is given to the phrase "are neither true nor false." A sentence with one or more symbols that may be replaced by the elements of a given set. Open sentences cannot be identified as true or false without insertion of additional information.

Examples: The statement "He is a teacher." Until "He" is replaced with a name of a person, the sentence is neither true nor false. $n < 5$ until "n" is replaced with a numeral it is an open sentence.

The important thing for the student in open sentences is an understanding of the replacement set.

(See general number chart section III-a for the set of natural numbers, the set of integers, and the set of rational numbers.)

F. Replacement set

The teacher should make this concept clear to the students as it will aid in the solution of an equation.

Have the students solve many examples with the solution set requiring examples of: (1) Set of natural numbers. (2) Set of integers. (3) Set of rational numbers.

G. Subsets and disjoint sets

The teacher may wish to use Venn diagrams to show relationships between sets. Give the students examples of different sets of numerals. Have them draw the diagrams showing the union, intersection, subset and disjoint sets.

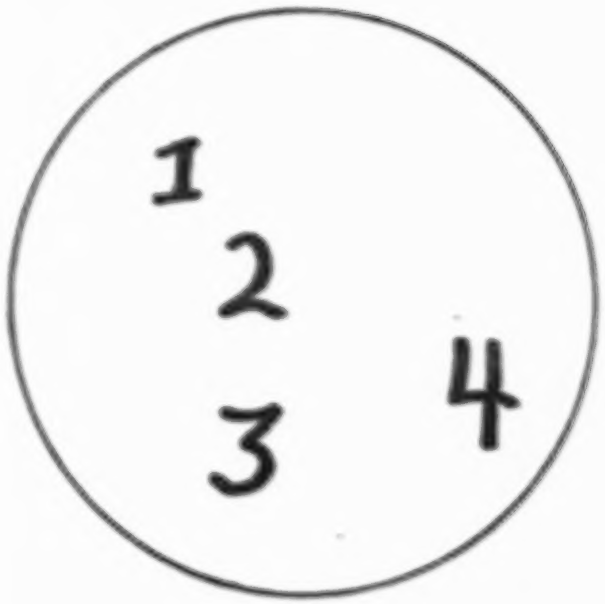
Subset: Set A is a subset of set B if set A is the empty set or if set A contains one or more elements, and only the elements, which belong to set B.

$A \subseteq B$ is read A is a subset of set B. Use the symbols to name the sets on the diagram.

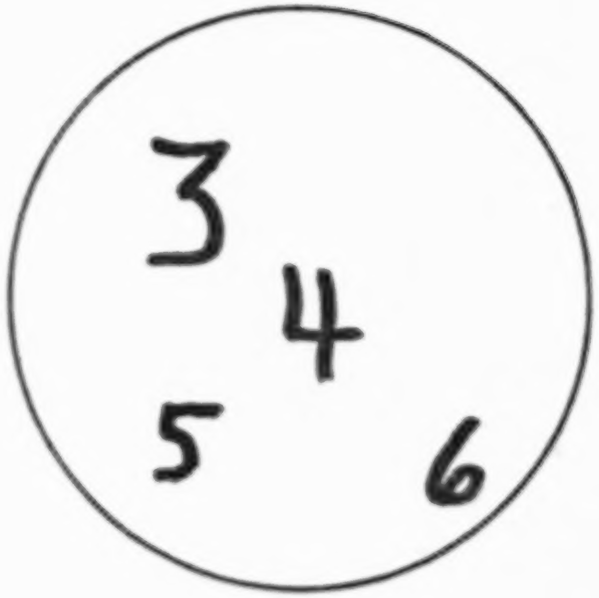
H. Solution set

The student may use the term solution set or truth set to describe the replacement set of a sentence which makes the statement true. Given set A and B.

Set A = 1, 2, 3, 4



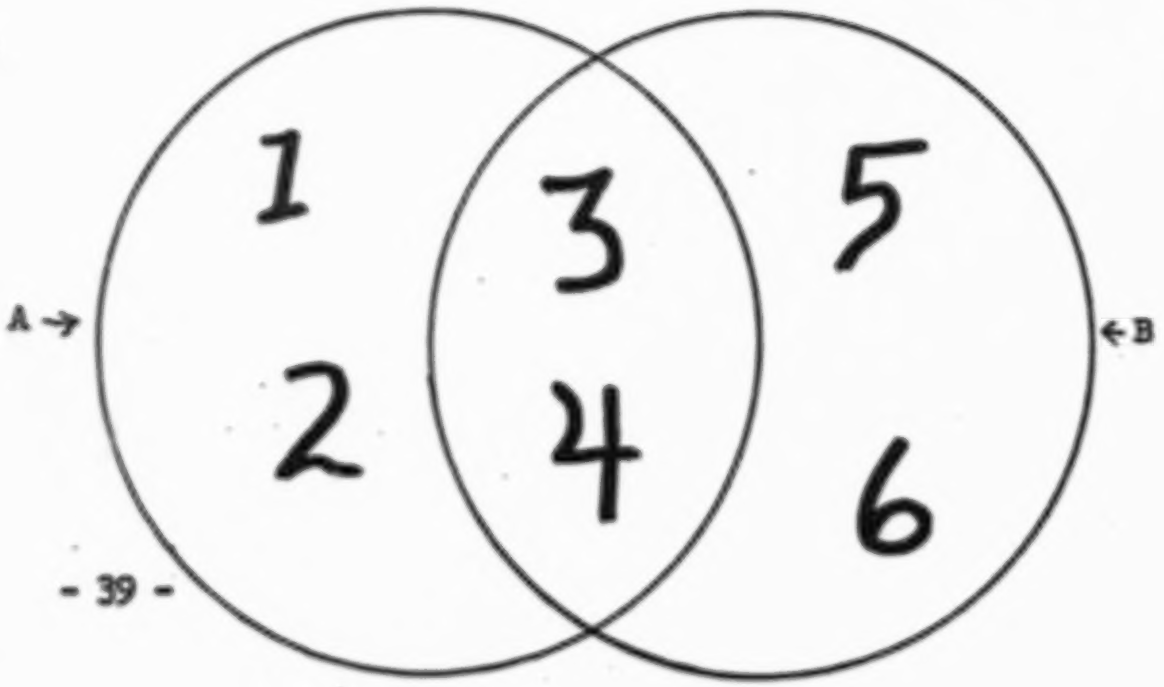
Set B = 3, 4, 5, 6



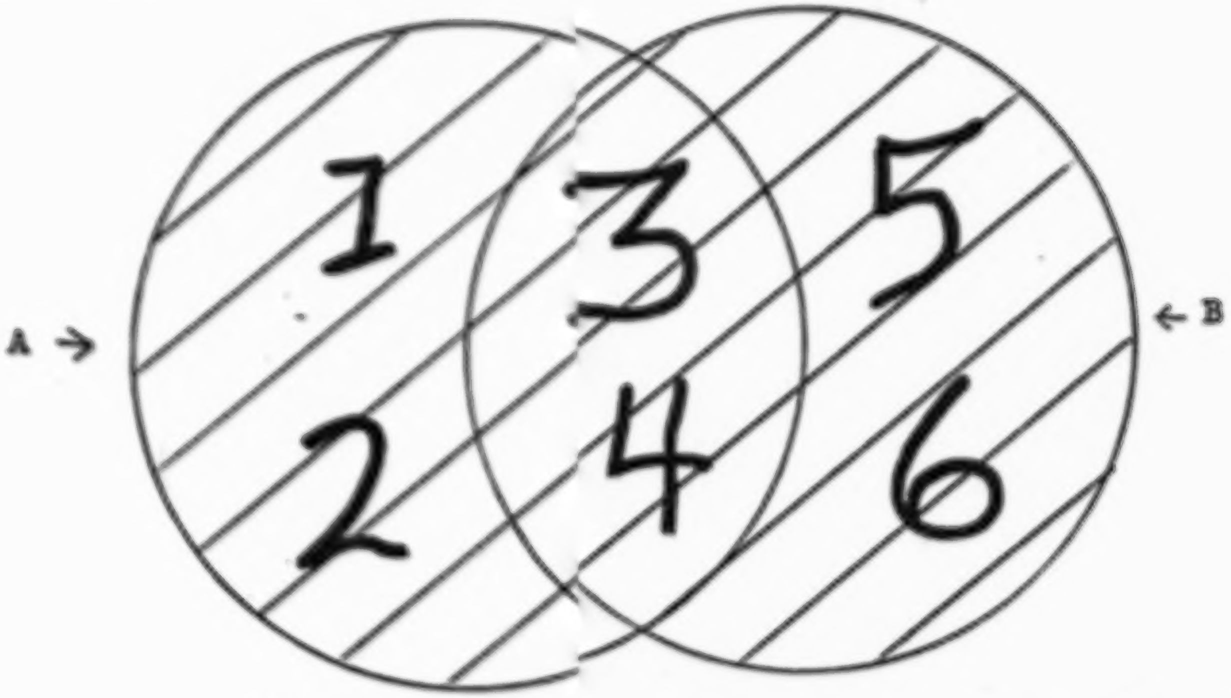
1. Elements common to both sets:

   $A \cap B = \{3, 4\}$

Note: The name for the symbol used to denote intersection is cap.



2. All the elements that are in either set: $A \cup B$ = 1, 2, 3, 4, 5, 6
(Read: A union B is the set containing elements 1, 2, 3, 4, 5, 6)

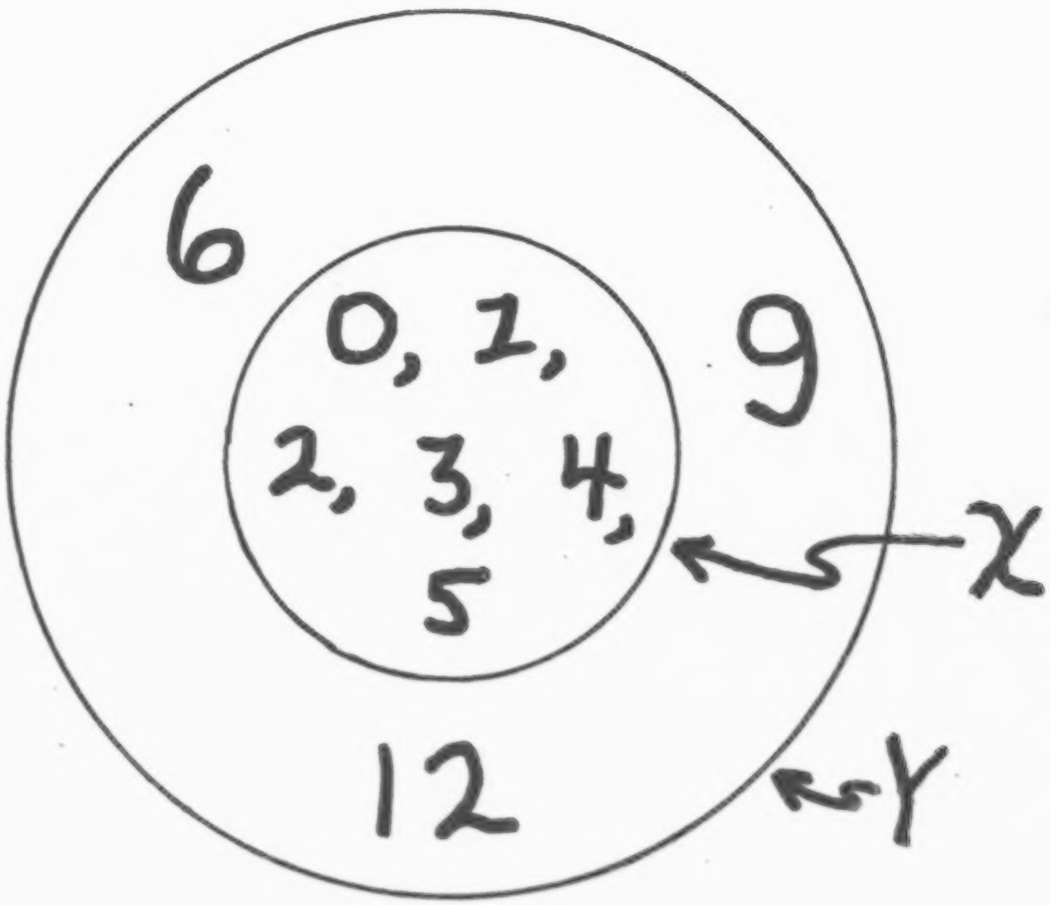


The union of two or more sets is the set whose members are in either of the sets or in all of the sets. ( )

$X = \{0, 1, 2, 3, 4, 5\}$ $Y = \{0, 3, 6, 9, 12\}$

$X \cup Y = \{0, 1, 2, 3, 4, 5, 6, 9, 12\}$

By Venn diagram



Note: One "name" for the symbol used to denote union is "cup."

3. Samples and examples:

a. The teacher may wish to develop this further by using a third set.

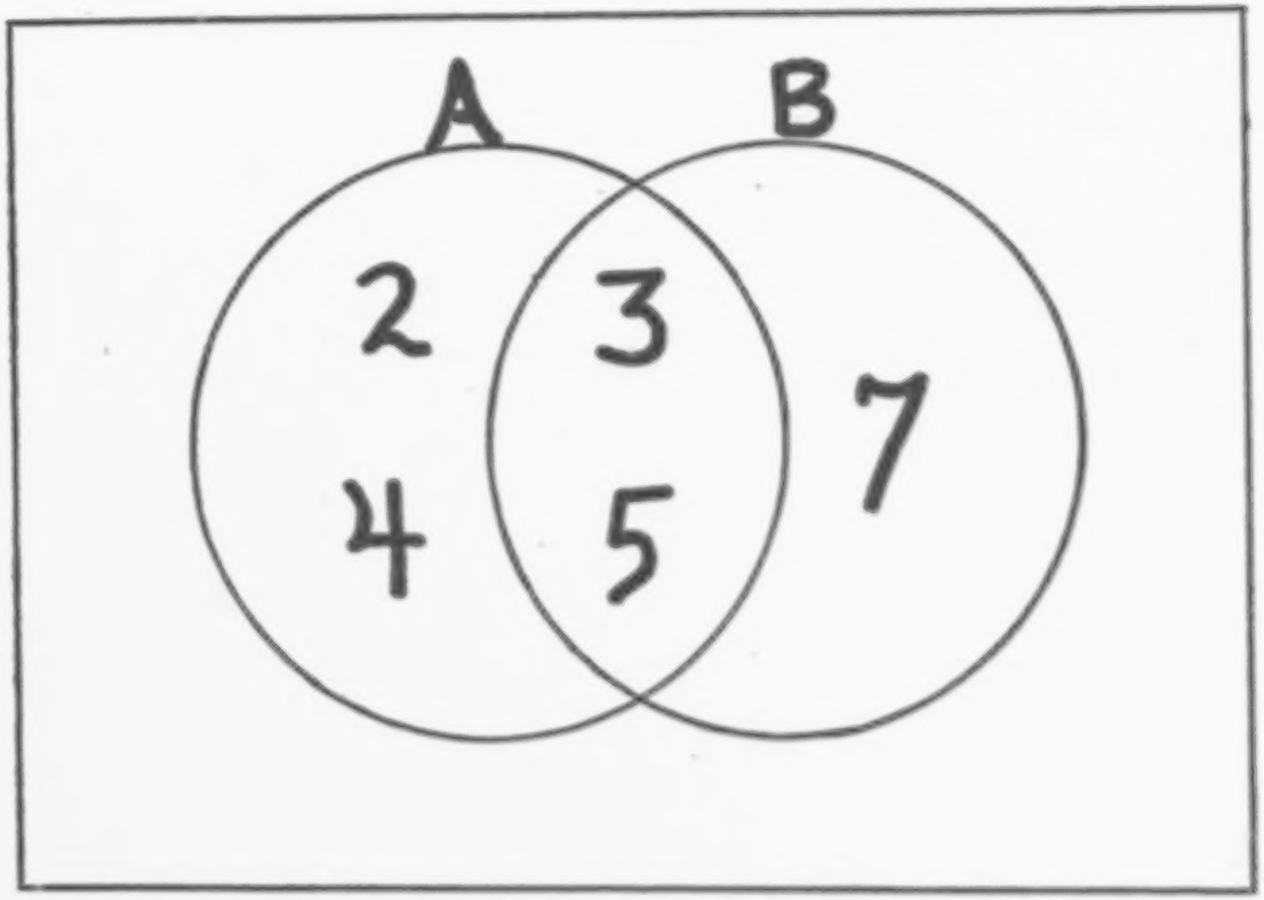
Set A = $\{2, 3, 4, 5\}$

Set B = $\{3, 5, 7\}$

Set C = $\{2, 4, 5\}$

$A \cup B = \{2, 3, 4, 5, 7\}$

$A \cup B$

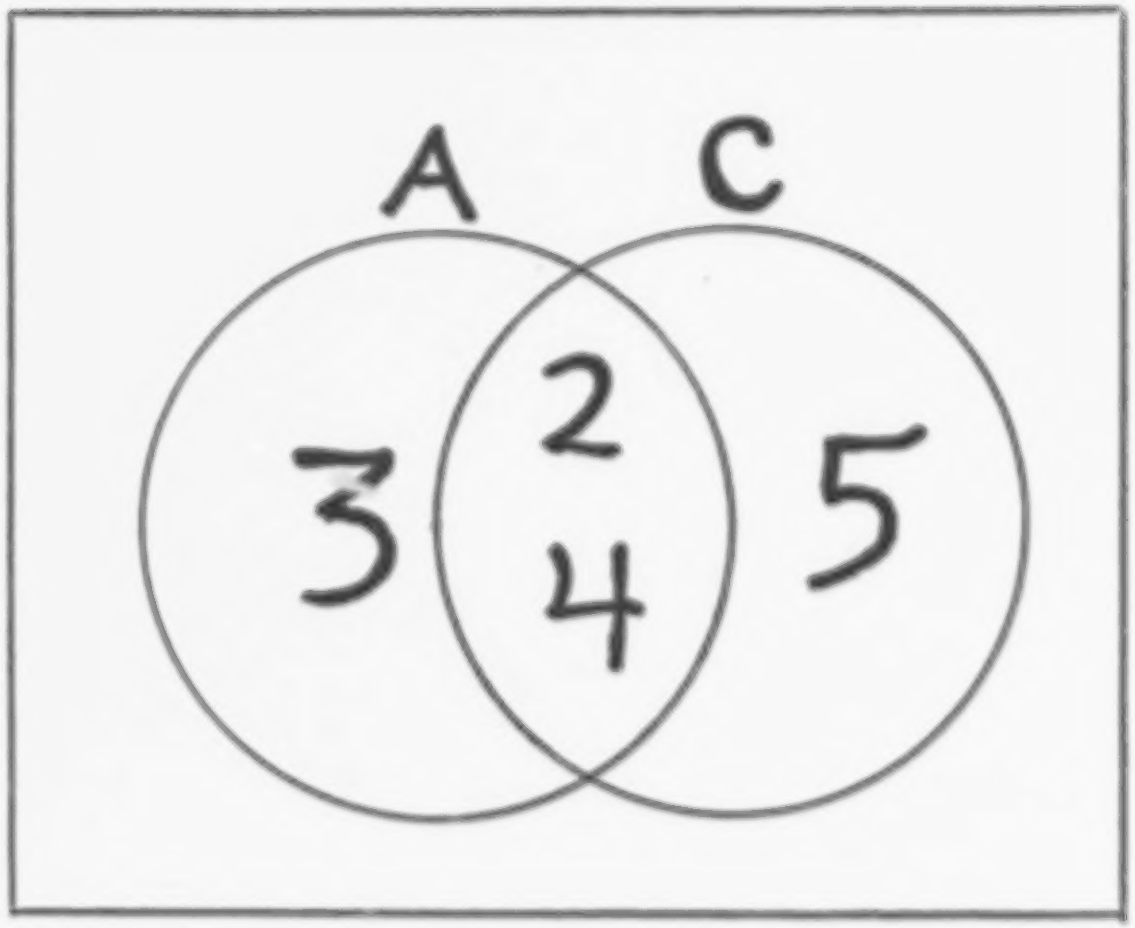


$A \cap B = \{3, 5\}$

$A \cup C = \{2, 3, 4, 5\}$

$A \cap C$

$A \cap C = \{2, 4\}$



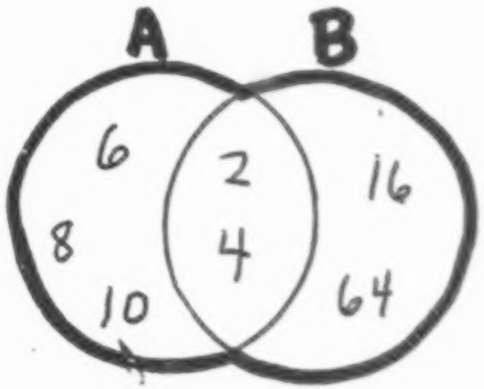
b. Here are some Venn diagrams with union and intersection of sets:

Using the Venn diagrams

$A = \{2, 4, 6, 8, 10\}$ $B = \{2, 4, 16, 64\}$

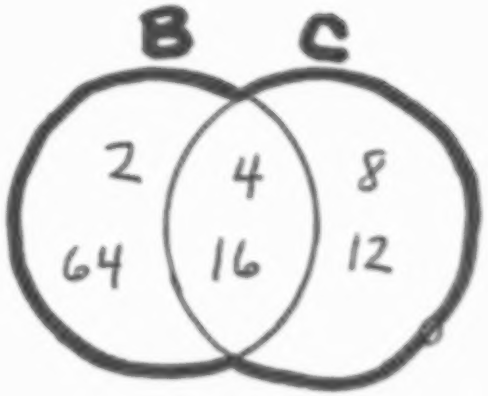
$C = \{4, 8, 12, 16\}$ List the sets below the diagram.

1. $A \cup B$



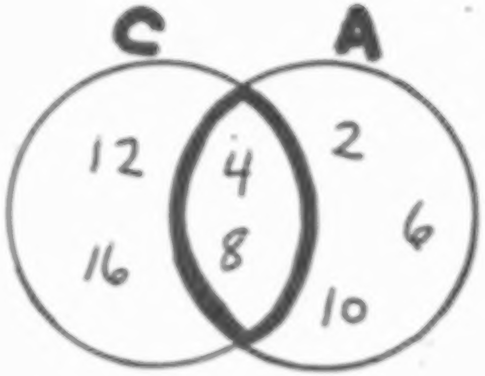
$\{2,4,6,8,10,16,64\}$

2. $B \cup C$



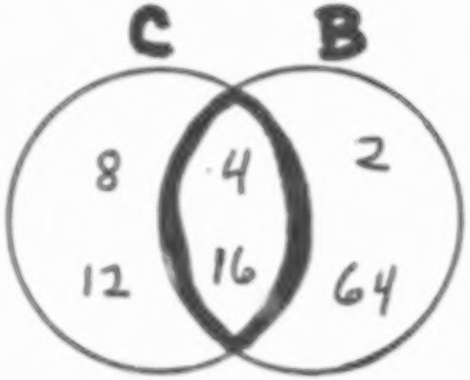
$\{2,4,8,12,16,64\}$

3. $C \cap A$



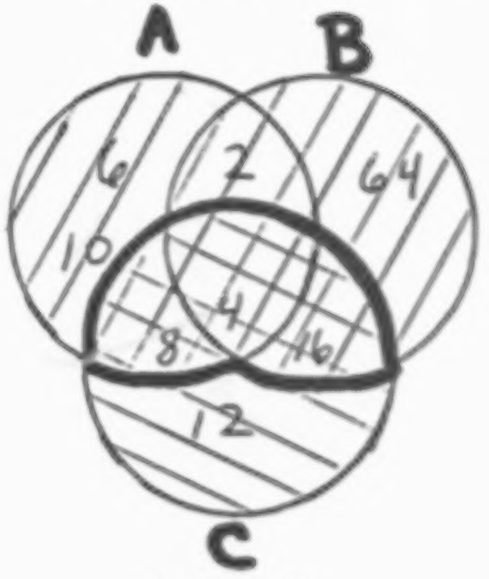
$\{4,8\}$

4. $C \cap B$



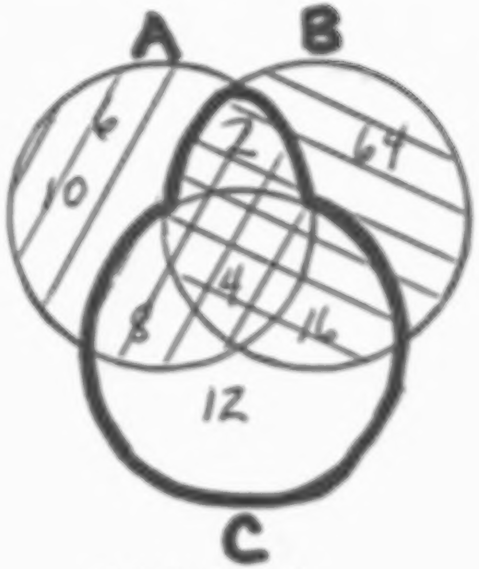
$\{4,16\}$

5. $(A \cup B) \cap C$



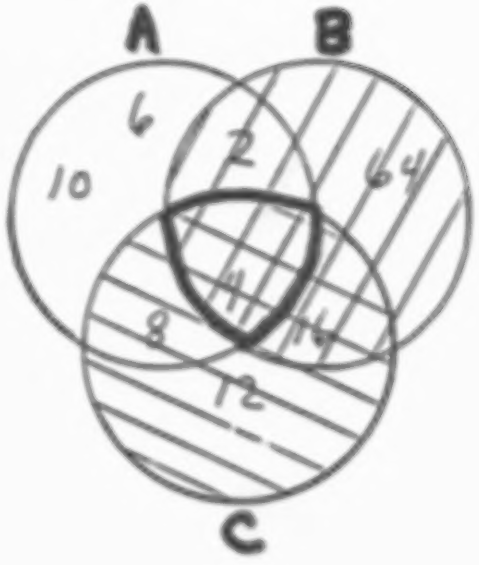
4, 8, 16

6. $(A \cap B) \cup C$



2,4,8,12,16

7. $A \cap (B \cap C)$



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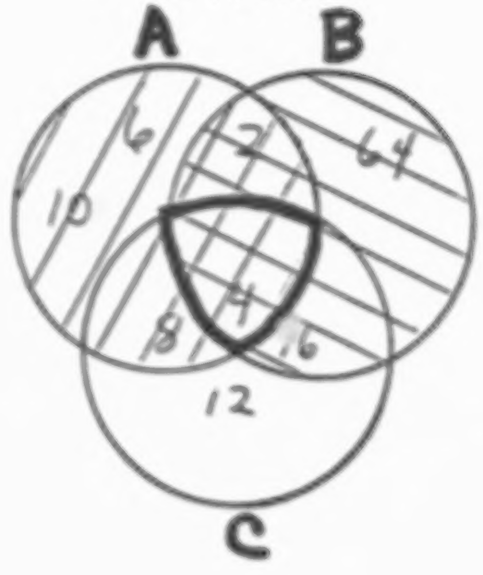
8. $(A \cap B) \cap C$



4

The solution set must contain no elements, one element, several elements, or an unlimited number of elements.

I. Use of addition property in equations

To find the root of the more complicated equations the student will realize a need for a systematic process. For all integers, a, b, and c
If $a = b$, then $a + c = b + c$
What is the solution set:

$y + 7 = 31$ (problem)
(if $a = b$, then $a + c = b + c$)
$(y + 7) + (-7) = 31 + (-7)$
(associative prop of addition)
$y + \quad 7 + (-7) \quad = 31 + (-7)$
(sum of opposites

$y + 0 = 24$

(add. prop. of zero)

$y = 24$

24

1. Check results

(original problem)

$y + 7 = 31$

(by replacement)

$24 + 7 = 31$

(an identity   24 is in the Solution Set)

$31 = 31$

2. Transpose

When the student has enough experience in the above method the transposing method can be used.

(problem)

$y + 7 = 31$

(unknown on left)

$y = 31 - 7$

$y = 24$

Note: The method of transposing is not recommended to be taught at this level of instruction. There is plenty of time for shortcuts which the students will learn on their own. The student will learn the "why" with the addition property much better if the teacher does not use this method of approach!

The explanation for each step is the same as the axiom method.

J. Use of multiplication properties in equations

The student will need to know how the knowledge of inverse operations enables us to determine which property of equations to use and which number to use. For all integers a, b and c.

If $a = b$ then $a \cdot c = b \cdot c$

K. Using the properties of equations

1. The student should be asked to give reasons for each step in a solution.

2. Point out the method of checking the solution by emphasizing that we replace the variable in the original equation only.

3. When two or more properties of equations are used, we have no general rule for which property to use first.

$2n - 1 = 5$
addition property
$2n - 1 + (1) = 5 + (1)$
associative property
$2n + (-1 + 1) = 5 + 1$
sum of opposites
$2n + 0 = 5 + 1$
sum of integers
$2n + 0 = 6$
sum of integers
$2n = 6$
identity element

$\frac{2n}{2} - \frac{6}{2}$

division property

$2/2 \cdot n = 3$

division

$1 \cdot n = 3$

rename 2/2

$n = 3$

identity element

$\frac{n}{4} = 5$ - problem

$\frac{n}{4} \cdot 4 = 5 \cdot 4$ (inverse operation) if $a = b$ then $ac = bc$

$n \cdot \frac{4}{4} = 20$ rewrite problem (commutative property of multiplication)

$n \cdot 1 = 20$ ($\frac{4}{4}$ is identity for 1)

$n = 20$

4. Division properties of equations: The division properties of equations may be explained as follows.

$3n = 15$ for all integers a, b, and c

$\frac{3n}{3} = 15/3$ if $a = b$ then $\frac{a}{c} = \frac{b}{c}$ if $c \neq 0$

$\frac{3}{3} . n = 5$ $(\frac{3}{3} = 1)$

$1 \cdot n = 5$ (identity for multiplication)
$n = 5$

VI. Factoring and Primes

It should be remembered that in this unit of study only the set of integers as factors will be covered. Certainly other factors will be covered in the student's later studies in mathematics. Restricting this unit of study to integers as factors will avoid confusion that might otherwise result, due to the broadness of factorization in various sets of numbers.

The word "factor" is defined to mean any of the numbers to be multiplied to form a product. A factor of a product is also a divisor of the same product, excluding zero. It should be pointed out that changing the order of the factors does not change the product, due to the commutative property of multiplication with respect to the set of integers. (e.g. $3 \cdot 4 = 12$, $4 \cdot 3 = 12$)

A. The smallest factors of a number: The number one as a factor

The student learns how to express a natural number as the product of its smallest factors.
$12 = 2 \cdot 2 \cdot 3$

Since the number 1 is a factor of every natural number and is a repeating factor we shall agree not to use 1 in expressing the factorization of a natural number.

$$6 = 2 \quad 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdots\cdots$$

Students should work a number of examples naming a number as a product of its smallest factors.

B. Prime numbers

A natural number is called a prime number if it is greater than one and its only factors are itself and one.

The teacher may point out that the number 1 does not satisfy the unique factorization properties of integers.

C. Composites

Any natural number greater than "one" that is not prime is a composite.

Every composite number can be expressed as the product of two or more prime numbers. Except for the order of the prime factors the factorization is unique - that is - there is only one set of prime factors for each composite.

Numbers such as 4, 6, 8, 9, ... which have the property that this number of objects or counters can be placed in a rectangular or square arrangement with an equal number in each row and columns and more than one row and more than one column, are composite numbers.

Teaching example: Have the students list the prime and composite numbers between one and two hundred.

Because of the student's need to express rational numbers in their simplest form - that is - where the numerator and denominator are relatively prime - it is most important that they be able to recognize composite numbers and to "prime-factor" them.

Example: $\frac{48}{60} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 5} = \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{2.2}{5} = 1 \cdot 1 \cdot 1 \cdot \frac{2.2}{5} = \frac{4}{5}$

Since every counting number greater than "one" which is not prime is a composite, and every composite is a product of primes, it follows that every counting number greater than one can be determined to be prime if no prime is a divisor of it.

Because two is the first prime - every second number after it is a multiple of it and therefore not prime. Because three is the next prime after two every third number after it is a multiple of it and therefore not prime. This pattern would continue with each succeeding prime found.

Example: Prime	Set
2	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...
3	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...
5	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...
Net:	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...

The above method of finding prime numbers is credited to the Greek mathematician Eratosthenes of some 2000 years ago.

The smallest prime is 2.

The largest known prime is $180 \times (2^{127} + 1)^2 - 1$. "Scientific American," February 1952 issue, calculated that it would take 1,809, 250×10^{68} columns of its magazine to print this prime using standard type.

Prime numbers have attracted the attention of mathematicians from the earliest days. The ancient Greek numerologists of the Pythagorean School considered primes endowed with mysterious powers. The odd primes below ten are 3, 5, and 7, and of these 3 and 7 are less familiar than 5. So 3 and 7 were believed to posses many mystic powers for good and evil. It is from this belief that we have three cheers, seven days of the week, seven seas, beginning to vote at the age of 3 x 7 years, and so on.

The teacher will wish also to illustrate Goldbach's conjecture, that every even number greater than 4, is the sum of two prime numbers.

Relatively prime: Two or more numbers are said to be relatively prime if their greatest common factor is "one."

Because of its being lengthy and quite often tiresome (if not impossible at times) the use of Eratosthenes's method in determining prime numbers is not recommended as a standard procedure.

The knowledge of a few divisibility checks will solve most problems for the student wishing to determine if a number is prime or not.

Because every 2nd number is a multiple of two - recognizing if a number is a multiple of two will solve 50% of all prime checks, etc.

Example:

Prime	Not Prime	% of natural numbers not prime and not counted by any other check
2	every 2nd	50%
3	every 3rd	16 2/3% - not 33 1/3 % - every second 3rd was a multiple of 2
5	every 5th	6 2/3% - only $\frac{1}{3}$ of every multiple of 5
7	every 7th	$3\frac{17}{21}$ % - only $\frac{4}{15}$ of every multiple of 7

With a recognizable check for eleven and the ones above, a student would be able to know if approximately 80% of all numbers were prime or not!

D. Divisibility

The student must have a clear understanding that the term "divisible by," as we are using it here, means to obtain a natural number for a quotient with a remainder of zero.

1. Divisibility checks

for 2 - A whole number is divisible by two if and only if the digit in the units place is a 0, 2, 4, 6, or 8.

for 3 - A whole number is divisible by three if and only if the sum of its digits is a multiple of three.

for 5 - A whole number is divisible by five if and only if the digit in the units place is zero or five.

for 7 - A whole number is divisible by seven if and only if, when the last digit (in units place) is doubled and subtracted from the preceding numeral, the remainder is a multiple of seven. If the resulting number cannot be immediately determined by sight as to whether or not it is a multiple of seven, then repeat the process the same as before and as many times as necessary to arrive at a simple enough form that the determination can easily be made.

Example: 1771

double the units digit $1 \cdot 2 = 2$, subtract from 177, $177 - 2 = 175$
double the units digit $5 \cdot 2 = 10$, subtract from $17 - 10 = 7$
7 is a multiple of seven and 177 is thus found to be divisible by 7

for 11 - A whole number is divisible by eleven if and only if the difference of the sum of the digits found in 10^1, 10^3, 10^5, etc. ("odd powers of ten") places and the sum of the digits found in the 10^0, 10^2, 10^4, etc. (even powers of ten") places is a multiple of eleven or zero.

Example: 4972 $9 + 2 = 11$ $11 - 11 = 0$, divisible by eleven
$4 + 7 = 11$

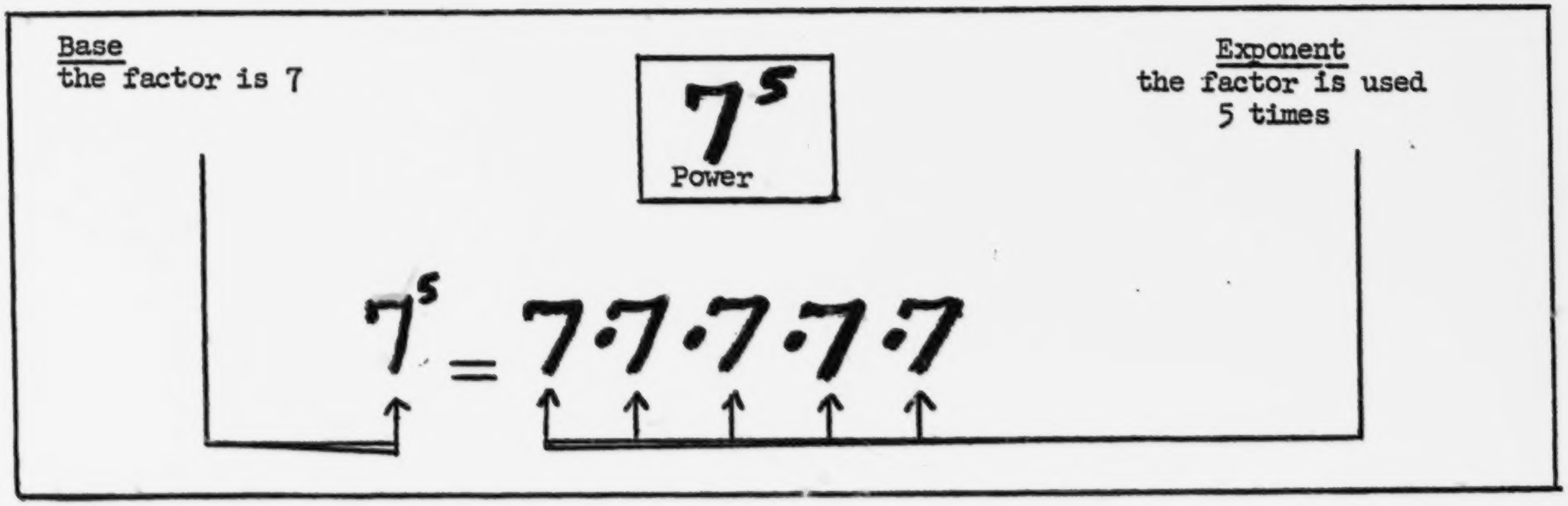
Note: Although the checks for seven and eleven may seem quite difficult or lengthy for many students - for many others it is a challenge which they will enjoy and use.

For the students to become proficient with primes, composites and factorization, it is necessary that they have a good deal of drill work in this area.

Hint: A student at this grade level is working well in this area when he or she can write from 2 to 100 and identify all the primes and prime-factor the remainder in approximately 30 minutes of time.

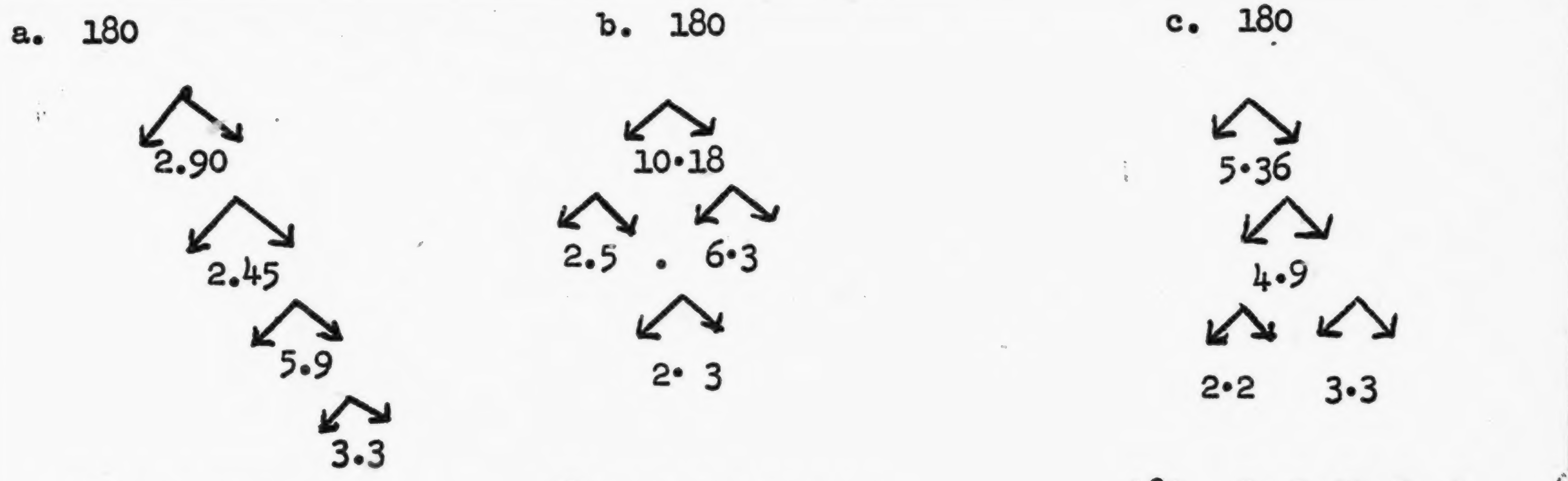
E. Using Exponents

The student should recall the meaning of the terms power, base, and exponent. The following chart may be helpful.



The exponent is used to avoid the writing of a factor many times.
(The teacher may be interested in knowing that the term "involution" means raising to a power.)

1. A factor tree. The student can express the factors of 180 as follows: (root type)



a. $$\begin{aligned}180 &= 2\cdot 2\cdot 5\cdot 3\cdot 3\\ &= 2\cdot 2\cdot 3\cdot 3\cdot 5\\ &= 2^2\cdot 3^2\cdot 5\end{aligned}$$

b. $$\begin{aligned}180 &= 2\cdot 5\cdot 2\cdot 3\cdot 3\\ &= 2\cdot 2\cdot 3\cdot 3\\ &= 2^2\cdot 3^2\cdot 5\end{aligned}$$

c. $$\begin{aligned}180 &= 5\cdot 2\cdot 2\cdot 3\cdot 3\\ &= 2\cdot 2\cdot 3\cdot 3\cdot 5\\ &= 2^2\cdot 3^2\cdot 5\end{aligned}$$

(For convenience, the factorization is usually written with the bases in ascending order)

F. Exponent one and zero

Any number is its own first power. $2^1 = 2$, $6^1 = 6$

We usually omit the exponent 1 when stating the first power. Point out to the student that $6^1 = 6$ does not mean 6×1.

When the exponent 0 is used it means the same as 1.

$6^1 \div 6^1 = 6^{1-1} = 6^0$

$6^0 = 1$ because $6 \div 6 = 1$

Example: $\frac{6^5}{6^5} = 6^{5-5} = 6^0 = 1$

$0^0 \neq 1$ since 0^0 is undefined

G. The greatest common factor (G.C.F.)

The teacher may explain that when two or more numbers have the same number as a factor, that factor is a common factor.

The largest number that is a common factor of two or more numbers is called the greatest common divisor or the greatest common factor.

Use the following steps in finding the greatest common factor of two or more numbers, such as 24 and 36.

(1) Write the prime factors of each number by using power notation.

$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \times 3$

$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \times 3^2$

(2) Pick out the smallest power of each common prime factor..

2^2 and 3

(3) The product of the smallest powers of the common prime factors is the greatest common factor or divisor.

$2^2 \cdot 3 = 2 \cdot 2 \cdot 3 = 4 \cdot 3 = 12$

$21 = 3 \cdot 7$, $6 = 2 \cdot 3$, 3 is a factor of 21 and 6. Have the students work some examples.

Which number is the greatest common factor of 36 and 48? That number is the Greatest Common Factor.

(1)

36	48
2	2
3	3
4	4
6	6
9	9
12	12
18	16
36	24
	48

(2) The student can also use prime factors to find the G.C.F.

$$36 = 2^2 \cdot 3^2$$

$$48 = 2^4 \cdot 3$$

Pick out the smallest power of each prime factor. $(2^2 \cdot 3)$ The product of these smallest powers of common prime factors is the Greatest Common Factor. $2^2 \cdot 3 = 12 =$ G.C.F.

An additional method used and most helpful when the numbers are large is the Euclidean Algorism.

About 300 B.C. Euclid wrote a unique way for finding the greatest common divisor or the highest common factor, as follows:

Divide the larger number by the smaller. Divide the divisor by the remainder. Continue dividing divisors by remainders until the remainder is zero. The last divisor used is the greatest common divisor of the two given numbers.

Example:

```
3136  4096
      3136     3
       960  3136
            2880      3
             256    960
                    768      1
                    192    256
                           192      3
                            64    192
                                  192
                                    0
```

Thus the greatest common divisor or factor of 3136 and 4096 is 64.

H. The least common multiple (l.c.m.)

The student may list the multiples of two or more numbers. Then by inspection you can see the least common multiple.

(1) 2 = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

3 = 3, 6, 9, 12, 15, 18, 21, 24

4 = 4, 8, 12, 16, 20, 24

(2) List the prime factors of each number.

```
24 = 2 · 12                        30 = 3 · 10
     1   ^                              1   ^
     2 · 2 · 6                          3 · 2 · 5
     1       ^
     2 · 2 · 2 · 3
---------------------
     2^3 · 3
```

Pick out the largest power of every factor in the number.

$(2^3 \cdot 3 \cdot 5)$

The product of the largest power of every factor is the Least Common Multiple.

$$2^3 \cdot 3 \cdot 5 = 8 \cdot 3 \cdot 5 = 120$$

An additional method of explaining how to find the l.c.m.

Use the following steps in finding the l.c.m. of two or more numbers, such as 12 and 15.

Step (1) Write the prime factors of each number by using power notation.

$12 = 2 \times 2 \times 3 = 2^2 \times 3$
$15 - 3 \times 5$

Step (2) Pick out the largest power of every prime factor which occurs anywhere in any of the factor products.

2^2, 3, and 5

Step (3) The product of the largest powers of the prime factors is the common multiple.

$2^2 \times 3 \times 5 = 4 \times 3 \times 5 = 60$

By applying the rules of divisibility (Sec. VI, C) the student may do factoring more easily and thus find the prime factors.

H. Factors of negative integers

The student is familiar with the multiplication of positive integers and negative integers as explained in part III.

The student can use the property of -1 and state any negative integer as the product of -1 and a positive integer.

$$\begin{array}{rl} -12 = & (-1) \cdot 12 \\ & \quad 1 \quad\;\; \wedge \\ & (-1) \cdot 2 \cdot 6 \cdot \\ & (-1) \cdot 2 \cdot 2 \cdot 3 \end{array}$$

The student is now ready to do examples of complete factorization using positive and negative numbers.

VII. Points, Lines, and Planes.

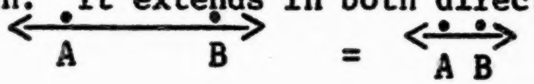
A. Points and lines

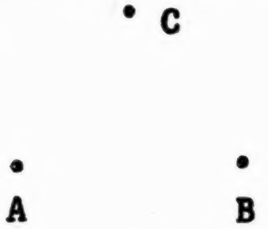
1. The student may be introduced to geometric ideas and thoughts by relating them to physical objects from his environment.

The point is an idea about an exact location; it has no dimensions whatsoever. It is represented by a dot or small cross (., +). Use letters of the alphabet to name geometric ideas. (A B C)

Related concept: Numeral is to number as dot is to point.

2. The line is an idea with one dimension, length. It extends in both directions indefinitely and is represented as follows,



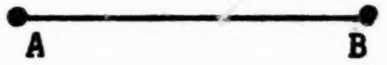


Point C lies between A and B if and only if $\overline{AC} \cup \overline{CB} = \overline{AB}$ (The bar indicates a segment).

In case the above point C is not between A and B, it is not collinear with both A and B. We consider space as a set of points. A line is then a subset of points in space. The term "line" is normally understood to be a straight line.

B. The line segment and ray

The student may be asked to measure the length of a line segment which is defined as two specific points on a line together with all the points between them.



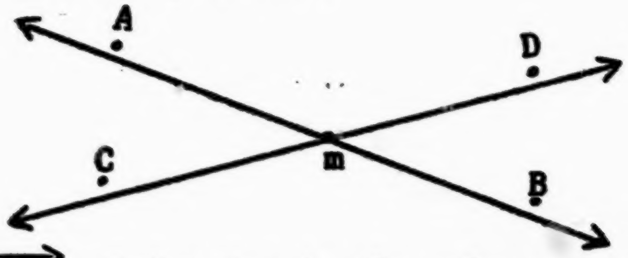
= $\overline{AB}$

Any point on a line separates the line into two half-lines. A ray is the union of a half-line and the point determining it (the end points of the ray).

C. Intersecting lines

Ask students to visualize a point space. How many distinct lines can be passed through the point (Answer: an indefinite number). Now draw two points on a paper, $\overset{\bullet}{A}$ $\overset{\bullet}{B}$. How many distinct lines pass through both of these points? (Answer: only one)

Draw two lines $\overleftrightarrow{A\ B}$ and $\overleftrightarrow{C\ D}$ intersecting at "m".



Point "m" is common to $\overleftrightarrow{A\ B}$ and $\overleftrightarrow{C\ D}$. Point "m" is the only common point, so two distinct lines can intersect in at most one point.

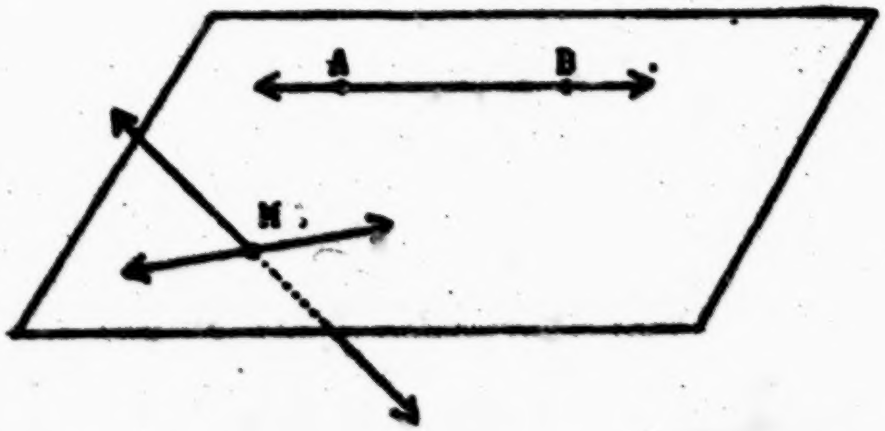
D. Planes

1. The points of a line on a plane

A plane is a set of points such that
a. the set contains three non-collinear points
b. If two points of a line are in the set, then all point of the line are in the set
c. Any two lines in the set either intersect or are parallel

To help the students to develop the concept of the plane on an intuitive basis, have them consider the surface of a window pane, the light from the sun coming through a crack in the wall, or a sheet of plastic. Point out repeatedly that a geometric plane is flat, has no thickness, and has no bounds.

Have the students construct a drawing representing a plane and place three points A, B, and M on the plane.

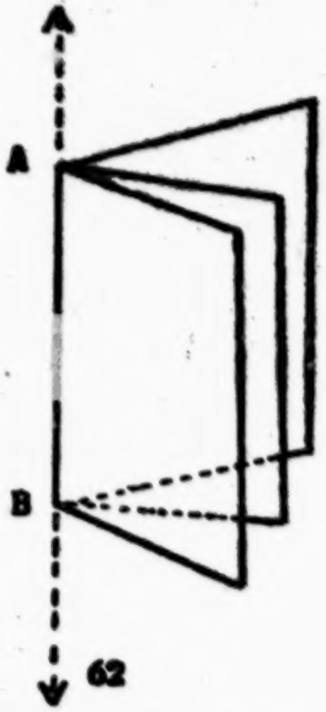


How many lines may pass through point M? B? A? (Answer: an infinite number)
How many points may be placed on the plane? (Answer: an infinite number)

Draw a line through $\overleftrightarrow{A\,B}$. Are all the points of A B on the plane? How many lines go through A B?

2. A plane through two points of a line

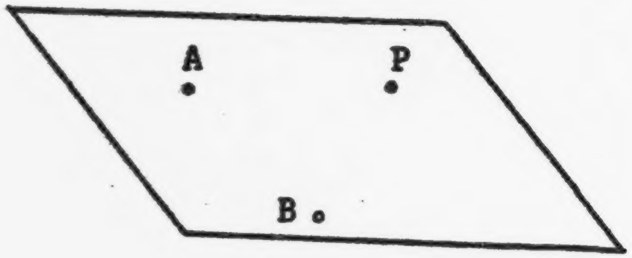
The student may think of a text book with the spine of the book as a segment of a line A B. Two points determine a line. How many pages of the text may pass through the line $\overleftrightarrow{A\,B}$? (Answer: as many as there are pages)



3. A plane is named by three points.

Now have the students place points (P), (A), and (B) on one of the pages. How many planes may pass through three (3) points which are not in the same straight line? (Answer: one and only one plane)

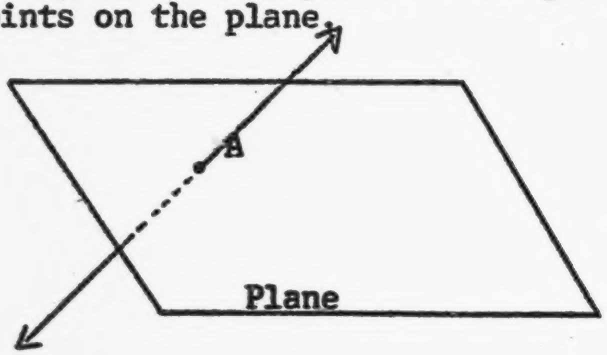
A plane can be named by three points when the points are not collinear.



4. Separation of space

A plane separates space into three distinct sets of points

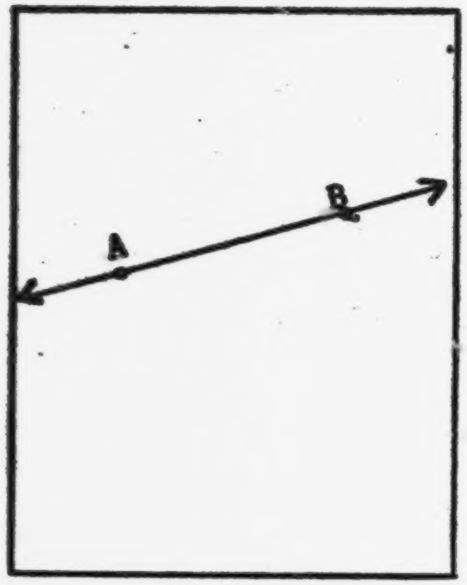
Use a sheet of plastic or a pane of glass with one point on the plane. The space is separated into two half-spaces and the set of points on the plane.



The plane is the boundary of each half space and a part of neither.

5. Separation of planes

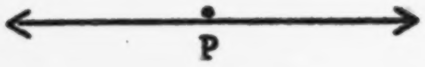
On a sheet of paper, a plane, draw a line, $\overleftrightarrow{AB}$.



Any line in a plane separates the plane into two half-planes. The lines is called an edge of both half-planes and is part of neither.

6. Separation of a line

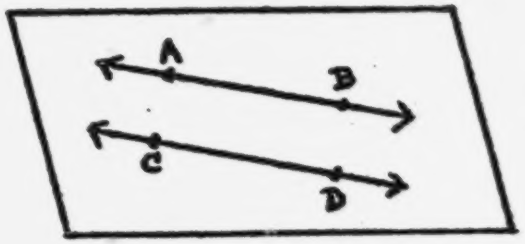
Construct a line with point P.

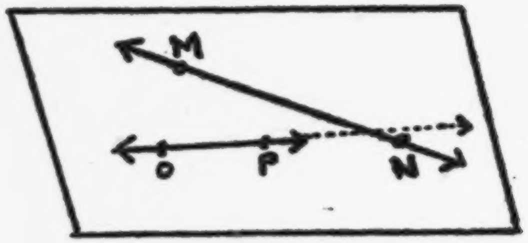


The line is separated into three sets of points: the set at P, and two sets of points called half-lines.

E. Parallel lines and planes

The student may construct the following and discover that two lines in a plane are either parallel or intersecting lines.



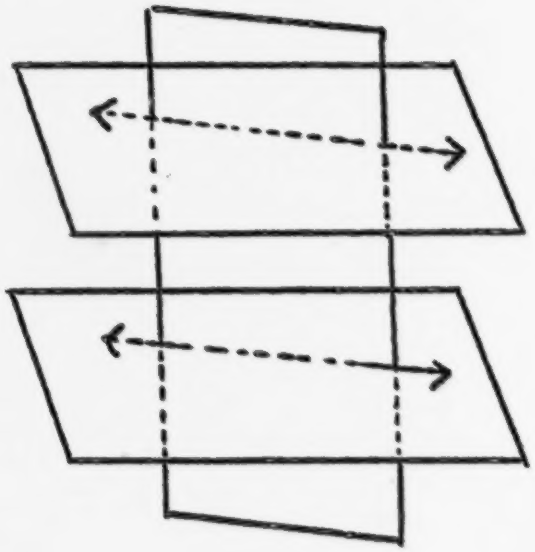


Explain that planes have no boundaries and lines are of infinite lengths in either direction.

If two planes are parallel, is every line on one of the planes parallel to the other plane?

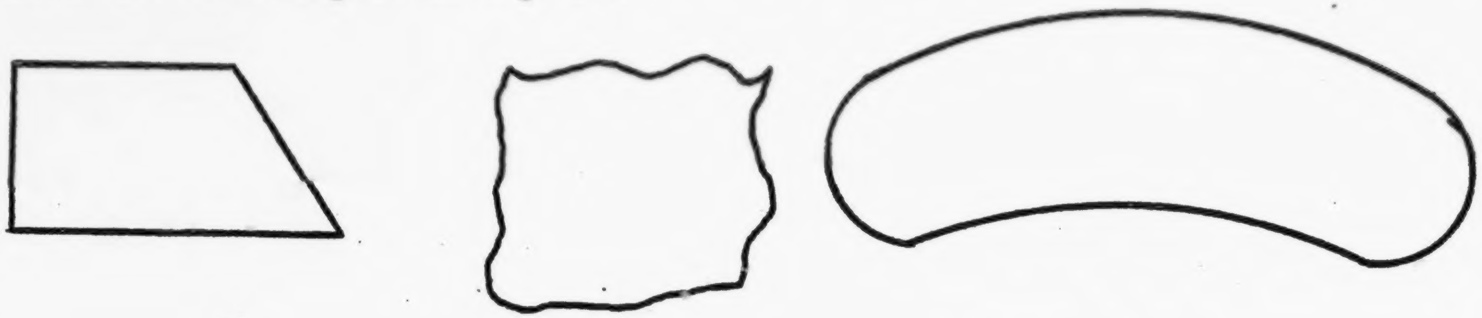
F. Polygons

1. Simple closed figures



Two planes which do not intersect are called parallel.

The student may experiment by constructing figures whose boundaries never intersect themselves and enclose part of a plane.

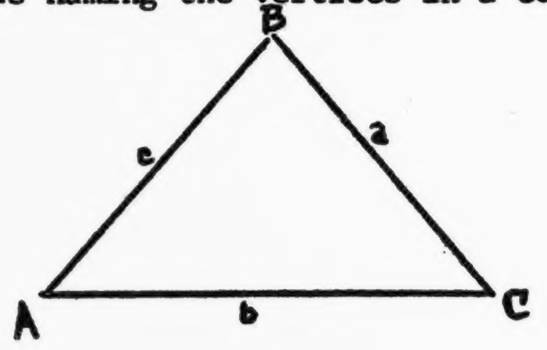


The simple closed figure separates a plane into three distinct sets; the figure, the interior and the exterior.

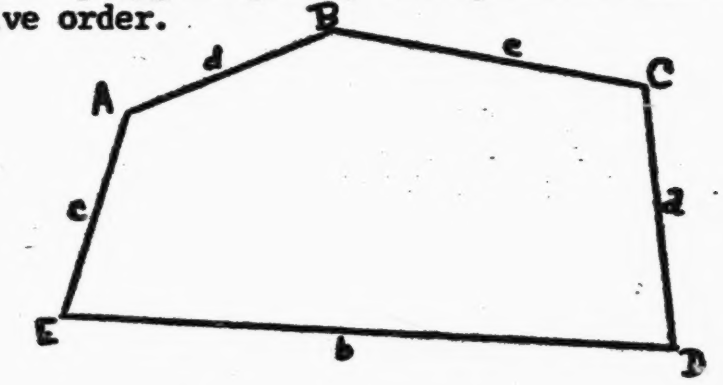
POLYGONS

No. of Sides	Vertices	Names
3	3	Triangle
4	4	Quadrilateral
5	5	Pentagon
6	6	Hexagon
7	7	Septagon
8	8	Octagon
9	9	Nonagon
10	10	Decagon
12	12	Dodecagon

The student should construct a number of polygons giving the special names as well as naming the vertices in a consecutive order.



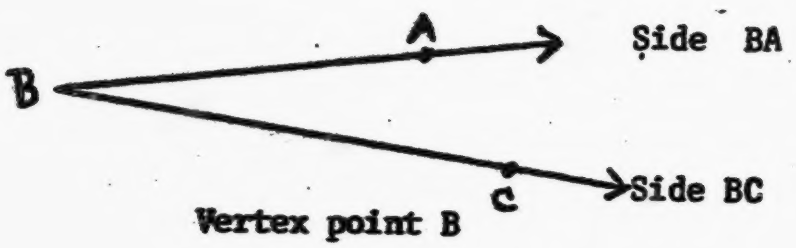
Triangle



Pentagon

The student should discover that the number of sides of a polygon is the same as the number of vertices.

The student may construct angles by forming two rays with a common vertex.

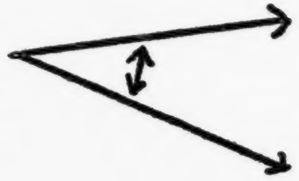


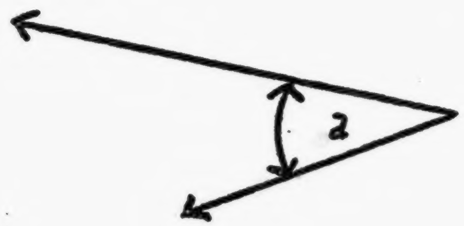
G. Angles

Angles are named by three points of the angle, the vertex or end point and a point on each ray.

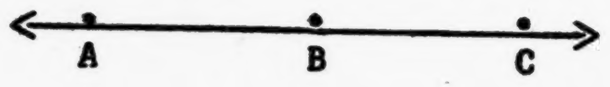
In ∠ABC, the vertex is named second. If there is no confusion as to which angle is referred to, one can simply name the vertex, B, etc.

Another convenient method:





Construct and name a straight angle.



In ∠ABC the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are considered to be an angle even though the rays are collinear.

H. Measuring angles

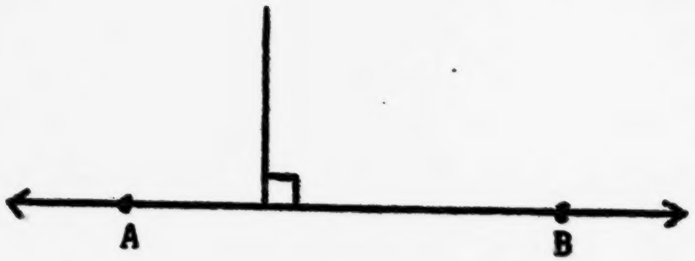
The student may be given some review on the use of the protractor.

This is an opportunity to explain that the measure of degress of a standard protractor is usually one half of the measure of degrees of a circle. (A circle has 360 arc degrees.)

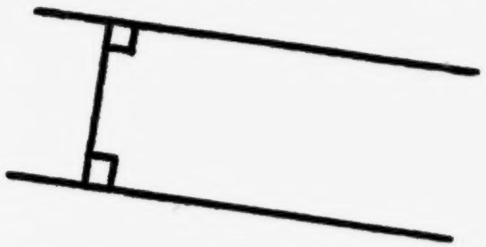
The protractor is divided into 180 arcs of equal size with each arc being one degree (1°).

Review the names of angles which are classified according to degrees or measure. (acute, obtuse, etc.)

Use the protractor to draw perpendicular lines and parallel lines.



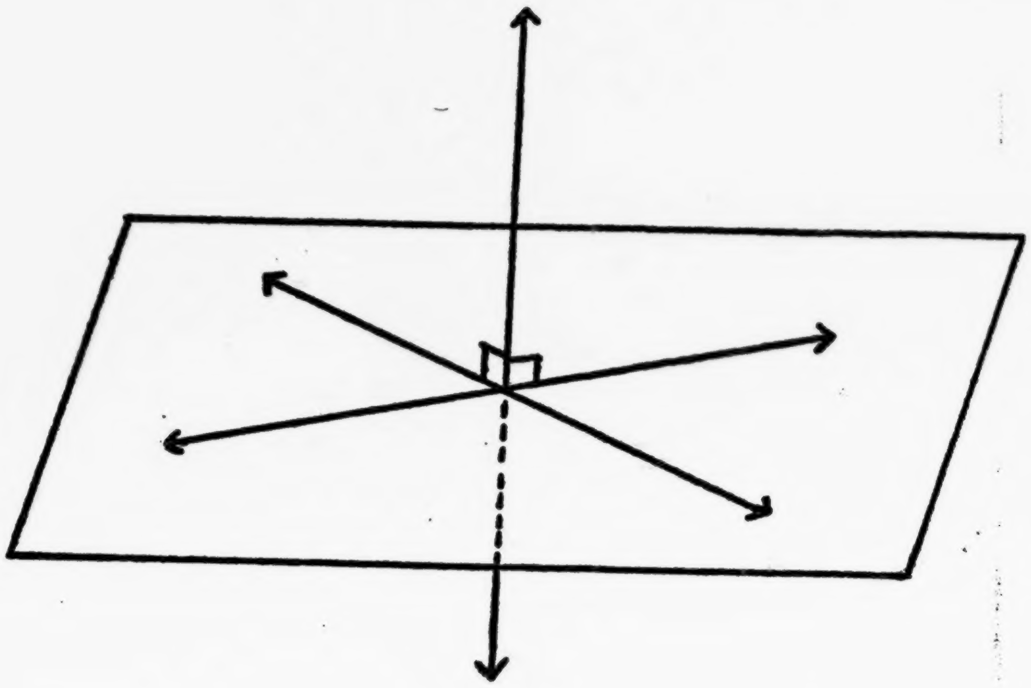
(a) Perpendicular lines



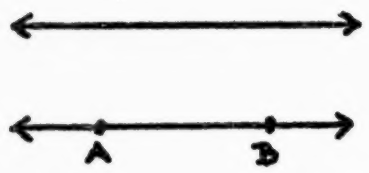
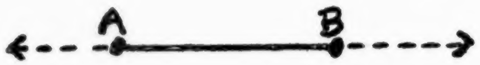
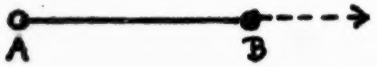
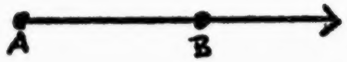
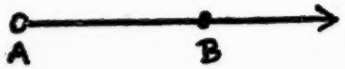
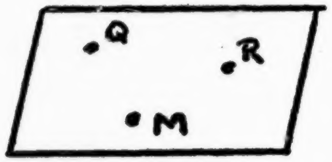
(b) Parallel lines

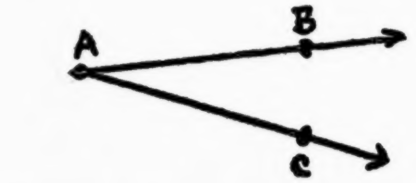
1. A line is perpendicular to a plane

   If a line is perpendicular to two lines on a plane through its foot, the line is perpendicular to the plane.



I. Summary of terms for Chapter 8

	Dot. used to represent a point. A point has no dimension (cannot be "moved"). An infinite number of points are between any two distinct points - basic concept for all geometric figures.	
	**Line.** an infinite set of collinear points with infinite length in both directions - named with the use of two points - a point on a line separates the line into 3 sets (two half lines and the point).	$\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$
	**Line Segment.** a subset of a line which has a finite length - named with the use of two points.	$\overline{AB}$ or $\overline{BA}$
	**Open Segment.** open at either end or both ends. Shown (open at A - Set consists of all points between A and B, and point B).	$\overset{\circ\!-}{AB}$ or $\overset{\circ\!-}{BA}$
	**Ray.** subset of a line which has an endpoint and has infinite length in only one direction - named with the use of two points.	$\overrightarrow{AB}$
	**Open Ray.** half line which does not include end point.	$\overset{\circ\!\rightarrow}{AB}$
	**Plane.** a two dimensional set which has no thickness or definite area. It is not bounded - plane may be named with the use of 3 non-collinear points. A line of a plane separates the plane into 3 sets (two half planes and the line). A plane Separates space into 3 sets (two half spaces and the plane).	

	Angle. the union of two rays with a common end point - end point named Vertex. An angle separates a plane into 3 sets (interior, exterior, and the angle).	∠BAC or ∠CAB sometimes ∠A
	Acute Angle. has arc measure greater than 0° and less than 90°.	
	Obtuse Angle. has arc measure greater than 90° and less than 180°.	
	Right Angle. has arc measure of 90°.	
	Straight Angle. has arc measure of 180°.	

VIII.. Measurements

Although many textbooks give much time to the development of certain concepts on measurement (Precision, Greatest Possible Error, Relative Error and Significant Digits, for example), it is suggested that there is an outstanding lack of knowledge about solving everyday problems of measurement. We believe that a more rigorous treatment of the concepts of measurement should be given as students are not proficient in fundamental operations of measurements or the concepts of changing from one unit to another.

The more complex measurements required in problems involving time, distance, rate, etc. are needed in solving algebraic sentences; the present 8th grade mathematics curriculum offers very little preparation for this type of problem.

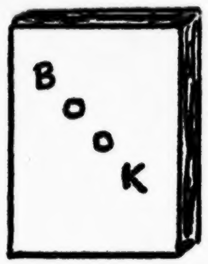
Examples:
(a) If a car goes 30 miles per hour, how many feet does it travel per second?
(b) If a plane uses 15 gallons of fuel per hour, how many quarts does it use per minute?
(c) How many years, months, weeks, and days old are you today?
(d) How much time elapses between 8:47 a.m. of one day and 5:51 p.m. of the next day?
(e) If a recipe calls for 7/8 of a cup of flour, how much would you use for half the recipe? for 3 times the recipe? (Hint: 16 tablespoons = 1 cup)

A. Assumptions and measurements

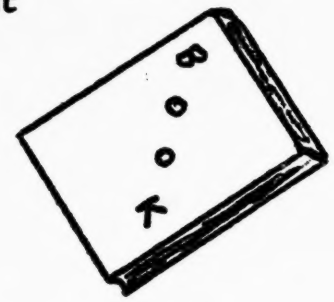
The teacher should perform some of the following demonstrations to make students aware of certain assumptions about measurements.

1. Movement

(1) Geometric models and other objects may be moved without changing size or shape.
Examples: Moving a ruler, a book, or an eraser from one location to another does not change its size or shape.

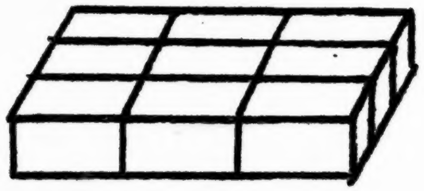


(1) Movement

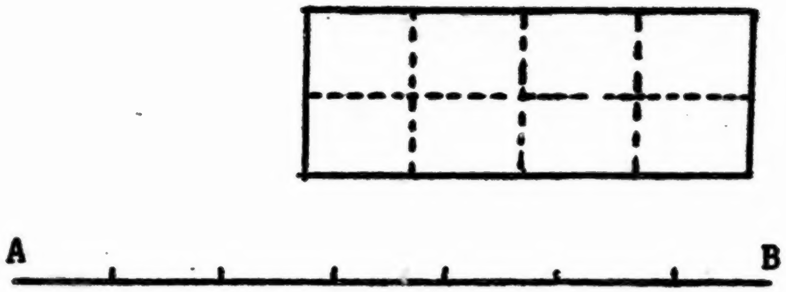


2. Subdivision

(2) A geometric figure (not a point, which is only a concept and has no dimensions) or physical object may be subdivided. Examples: A pound of butter, a city block, a pint of water, a line segment.

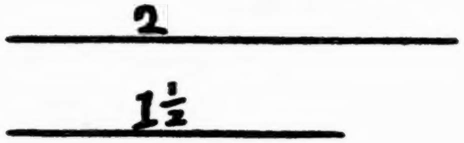


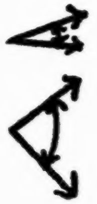
(2) Subdivision

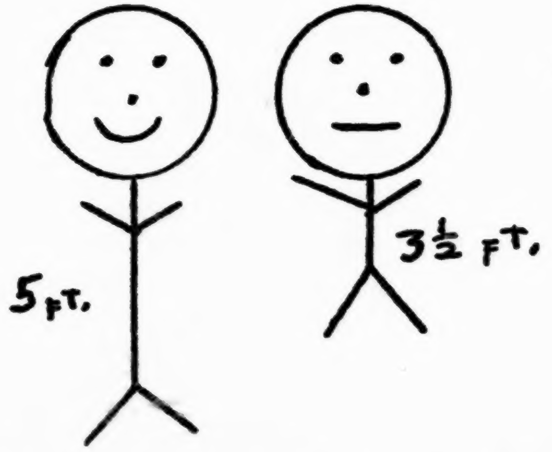


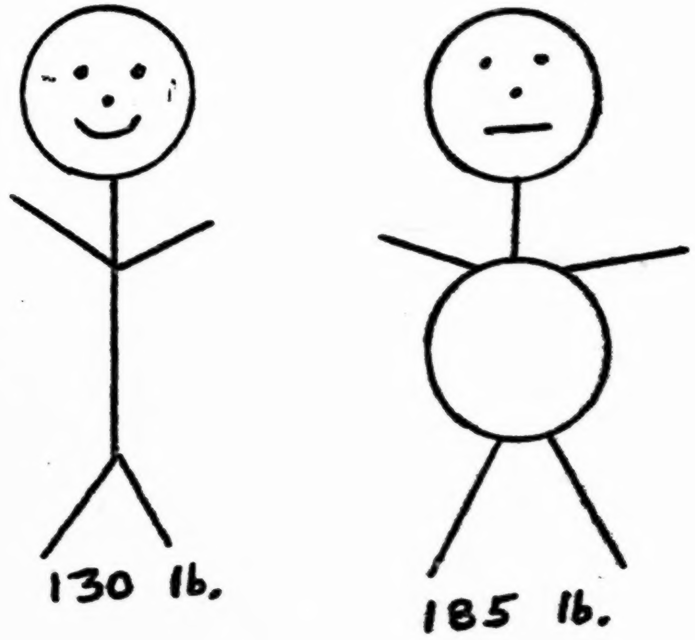
3. Comparison

(3) Two geometric figures or physical objects may be compared if they have a common characteristic.
Example: The length of two lines, the height or weight of two boys, the size of two angles.








130 lb.
185 lb.

The teacher may draw line segments on the board or give a dittoed set of line segments to each student and ask students to measure each segment to the (a) nearest 1/4 inch, (b) nearest 1/8 inch, (c) nearest 1/2 inch, and to write each measurement.

B. Approximation, Precision, Accuracy

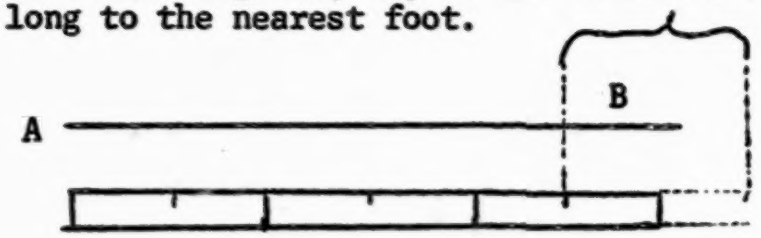
Students should be reminded of the concept learned in previous grades that measurement is never exact, only approximate. A ruler is a model used to compare lengths.

1. Precision

Because a smaller unit of measure is used (1/16"), $1\frac{14}{16}$" is more precise or has greater precision than 1 7/8" (unit = 1/8"). It is very important to future work in this unit that students understand the concept that the smaller the unit of measure, the greater is the precision.

2. Greatest possible error (GPE or gpe)

The teacher may draw a line segment, $\overline{AB}$, on the board as below. The line segment is said to be 3 feet long to the nearest foot.



But from work on measurements done in the 7th grade the students should know that the length of $\overline{AB}$ might be about 1/2' less than 3' or 1/2' more; that is, B might be any point on the line segment in the brace (above). In other words, the greatest possible error is 1/2 foot, which is equivalent to one-half the unit of measure used. The measurement of the line segment $\overline{AB}$ may be written as $(3 \pm \frac{1}{2})$ feet and read "3 plus or minus ½ foot."

From this demonstration the student may form the concept that the greatest possible error of a measurement is equivalent to ± one-half of the unit of measure being used.

The teacher should now give the class a series of measurements and ask them to give the unit of measure, the greatest possible error, and to write it using the ± symbol.

Measurement	Unit of Measure	G.P.E.	Written
$7\frac{2}{3}$ inches	$\frac{1}{3}$ inch	$\frac{1}{6}$ in.	$(7\frac{2}{3} \pm \frac{1}{6})$ in.
1.5 hours	.1 hour	.05 hr.	(1.5 ± .05) hr.
$5\frac{3}{4}$ miles	$\frac{1}{4}$ mile	$\frac{1}{8}$ mile	$(5\frac{3}{4} \pm \frac{1}{8})$ mi.

Other measurements should be given in order to give students sufficient practice.

3. Decimal notation and precision

Students will find that in selecting the unit of measure in the metric system, the problem is a little more difficult than in fractional forms. In order to clearly state the unit of measure in a measurement of 1800 feet, for example, a zero may be underlined to show the unit of measure, or precision. For example:

$180\underline{0}$' means that 1' is the unit of measure
$18\underline{0}0$' means that 10' is the unit of measure
1800' means that the unit of measure is 100 feet

On the board write: 1 mi. = 1.61 K. Then students will readily see that .01 is the unit of measure. Then G.P.E. is (.01 x .5) = .005 and (1.61 ± .005) kilometers means that the length of a mile is between 1.605 and 1.615 kilometers. If students learn, in working with decimal notation, to multiply the unit of measure by .5 to find the G.P.E. the problem is simple, and concepts used in fractional notation apply to decimal notation.

Precision of a number is specified by the place value of the right most significant digit.

4. Relative error and accuracy

Students should know that measurement is only an approximation, so it is important that they also know just how exact or how accurate the measurement really is. Explain to them that we obtain an idea of the importance of the greatest possible error by comparing it with the actual measurement. This is called the relative error, and relative error = $\frac{\text{Greatest Possible Error}}{\text{measurement}}$. For example, the measurements of $2\underline{0}$ feet and 4 feet are made with the same precision, G.P. E. is .5 ft.

(a) 20 ft. ± .5 ft (b) 4 ft. ± .5 ft.

Dividing by the measurement in each we get:

(a) $\frac{.5}{20} = .025 = 2.5\%$ (b) $\frac{.5}{4} = .25 = 12.5\%$

So in (a), relative error is 2.5% and in (b) it is 12.5%.

So from problems (a) and (b) and other similar examples, the concept is developed that the smaller the relative error, the greater the accuracy.

Students should have practice in finding G.P.E. and relative error, and may tabulate the results in a table similar to the one below, but using their own measurements in addition to those given. Relative error need not be in percent, but usually is.

5. Summary of concepts

The student should now be reminded of a few important concepts:

1. Every measure is approximate
2. A measurement is a number together with a unit (7 feet) and a measure is a number (7)
3. As soon as a standard unit of measure is chosen, we know how to find the G.P.E.
4. The notation (±) indicates greatest possible error but does not necessarily indicate operations to be performed.

C. Significant digits

The student should have it called to his attention the meaning of significant digits in measuring. Significant digits tell how many times the unit of measure is contained in a measurement. This chart may aid students in recognizing significant digits.

Measurement	Unit of Measure	Number of Units	Number of Signif. digits
283 lb.	1 lb.	283	3
6900 yd.	100 yd.	69	2
50.47 mi.	.01 mi.	5047	4
.0004 ft.	.0001 ft.	4	1

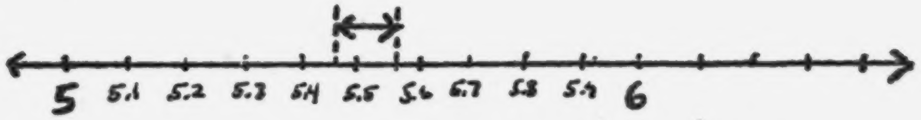
From observation, the student may understand the following concepts about significant digits:

(a) every non-zero digit is significant
(b) every zero between non-zero digits is significant
(c) the more significant digits in a numeral for a measurement, the more accurate is the measurement.

D. Operations on measures

1. Adding and subtracting measures

Students will now understand the concept that the sum or difference of measures cannot be more precise than the least precise measure involved.
For example: The measure named by the numeral 5.5 may be represented on a number line as follows:



The unit of measure is .1, so the segment shown ($\longleftrightarrow$) represents a set of numbers, 5.45 $\longleftrightarrow$ 5.55, the difference of which is the smallest unit that can be read (.1 = .05 + .05).

The following examples should be explained to the students before a final concept for addition or subtraction of measures is formed:

Addition

$$\begin{array}{r l}
26.2 & \text{(Precision is .1)} \\
9.57 & \text{(Precision is .01)} \\
+\ 13.093 & \text{(Precision is .001)} \\
\hline
48.863 & = 48.9 \quad \text{(nearest .1)}
\end{array}$$

Subtraction

$$\begin{array}{r l}
29.382 & \text{(Precision is .001)} \\
-15.87 & \text{(Precision is .01)} \\
\hline
13.512 & - 13.51 \text{ (nearest .01)}
\end{array}$$

From this, the student will formulate a plan for these operations: add or subtract measures as if they were named numbers, then round off the sum or difference to the precision of the least precise measures involved.

2. Multiplying measures
   a. Multiplication of a measure by an exact number

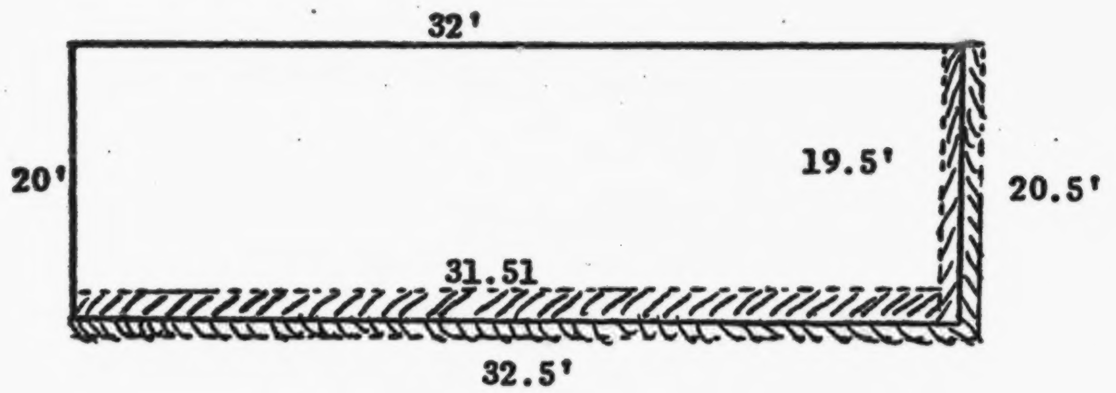
   The student may easily see that if one factor in a multiplication results from measurement and the others is an exact number, the multiplication is a repeated addition.
   Example: Find the weight of 3 chairs if each weighs 10 ½ lbs.

$$3 \times 10\tfrac{1}{2} = 10\tfrac{1}{2} + 10\tfrac{1}{2} + 10\tfrac{1}{2} = 3 \times 10\tfrac{1}{2} =$$

$$3\ \frac{21}{2} = \frac{63}{2} = 31\tfrac{1}{2} \text{ lbs.}$$

   b. Multiplication of a measure by another measure

   However, another type of problem in which both factors are measures, also is common.
   Example: Find the area of a rectangle having dimensions of 32 ft. by 20 ft.



Using g.p.e. we find that the rectangle could have maximum measurements of 32.5 feet by 20.5 feet and the minimum measurements of 31.5 feet by 19.5 feet. So the areas would be:

Smallest area	Given area	Largest area
31.5	32	32.5
x 19.5	x 20	x 20.5
614.25 sq. ft.	640 sq. ft.	666.25 sq. ft.

The given area (32 x 20 = 640 sq. ft.) is about 26 sq. ft. larger than the smallest area and about 26 sq. ft. smaller than the largest area. The student will see that 640 sq. ft. is not correct to the nearest square foot but only to approximately 26 sq. ft., so we may express the area as (160 ± 26) square feet.

3. Division of measures

Students have the concept that division is the inverse of multiplication so they may follow the procedure for multiplication when dividing measures.

4. Use of the concept of significant digits in multiplying and dividing measures

Students should be told that it is acceptable to round off the measures in the following way in order to save unnecessary writing and computation:

(1) In multiplication, if one of the two measures contains more significant digits than the other, we round off the measure that has more significant digits so that it contains only one more significant digit than the other.

Example: 5.63794062 (9 significant digits)
x 72.8 (3 significant digits)

may be rounded off thus:

$$\begin{array}{r} 5.64 \\ \times\ \underline{72.8} \end{array}$$

The same situation holds for division.

E. Sample Test

Measurement	(a) Unit of Measure	(b) Number of Unit of Measure	(c) Greatest Possible Error	(d) Written	(e) Number of Significant Digits	(f) Relative Error
1. 142.6 inches	.1 in.	1426	.05 in.	(142.6 ± .05) in	4	$\frac{.05}{142.6} \cong .035$ %
2. 30400 yards						
3. .0080 miles						
4. 6 3/8 feet						
5. .0015.0 gal.						
6. .50 oz.						
7. 160.006 rods						
8. 2 1/7 cm.						

9. Most precise of the above. ________________

10. Least precise of the above. ________________

11. Most accurate of the above. ________________

12. Least accurate of the above. ________________

13. Add the following measurements and give the best answer. ________________
    13.04' + .0064' + 2.3' + 146.8' ≅

14. Subtract the following measurements and give the best answer. ________________
    29.007' - 6.8184926' ≅

15. Multiply the following measurements and give the best answer. ________________
    (27.00814639)(.000023) ≅

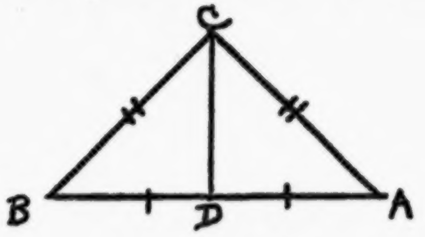
16. Divide the following measurements and give the best answer. ________________
    19.0517565 - .005 ≅

IX. Construction and Congruence

It may be explained to the student, as a point of interest, that the ancient Greek geometers usually restricted the instruments to be used in all geometric construction to the straightedge and compass. This part of Euclidian geometry is still given consideration today. The straightedge is not to be used as a measuring device.

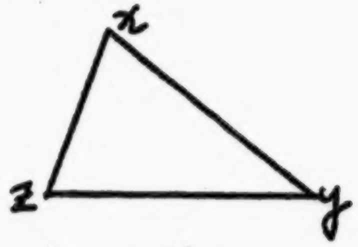
Congruence ($\cong$) generally refers to the relation of being the same size and shape. In geometry, congruent figures are figures which coincide in all corresponding parts. As an example--when we say two triangles are congruent, we mean that they have exactly the same size and shape--a tracing of one will fit exactly over the other. It cannot be said that two triangles or planes are equal because equal means "the same" and two planes or triangles occupy different sets of points in space.

Congruence is expressed in the following example of congruent triangles.



$\triangle BCD \cong \triangle ACD$

or



$\triangle xyz \cong \triangle xyz$

A. Congruent line segments

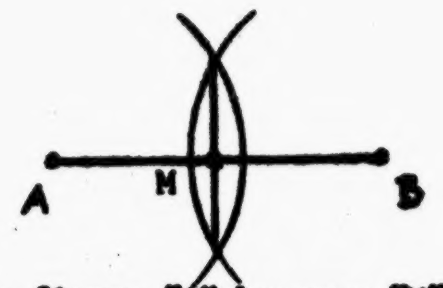
Line segments are congruent if a tracing of one fits exactly along the other. Page #185 - Mathematics 8 - (see bibliography) gives specific directions for constructing congruent line segments by using a straightedge and compass.

$\overline{AB} \cong \overline{CD}$ are symbols used to show congruence.

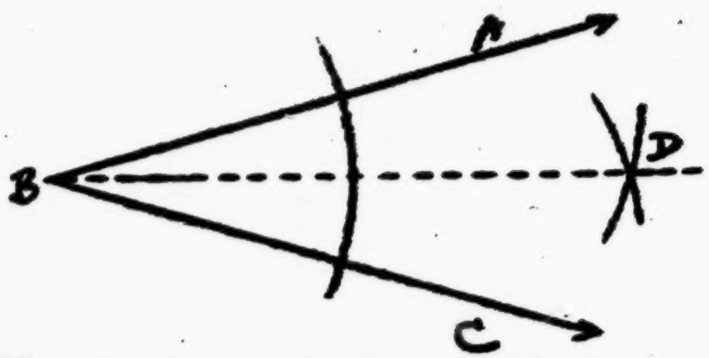
B. Bisecting line segments and angles

The student should be familiar with the term bisect which means to separate into two parts of equal size. Have them construct a number of segments and angles. Bisect each figure as illustrated on the next page.

(a) Given $\overline{A\ B}$



(b) Given $\angle A\ B\ C$



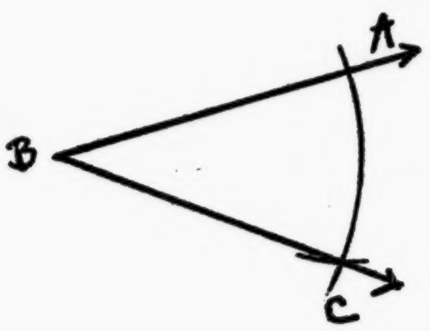
In figure "a" because "M" is the midpoint, $\overline{AM} \cong \overline{MB}$. Check this construction by using a ruler. In figure "b" the measure of the angles is symbolized as follows:

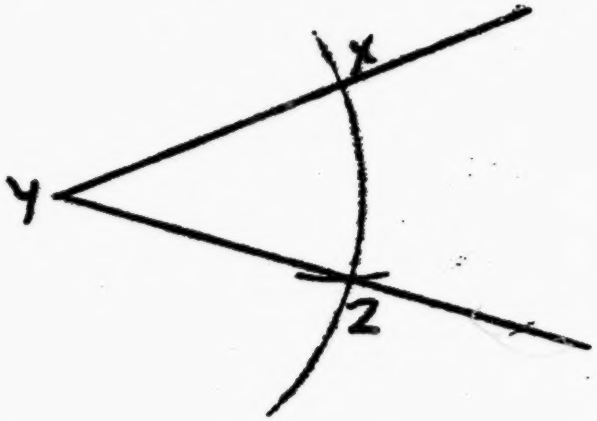
$m\angle ABD \neq m\angle CBD$; measures are not congruent. Sets of points are. $\angle ABD \cong \angle CBD$ and $\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. However, $m\angle ABD = m\angle CBD$.

C. Constructing congruent angles

Two angles are congruent if the vertex of one can be mapped exactly into the vertex of the other and the rays of the first lie exactly along the rays of the second. Congruent angles are expressed as $\angle XYZ \cong \angle ABC$.

Give the student the measure of $\angle ABC$ and require a construction for $\angle XYZ$ such that both angles are congruent.





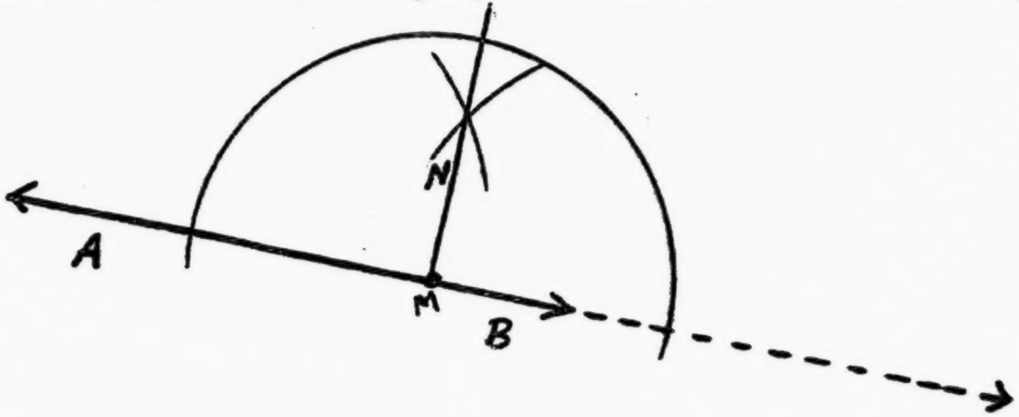
$m\angle ABC = m\angle XYZ$. Then we may say that the two angles are congruent, $\angle ABC \cong \angle XYZ$.

D. Constructing perpendicular lines

Two lines are perpendicular to each other if, and only if, they intersect at right angles. The common symbol is "$\perp$." The two expressions "perpendicular" and "at right angles" are equivalent.

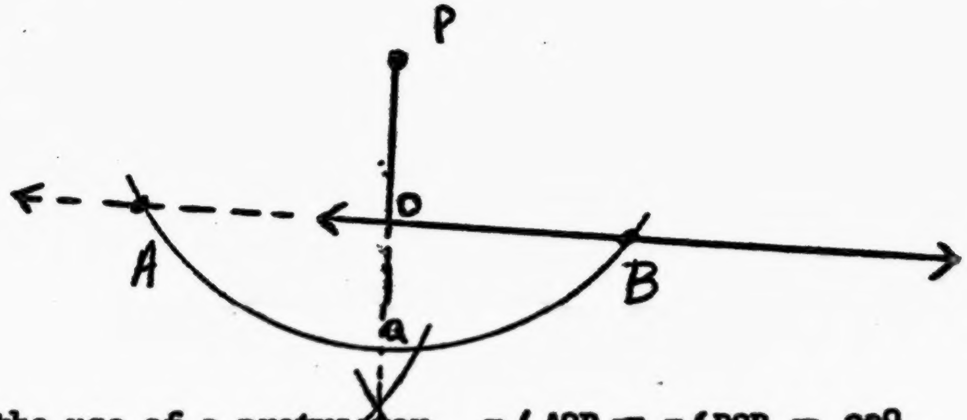
Motivate the student to perform more construction by giving some partial examples and ask that they complete them.

Give the student a line $\overleftrightarrow{AB}$ and ask that a perpendicular be constructed at point M on $\overleftrightarrow{AB}$.



Use a protractor to check that m$\angle$AMN = 90°

Give the student a point P not on $\overleftrightarrow{AB}$. Construct a $\perp$ to $\overleftrightarrow{AB}$ through point P.



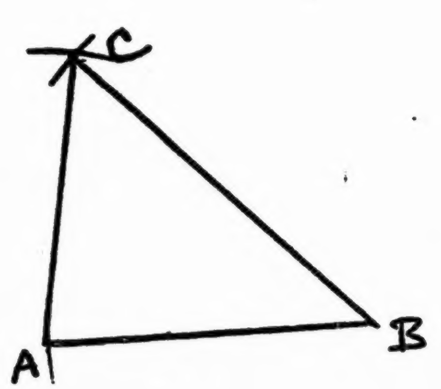
check by the use of a protractor. m$\angle$AOP = m$\angle$BOP = 90°.

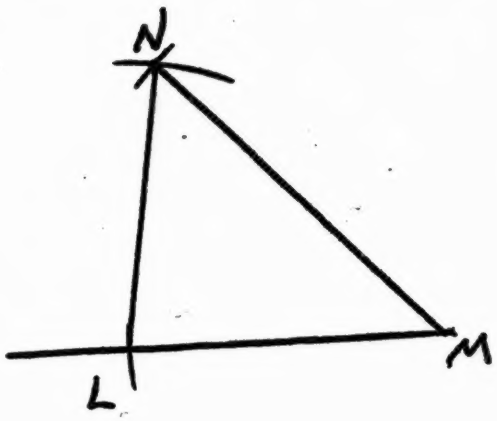
E. Congruent triangles

When two triangles are congruent, the corresponding angles and sides are congruent.

The student should now know that, in general, if one geometric figure coincides with another they are congruent. But, is it necessary to know that all corresponding parts are congruent before being able to state that the figures are congruent?

Construct the sides of $\triangle$ ABC congruent to the corresponding sides of $\triangle$ LMN.





Have the student check the measure of the angles formed and also cut out the figure LMN so that it may be superimposed on $\triangle$ ABC. The student may now assume that two triangles are congruent if three sides of one triangle are congruent respectively to three sides of another triangle.

By having the class do additional drawings and experiments the student may discover further assumptions about congruent triangles.

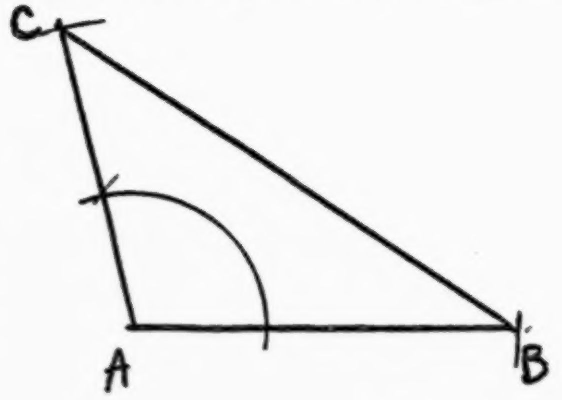
Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other triangle. (S.A.S. $\cong$ S.A.S.)

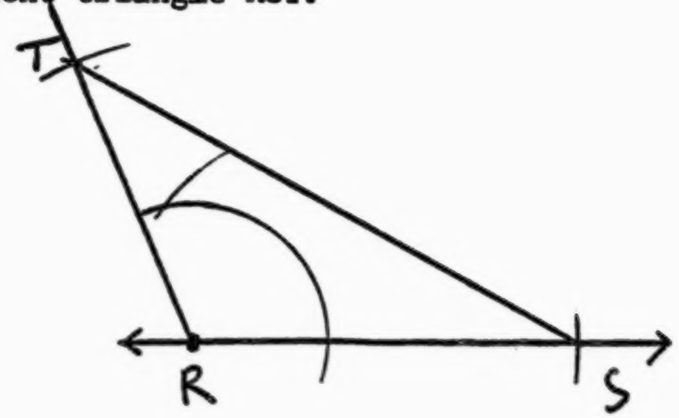
That two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other triangle. (A.S.A. $\cong$ A.S.A.)

F. Constructing congruent triangles

The student may now construct simple congruent polygons and name the corresponding parts in the congruent figures. He should be able to tell why the figures are congruent.

Given a triangle ABC, construct a congruent triangle RST.



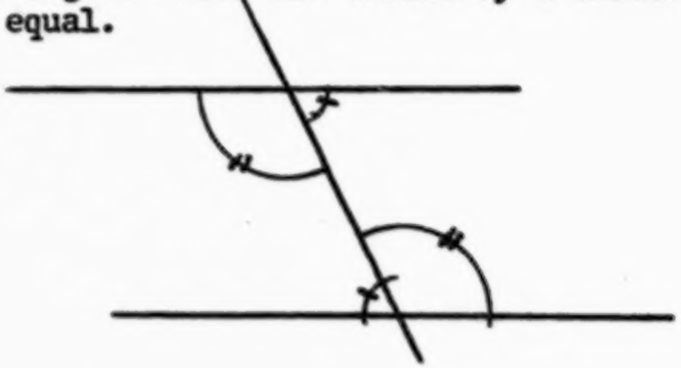


Construct $\overline{RS}$ congruent to $\overline{AB}$ so that $\overline{AB} = \overline{RS}$. At end point R on $\overline{RS}$ construct an angle congruent to $\angle A$. Construct $\overline{RT}$ so that $\overline{RT}$ is congruent to $\overline{AC}$. Connect points S and T with $\overline{ST}$.

G. Parallel lines

In a plane, two lines are parallel if, and only if, their intersection is the empty set. In space two planes are said to be parallel if, and only if, their intersection is the empty set. The student should know the meaning of the following terms: alternate interior angles, alternate exterior angles, transversal.

The student may construct and discover a pattern of angles which are formed by a transversal and parallel lines. The alternate interior angles are equal.



This concept may be developed further by the teacher and text.

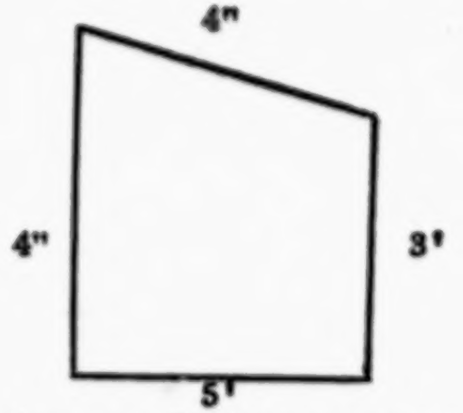
X. Perimeter, Area, and Volume

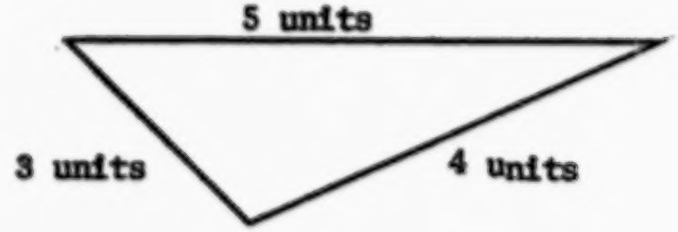
The students may reinforce their understanding of terms and formulas.

A. Perimeter

The perimeter is the measure of the entire polygonal path (The measure of the union of its segments). The perimeter will always be a linear measure.

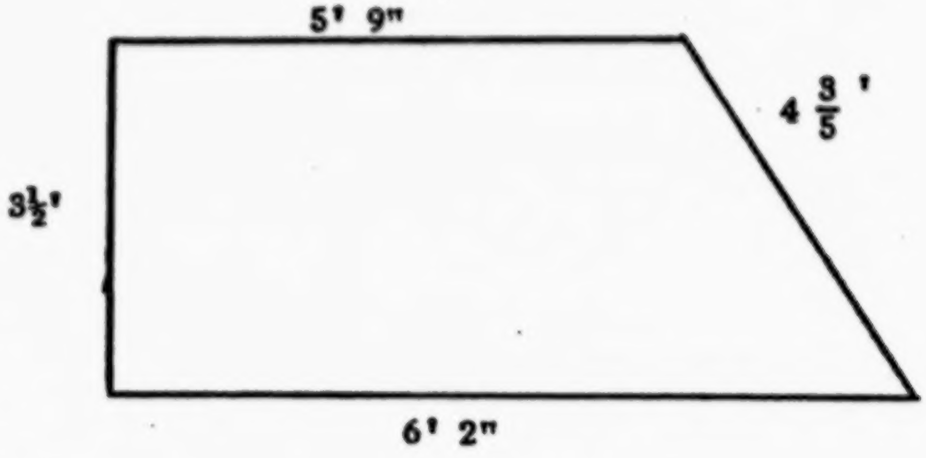
Finding the perimeter of any polygon is based on the addition of measures.





The teacher may find it necessary to mix units of measure to check mathematical computation.

Find the perimeter:



The student may use the rule for adding numbers which represent measurements of different precisions (see section 9B) when determining a perimeter.

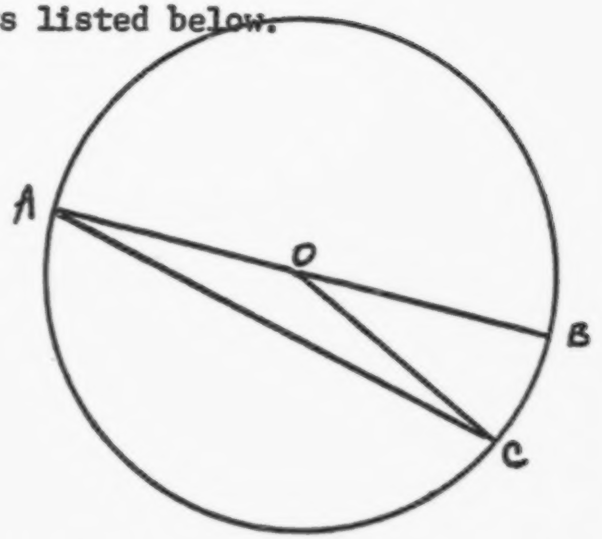
Definition of a circle: A circle is the locus of points in a plane each of which is equidistant from a given point.

B. Circumference (perimeter of a circle)

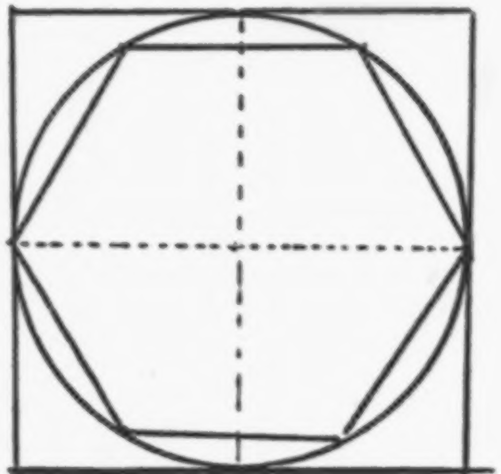
The student may construct a circle and define the parts listed below.

(1) Circle
(2) Center
(3) Radius
(4) Chord
(5) Diameter
(6) Arc

All of the above describe a set of points.



The student may develop an understanding of circumference through a practical development of the formula $C = \pi d$. Emphasize that π represents a constant ratio of the circumference to the measure of the diameter of the circle. $\pi = \frac{c}{d}$



Perimeter of square = 4D
Perimeter of hexagon = 3D

$C < 4D \cdot C > 3D$
$\therefore C = 3.14D$

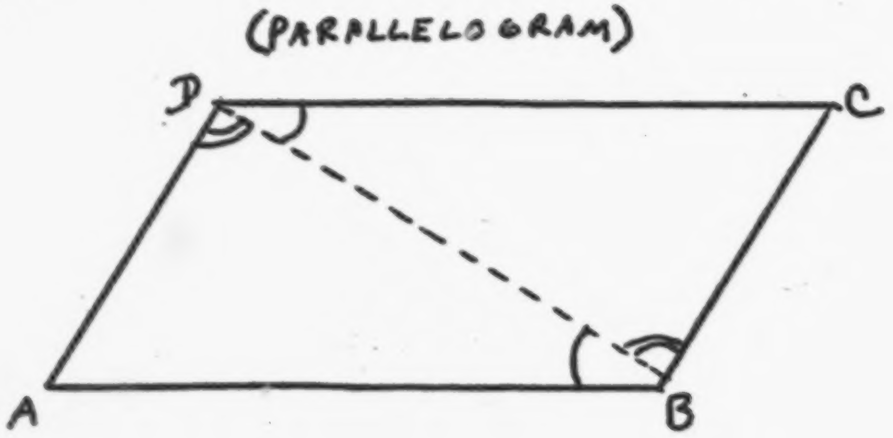
1. Formula $C = \pi d$
2. Limit concept

For the more advanced student the circumference of a circle may be defined as the limit of the perimeter of an inscribed regular polygon.

This method is explained in depth in most eighth grade texts.

C. Quadrilaterals

The student may define the subsets of quadrilaterals by what has been learned about congruence.



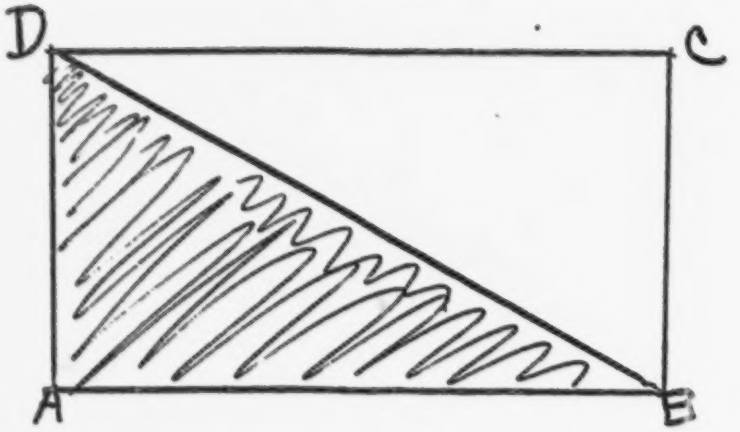
The more advanced student may construct a diagonal and prove triangles congruent; then use the alternate interior angles congruent to prove opposite sides parallel.

D. Area of a rectangle

The student is familiar with the method of finding the area of a rectangle; however, the concept of area being the area of a figure consisting of a rectangle and its interior may be discussed.

E. Area of a right triangle

The area of a right triangle may be derived from the area of a rectangle and then the formula of any triangle may be derived from the right triangle.



Given a rectangle
ABCD whose area
formula is $A = \ell w$ or $A = bh$.

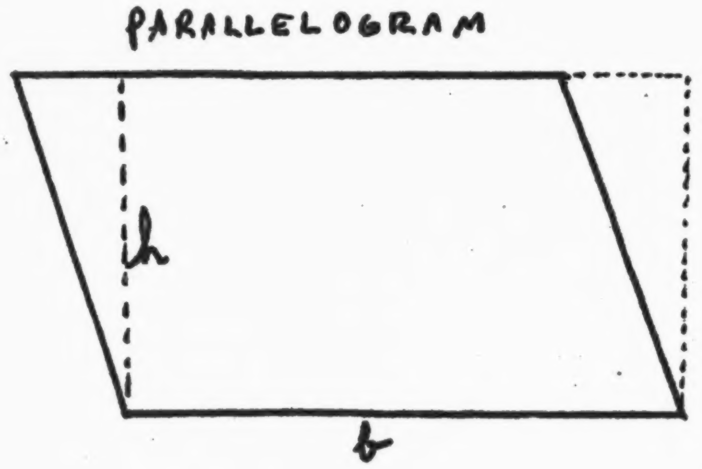
Construct a diagonal in the rectangle. One of the resultant triangles formed is one half the rectangle. Then $\frac{1}{2}\ell w$ or $\frac{1}{2}bh$ is the area of the triangle which is written as $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$.

The student may prove these two triangles congruent

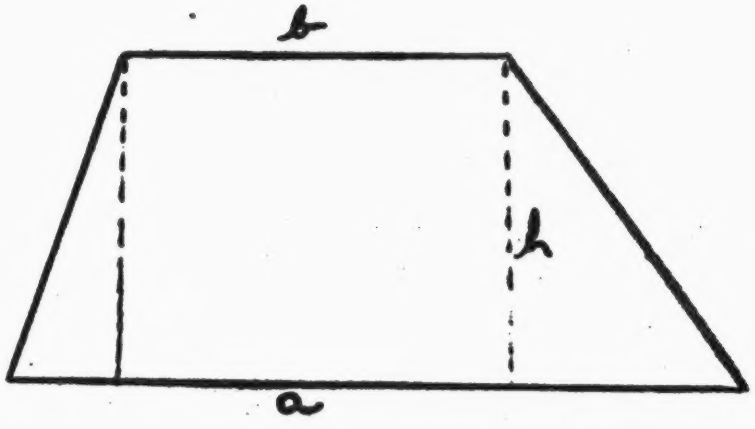
The area of any triangle may be found by extending lines and using the right triangle formula.

F. Area of a Parallelogram and Trapezoid

The student may discover that any side of the parallelogram or the parallel sides of the trapezoid may be defined as the base. With the construction of the right triangle the formula and area are derived.



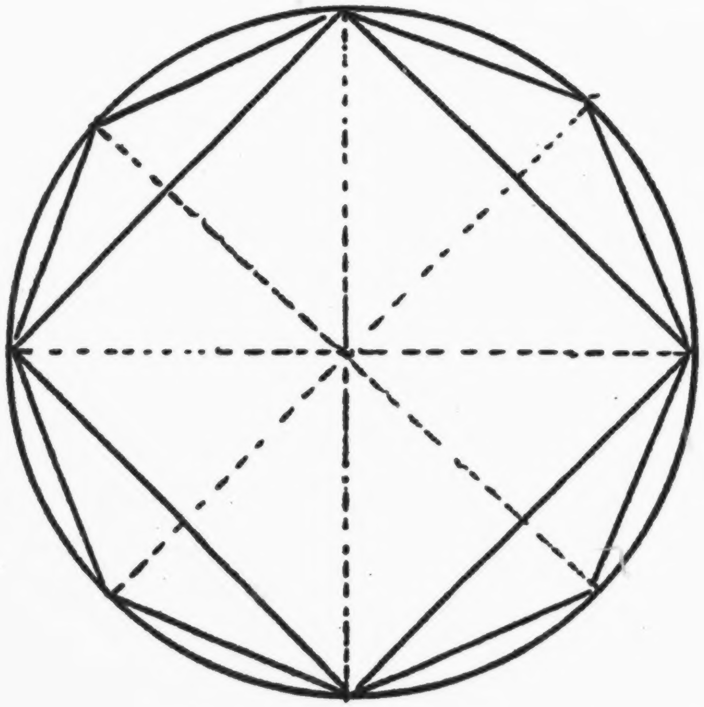
$$A = bh$$



$$A = \left(\frac{a+b}{2}\right)h$$

G. The area of a circle

The student may inscribe a regular polygon in a circle. By doubling the number of sides of the inscribed polygon observe what is happening.

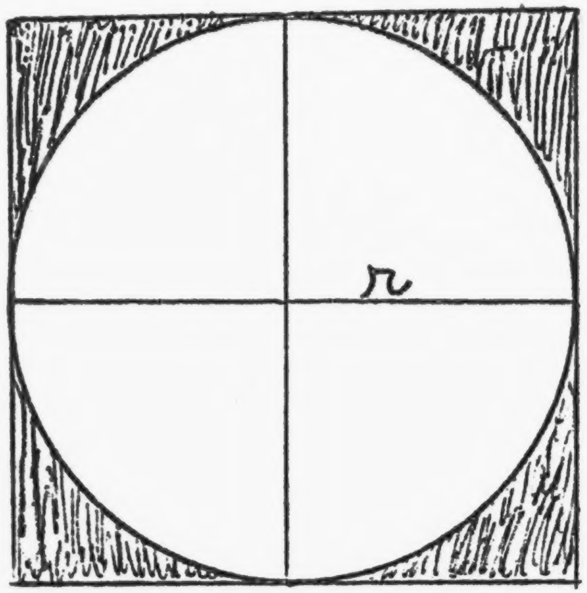


Use the radius to step
off the first polygon
on the circumference of
the circle.

The eighth grade student may write out and understand each step of the inscribed polygon method of developing the formula of a circle.

$$\begin{array}{llll} \text{(approx.)} & A = n(\frac{1}{2}bh) & \text{Since} & C = 2\pi r \\ & A = \frac{1}{2}(nb)h & & A = \frac{1}{2}Cr \\ & A = \frac{1}{2}Cr & & A = \frac{1}{2}(2\pi r)r \\ & & & A = \pi r^2 \end{array}$$

Another way of intuitively arriving at the idea of the area of a circle:



$A = r^2$
Total Area = $4r^2$

then the area, A of the circle $< 4r^2$.
But, also, $A > 3r^2$.
$3r^2 < A < 4r^2$ and
$3 < \pi < 4$
then A can be seen to be πr^2 more easily.

H. Prisms

The student is introduced to the concept of the geometric solid by a discussion of prisms. In this lesson we consider only prisms whose lateral edges are perpendicular to the planes of the bases. This kind of prism is called a right prism. All the lateral faces of a right prism are rectangular. Two of the faces are congruent polygonal cells in a parallel plane, and the remaining edges are parallel to one another. The two parallel faces of the prism are called bases.

The teacher may use a box, right rectangular prism, to aid in the development of concepts and terms pertaining to prisms.

The student should be given examples of right triangular and right rectangular prisms, which require solutions to problems involving area and volume.

I. Pyramids, Cylinders, and Cones.

The student is introduced to the terms, definitions and formulas of these geometric figures in most texts.

Pyramid	$V = \frac{1}{3} Bh$	$S = \frac{1}{2} nal$
Cylinder	$V = \pi r^2 h$	$S = 2\pi r^2 + 2\pi rh$
Cone	$V = \frac{1}{3}\pi r^2 h$	$S = \pi r \sqrt{r^2 + h^2}$

J. Sphere

The sphere is considered a topic for the eighth grade student. Definitions should be emphasized as well as the proper use of the formulas $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

This topic is covered very well in most eighth grade texts.

XI. Ratio, proportion, and percent

A. Meaning of terms

Students have been introduced to these terms before eighth grade, but to reinforce their concepts the teacher should discuss the following terms with the class:

(1) Emphasize that "ratio" and "fraction" are two names for the same thing.

(2) Ratio is a relation between numbers in a definite order.
(3) The equality of two ratios states a proportion.
(4) Percent is a special type of ratio which may be expressed as a common fraction or a decimal fraction.

B. Ratio

1. Expressing a ratio

Ratio expresses relationship between the sets but does not state how many objects are in each set. So in the ratio $\frac{2}{5}$, the first set contains 2k objects and the second set contains 5k objects. If k = 1, the set contains 2 and 5 objects; if k = 2, the set contains 4 and 10 objects, etc.

Any fraction is a ratio. Ratios usually are related to something. For example, students may see that the ratio of the total number of states in the U. S. to both hands is 50:10, or $\frac{50}{10}$.

2. Ratios in simplest form

Reducing this fraction to lowest terms is the simplest form of a ratio; so 50:10 = $\frac{50}{10} = \frac{10}{2} = \frac{5}{1}$.

Students should have a little practice in writing ratios in their simplest forms.

3. Ratio and measurements.

When measurements are expressed as a ratio they should be in the same unit of measure, otherwise the students continually make mistakes.

Example: Express the ratio of 6 ft. to 12 in. Both terms must be expressed as feet (6' to 1') or as inches (72" to 12").

4. Ratio must be expressed in a common unit of measure.

So the student may see that if measurements can be expressed in the same units of measure a ratio may be expressed.

Example: 6 oz.:2 lbs.

Changing pounds to ounces we have 6 oz.:32oz. = $\frac{6}{32} = \frac{3}{16}$, which is the ratio in simplest form.

b. If no common unit of measure exists, the relationship is called a rate.

The student is aware, however, that measurements often cannot be expressed in the same units of measure. For example, we cannot have a ratio of 18 feet to 1 second. Such a number pair, $\frac{18 \text{ ft.}}{1 \text{ sec.}}$ is a rate, not a ratio, because distance and time have no common unit of measure.

Example: $60 for 4T. is the same rate as $15 for 1T.
Also, 500' in 5 sec. is the same rate as 100' per second.

C. Proportion

1. Means and extremes

Students may easily understand that the statement of equality of two ratios is called a proportion. Also they should begin to use the terms "means" and "extremes." The teacher should write a proportion on the board in two forms:

(a) 8:20 = 12:30 and
(b) $\frac{8}{20} = \frac{12}{30}$

Tell the class that in (a) 8 and 30 are called the "extremes" and 20 and 12 are the "means." The class should then notice the position of the means and extremes in (b).

$$\underbrace{8:\overbrace{20 \;=\; 12}^{\text{means}}:30}_{\text{extremes}} \qquad\qquad \frac{(8)}{20} = \frac{12}{(30)}$$

8 and 30 are extremes, 20 and 12 are means.

2. True or false proportion?

The more aware students have possibly developed a concept for testing for a false or true proportion. If not, suggest that the students rename the ratios in the proportion so that they have a common denominator; then if the numerators are equal the proportion is true, but if not it is false.

Example: $\frac{12}{3} = \frac{48}{12}$ and $\frac{12}{3} \times 1 = \frac{48}{12} \times 1$

$\frac{12}{3} \times \frac{12}{12} = \frac{48}{12} \times \frac{3}{3}$

$\frac{144}{36} = \frac{144}{36}$ The proportion is true.

Example: $\frac{6}{7} = \frac{28}{30}$ and $\frac{6}{7} \times 1 = \frac{28}{30} \times 1$

$\frac{6}{7} \times \frac{30}{30} = \frac{28}{30} \times \frac{7}{7}$

$\frac{180}{210} \neq \frac{196}{210}$ The proportion is false.

Students should test for truth or falsity of various proportions.

3. Another test for true proportion: Product of extremes in a true proportion.

Explain to students that the product of the means equals the product of the extremes, and that the explanation for this statement is:

Example: $\frac{a}{b} = \frac{c}{d}$

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$$

$$\frac{a(bd)}{b} = \frac{c(bd)}{d}$$

$$\frac{a(db)}{b} = \frac{c(bd)}{d}$$

$$\frac{(ad)b}{b} = \frac{(cb)d}{d}$$

$$ad \cdot 1 = cb \cdot 1$$

$$ad = cb$$

This is a good opportunity to justify the operations at the left.

So now, as students know that the product of the means equals the product of the extremes, they may apply this concept in checking the truth or falsity of proportions.

4. Solving a proportion

Students should easily solve problems to find the unknown part of a proportion by using the concept that the product of the means equals the product of the extremes.

Example: $\frac{a}{7} = \frac{15}{35}$

$$a \cdot 35 = 15 \cdot 7$$

$$\frac{a \cdot 35}{35} = \frac{15 \cdot 7}{35}$$

$$a \cdot 1 = \frac{105}{35}$$

$$a = 3$$

Check: $\frac{3}{7} = \frac{15}{35}$; $105 = 105$

Have students find the unknown part of proportions in equations until they acquire speed and accuracy.

5. Problems about rates

Students discover that it is more difficult to change the units of a rate from one combination of units to another than to change to the common unit of measure as used in ratios.

Explain to the class that to change the name of a rate from one combination of units to another we must multiply each part of the fraction by the proper conversion factor and express the result as a fraction with denominator of 1.

Example: When a car is traveling at a speed of 60 m.p.h., how can we express its speed in feet per second?

(ft. per mi.)

$$\frac{60 \times 5280}{60 \times 60} = 1 \cdot \frac{5280}{60} = 88 \text{ ft. per sec.}$$

(sec. per. hr.)

D. Percents

Students have studied percent in previous years. They should recall that percent means per hundred. When students are making computations with percent it is often easier to change the percent to a fractional or decimal equivalent first.

1. Percent formula

A percent is a ratio in which the second number of the ordered pair is 100. This ratio, which may be written as percent (example: $\frac{40}{100} = 40\%$) is often called the rate, and represents the ratio of percentage to base. So we use "p" for the percentage, "b" for base and "r" for rate in the percentage formula: $\frac{p}{b} = r$ or $p = rb$.

2. Percent and proportion

Most percent problems may be solved by the percentage formula. The three usual types of precent problems are extensively discussed in most 7th and 8th-grade texts.

3. Equivalent fractions and per cents.

It will be very convenient for students to know the fractional equivalents of percents before working percent problems involving $33\frac{1}{3}$ %, $66\frac{2}{3}$ %, $16\frac{2}{3}$ %, etc.

However, students may learn to find the fractional equivalents by rewriting the per cent as hundredths and treating it as fractional divisions.

Example: $87\frac{1}{2}\% = \frac{87\frac{1}{2}}{100} = \frac{175}{2} \times \frac{1}{100} = \frac{175}{200} = \frac{7}{8}$

If the percent may be written as a terminating decimal the students will find it accurate to compute using the decimal form.

Example: Find $62\frac{1}{2}\%$ of 23 $p = rb$

$62\frac{1}{2}\% = .625$

$p = .625 \times 23$

$p = 13.375$

But if the percent, when written as a decimal, becomes a repeating decimal, it may be easier and more accurate to solve the problem using the fractional equivalent.

Example: Find $33\frac{1}{3}$ % of 90 $p = rb$

$33\frac{1}{3}\% = \frac{1}{3}$

$p = \frac{1}{3} \cdot 90$

$p = 30$

Suggestion: Practice of problems of discount, taxes, commission, etc. should be carried out to reinforce the concepts of ratio, proportion, and percent.

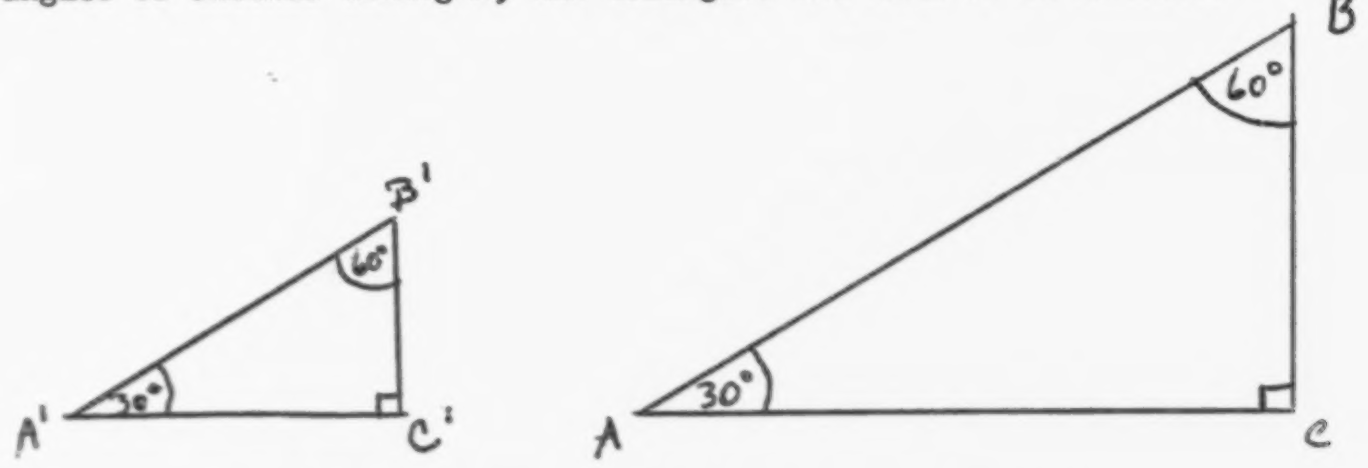
XII. Similarity of Geometric Figures

A. Similar figures

1. Triangles

The students have studied figures which are congruent and have seen that these figures are alike in size and shape and can be made to coincide.

Similar figures are alike in shape, but no necessarily in size. If the measure of the angles of one triangle, for instance, is equal, respectively, to the measure of the angles of another triangle, the triangles are said to be similar.

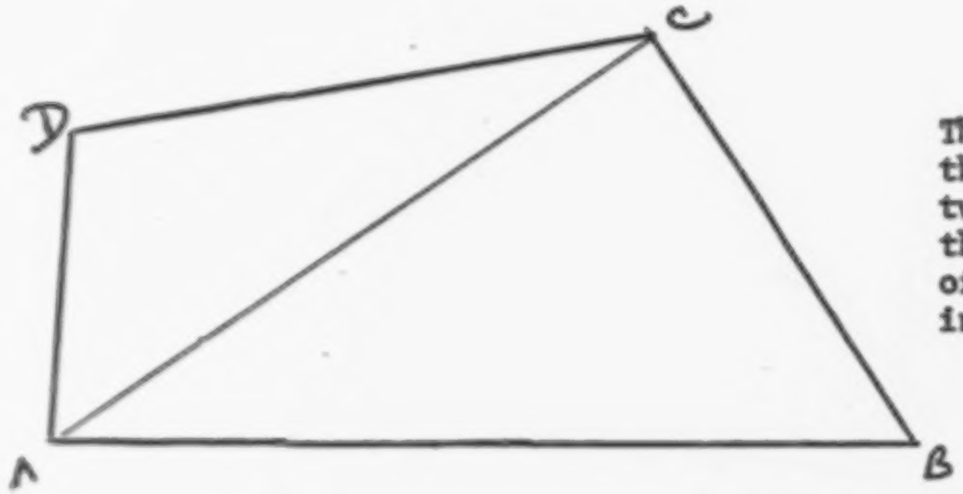


$\triangle A'B'C' \sim \triangle ABC$ (The sign for similar is $\sim$)

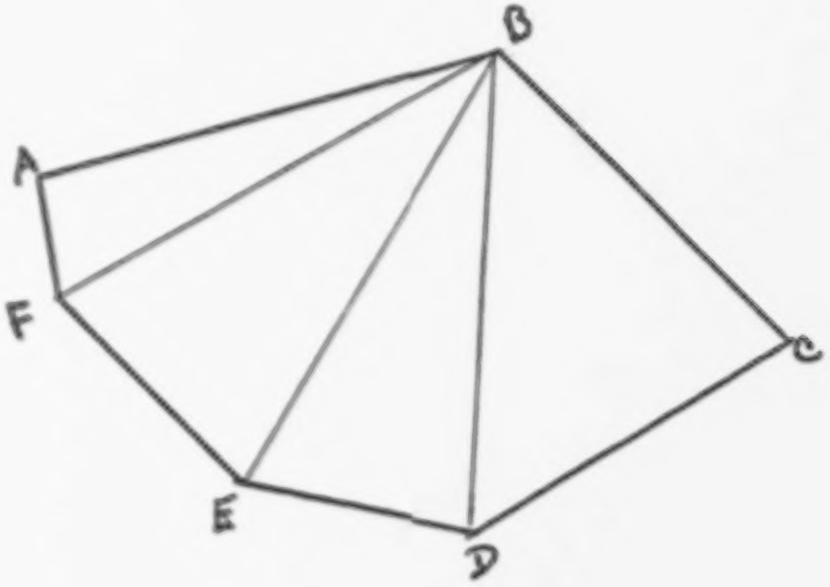
The sum of the measures of the angles of a triangle is 180°.

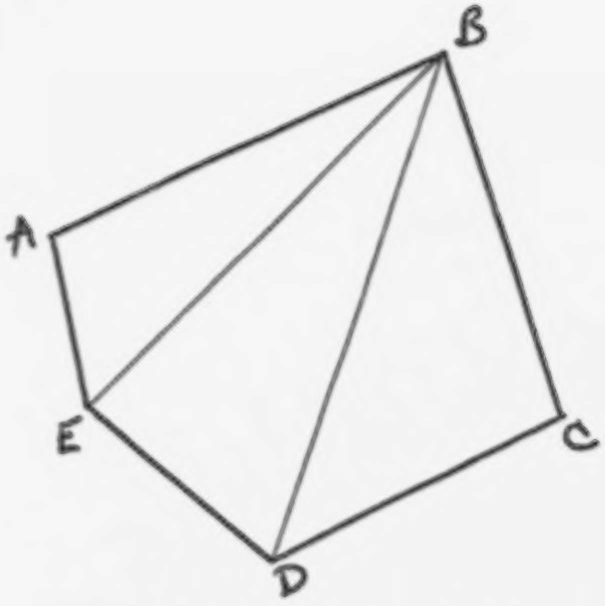
2. Polygons

When the students know that there are 180° in the sum of the measures of the angles of a triangle, they can find the sum of the measures of the angles of any polygon.



The diagonal $\overline{AC}$ separates the quadrilateral into two triangles. Therefore, there are 360° in the sum of the measures of the interior angles of ABCD.





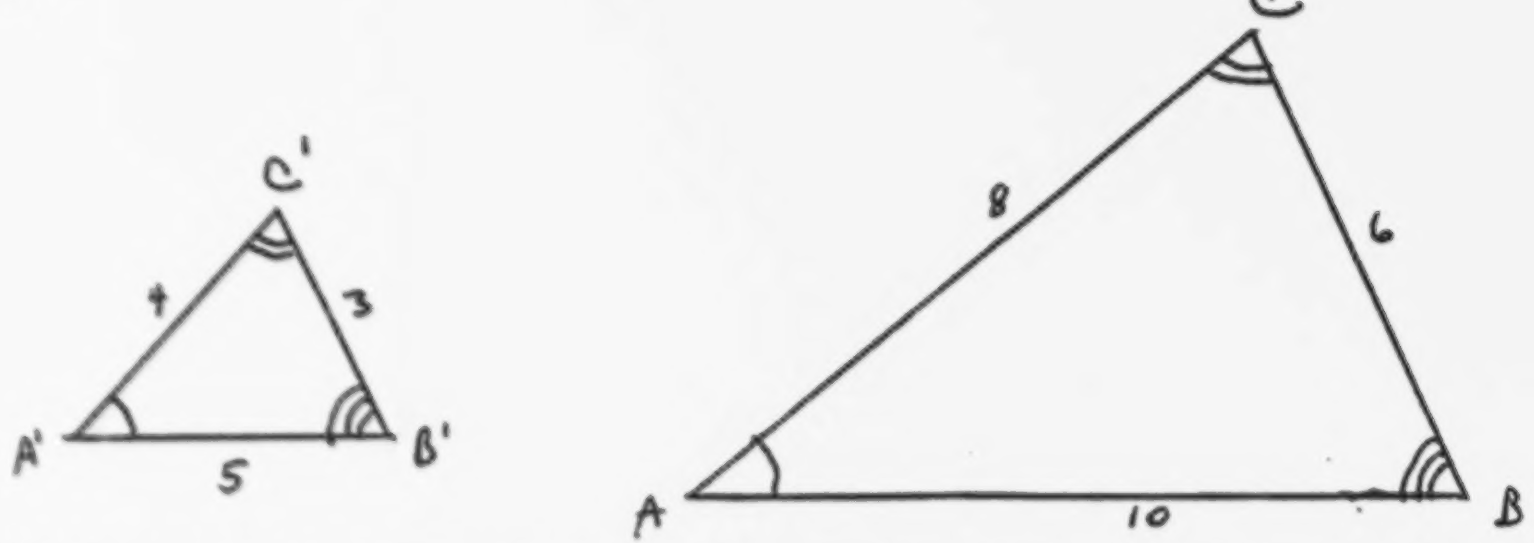
The six sided figure ABCDEF (Figure A - a hexagon) can be separated into four triangles, and so there are $4\ (180^{\circ}) = 720^{\circ}$ in the sum of the measures of the angles of a hexagon. In the same manner the student can see that the five sided figure ABCDE (Figure B - pentagon) has $3\ (180^{\circ}) = 540^{\circ}$ in the sum of the measures of its angles.

By using several of these examples, the students should be able to discover that if a polygon has n sides, the sum of the measures of its angles is $(n - 2)\ 180^{\circ}$.

3. Proportional sides of similar figures

Have the students draw examples of similar polygons, using a protractor to measure the angles and a ruler to measure the length of the sides. (The corresponding angles must be = .)

Example:



Have the students write the ratios of the corresponding sides. The student should discover that if two geometric figures are similar, their corresponding sides are proportional.

The students should have extensive practice in drawing similar triangles, rectangles, squares, trapezoids, etc., in measuring the degrees in the angles of similar triangles, and in measuring the lengths of the corresponding sides.

B. Practical problems

After the student has become adept at drawing and measuring similar figures, problems of the following types should be worked in sufficient number so that the student will retain the knowledge he has gained.

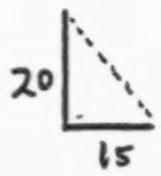
Examples:

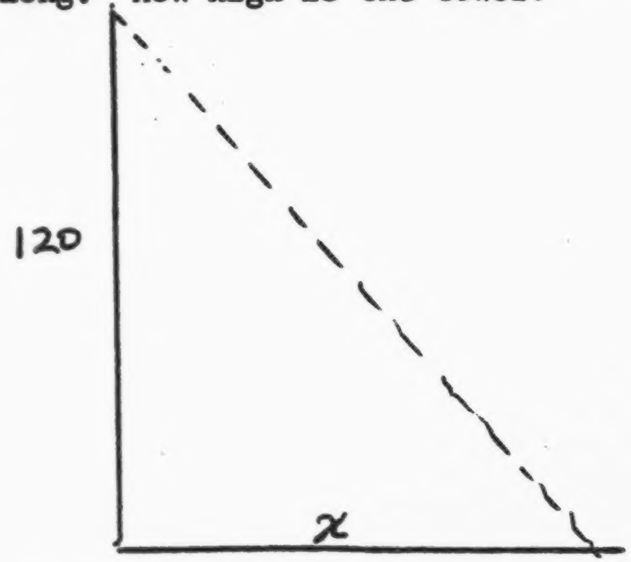
1. A tower casts a shadow 120 feet long when a 20 foot telephone pole casts a shadow 15 feet long. How high is the tower?

$$\frac{20}{15} = \frac{x}{120}$$

$$15x = 2400$$

$$x = 160$$





2. The sides of one polygon are 3, 7, 6, 8, and 15. The longest side of a similar polygon is 20. Find the lengths of the other sides of the second polygon.

$$\boxed{\begin{aligned} \frac{3}{x} &= \frac{15}{20} \\ 15x &= 60 \\ x &= 4 \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{6}{x} &= \frac{15}{20} \\ 15x &= 120 \\ x &= 8 \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{7}{x} &= \frac{15}{20} \\ 15x &= 140 \\ x &= 9\frac{1}{3} \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{8}{x} &= \frac{15}{20} \\ 15x &= 160 \\ x &= 10\frac{2}{3} \end{aligned}}$$

Students should make sketches of all geometric problems where possible.

XIII. Right Triangles

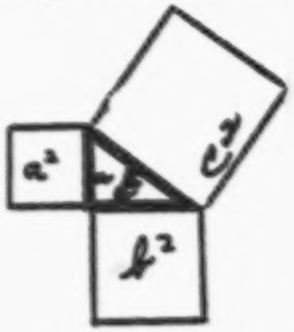
A. The Pythagorean Property

A class discussion of the use of right angles could introduce this unit. The students will point out that the walls of the room meet at right angles and that the shelves in the room, the windows, etc. meet at right angles.

The "Rule of Pythagoras" is probably the most well remembered property of the right triangle. (Even the parents will remember the name of the property)

This rule states that the square upon the hypotenuse of a right triangle is equal to the sum of the squares on the two other sides.

When the squares are constructed on the sides of the triangle the area of square a is a^2, the area of square b is b^2 and the area of square c is c^2.



$$a^2 + b^2 = c^2$$

The students should construct several right triangles, and then construct the squares on the sides of the right triangle. This should be sufficient proof of the Pythagorean property for students this age.

B. Square Root, a short method

It is more important to be able to find the measure of the length of one of the sides of a right triangle given the measure of the two other sides than it is to know the areas of the squares on the sides.

When the "Rule of Pythagoras" is used, the algebraic equation $a^2 + b^2 = c^2$ can be solved three ways.

$$\begin{array}{lll} a^2 = c^2 - b^2 & c^2 = a^2 + b^2 & b^2 = c^2 - a^2 \\ \sqrt{a^2} = \sqrt{c^2 - b^2} & \sqrt{c^2} = \sqrt{a^2 + b^2} & \sqrt{b^2} = \sqrt{c^2 - a^2} \\ \therefore\ a = \sqrt{c^2 - b^2} & \therefore\ c = \sqrt{a^2 + b^2} & \therefore\ b = \sqrt{c^2 - a^2} \end{array}$$

Most textbooks have tables of squares and square roots, and the students should understand how to read them, but it is also important to be able to "find" the square root if a table is not available.

A short method of finding the square root of numbers which contain a square as one of two factors should be explained to the students.

Examples: 20, 18, 12, 27, 8, 32, etc.

$$\sqrt{20} \rightarrow \sqrt{4 \cdot 5} \rightarrow 2\sqrt{5}$$

$$\sqrt{18} \rightarrow \sqrt{9 \cdot 2} \rightarrow 3\sqrt{2}$$

$$\sqrt{12} \rightarrow \sqrt{4 \cdot 3} \rightarrow 2\sqrt{3}$$

The answer to most problems can be left in this reduced form, but sometimes it will be necessary to use this answer to continue work in a problem.

$$2\sqrt{5} \doteq 2 \times 2.236 \qquad 3\sqrt{2} \doteq 3 \times 1.414$$

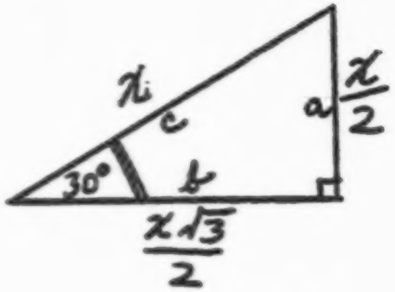
$$2\sqrt{5} \doteq 4.472 \qquad 3\sqrt{2} \doteq 4.242$$

Have the students work a number of these problems until they become efficient in this method of "finding" the square root of a number.

Most textbooks give other methods of extracting square roots.

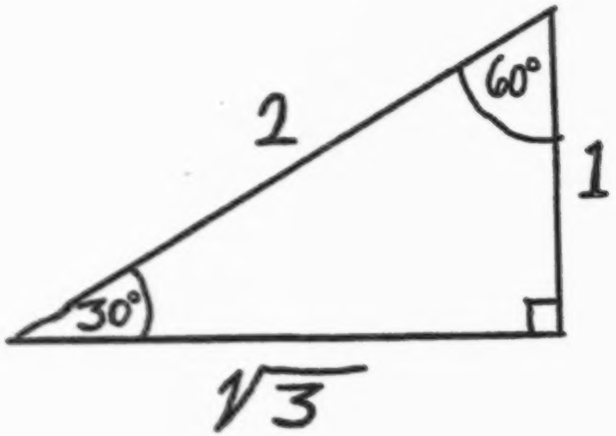
C. The 30^o - 60^o Right Triangle

The 30^o - 60^o right triangle is a special and very useful right triangle, and it has a special property. The side opposite the 30^o angle is equal in length to one-half the length of the hypotenuse.



If $a = \frac{c}{2}$, then $c = 2a$.

After students have worked several problems of this type, it is hoped that they will discover that the ratio of the sides of this special triangle are:



Example: If $a = 5$, find c and b.

If $a = 5$, $c = 10$
$a^2 + b^2 = c^2$
$b^2 = c^2 - a^2$
$b = \sqrt{c^2 - a^2}$

$5^2 + b^2 = 10^2$
$b = \sqrt{100 - 25}$
$b = \sqrt{75}$
$b = \sqrt{25 \cdot 3}$
$b = 5\sqrt{3}$

Example: If $c = 12$, find a, b.

If $c = 12$, $a = 6$
$b = \sqrt{c^2 - a^2}$
$b = \sqrt{144 - 36}$
$b = \sqrt{108}$
$b = \sqrt{36 \cdot 3}$
$b = 6\sqrt{3}$

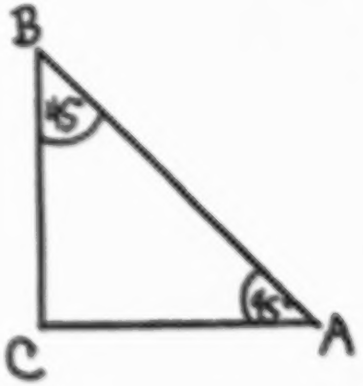
Have the students work a good number of these problems.

D. The 45° Right Triangle

The 45° right triangle is called an isosceles right triangle, and it has special properties.

"The side opposite the 45° angle of a right triangle is equal to one-half the hypotenuse times the square root of 2."

Example:



Let $\overline{AB} = 12"$

(The students should recall, or have it pointed out that if two angles of a triangle are equal, the sides opposite these angles are equal.)

$$x^2 + x^2 = 12^2$$
$$2x^2 = 144$$
$$x^2 = 72$$
$$x = \sqrt{72}$$
$$x = \sqrt{36 \cdot 2}$$
$$x = 6\sqrt{2}$$

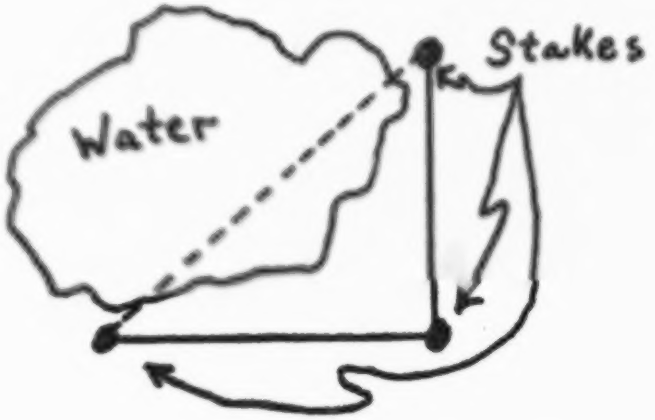
Have the students work several examples.

E. Practical Problems.

Divide the students up into sets of three and have them measure the basketball courts.

Set up special problems on the ball field and have the students set problems up for each other.

Example:



XIV. Scientific Notation and the Metric System

A. Naming a large number

The teacher may motivate the students with a discussion on the need of a larger and a smaller numeration unit.

In expressing very large numbers, use the product of a number between one and ten and proper power of ten.

$$2000 = 2 \cdot 10^3$$

$$368{,}000 = 3.68 \cdot 10^5$$

(A number expressed this way is said to be written in scientific notation.)

A number is expressed in scientific notation if it is written as a product of a number between one and ten and the proper power of ten.

If the first number is a power of ten, the first factor is one and is usually not indicated.

$$1000 = 10^3$$

$$1{,}000{,}000 = 10^6$$

B. Computing with large numbers

The student may practice computing some very large numbers.

$$1000 \times 100{,}000 = 10^3 \cdot 10^5 = 10^8$$

This computation in a formula may be written as follows:

For all positive integers a and b, $10^a \cdot 10^b = 10^{a+b}$.

Some examples for the student may now appear as the following:

$$\begin{aligned} 23{,}500 \times 1000 &= 2.35 \cdot 10^4 \cdot 10^3 \\ &= 2.35 \cdot 10^7 \end{aligned}$$

Similarly, some numbers may be expressed as follows:

$$55{,}000 \times 23{,}500 =$$
$$(5 \cdot 5 \cdot 10^4) \cdot (2.35 \cdot 10^4) =$$
$$(5 \cdot 5 \cdot 2.35) \cdot (10^4 \cdot 10^4) = 12.925 \cdot 10^8$$

$12.925 \cdot 10^8$ may be written in scientific notation as
$1.2925 \cdot 10^9$

Have the student compute examples with the multiplication of factors expressed in scientific notation.

C. Computing with small numbers

The student may find it interesting to compute with very small numbers. This lesson will also reinforce their understanding of using integral exponents.

Explain that $.1 = \frac{1}{10} = 10^{-1}$

$$.01 = \frac{1}{100} = 10^{-2}$$
$$.0045 = 4.5 \cdot \frac{1}{1000} = 4.5 \cdot 10^{-3}$$

1. Multiply

The student may multiply some very small numbers.

$$.036 \times .00054 = \triangle$$
$$(3.6 \cdot \frac{1}{100}) \cdot (5.4 \cdot \frac{1}{10{,}000}) = \triangle$$
$$(3.6 \cdot 10^{-2}) \cdot (5.4 \cdot 10^{-4}) = \triangle$$
$$(3.6 \cdot 5.4) \cdot (10^{-2} \cdot 10^{-4}) = \triangle$$
$$19.44 \cdot 10^{-6} = \triangle$$

2. Divide

Dividing is equivalent to multiplying by its reciprocal.

$$.0039 \div .013 = \triangle$$

$$(39 \cdot 10^{-4} \div 13 \cdot 10^{-3} = \triangle$$

$$\frac{39 \cdot 10^{-4}}{13 \cdot 10^{-3}} = \triangle$$

$$3 \cdot 10^{-4-(-3)} = 3 \cdot 10^{-1}$$

D. Linear Metric measure

The student can review his experience with the metric system. He is familiar with its consistent pattern and relationships based on the number 10. The use of scientific notation is especially useful in dealing with metric units.

The student may construct a chart of linear metric units and equivalent scientific notations.

Linear Metric Measure

Unit	Abbreviation	Meters	Scientific Notation
1 kilometer	1 Km.	1000m	10^3m
1 hectometer	1 hm.	100m	10^2m
1 dekameter	1 dkm.	10m	10^1m
1 meter	1 m.	1m	1m
1 decimeter	1 dm.	$\frac{1}{10}$ m	10^{-1}m
1 centimeter	1 cm.	$\frac{1}{100}$ m	10^{-2}m
1 millimeter	1 mm.	$\frac{1}{1000}$ m	10^{-3}m

The student may discover English words similar to the metric units.

The teacher may point out in a class discussion that the standard unit is the meter and that 39.37 inches is the English approximation.

1. Conversion within the system

   The teacher may point out the pattern that is established when changing from one unit of measure to another unit of measure within the metric system. (.1m = 10 cm.)

2. Conversion from the metric to British Engineering System

   The student may do a number of examples using this method:

$$1 \text{ m} = 39.37 \text{ in.}$$

change 1Km to miles

$$1 \text{ m} = \frac{39.37}{12} \text{ ft.}$$

$$1 \text{ m} = \frac{39.37}{(12)(5280)} \text{ miles}$$

$$1000 \text{ m} = \frac{(39.37)(1000)}{(12)(5280)} \text{ miles}$$

$$1 \text{ Km} = .62 \text{ miles}$$

E. Metric units of volume

In this study, the student reinforces their understanding of the metric units of measure. They also learn that the concept of volume holds regardless of the unit of measure being used.

The unit of volume may be defined as a cube whose edges all measure a given metric linear unit. The unit is then expressed as a cubic unit.

F. Mass, weight, and capacity

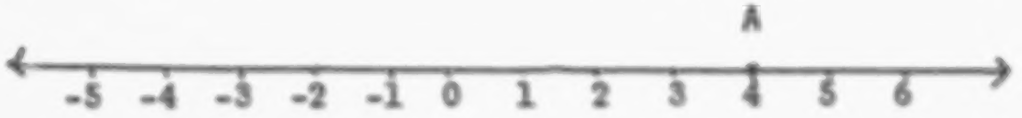
These units follow the same pattern as has been established and need not be covered here. They are covered in the eighth grade texts however.

XV. Cartesian Coordinate Systems in Graphing

A. Coordinates on a line

Explain to the students that the coordinate of a point on a number line denotes two things--the distance from the origin (the 0-point on the graph) and the direction from the origin.

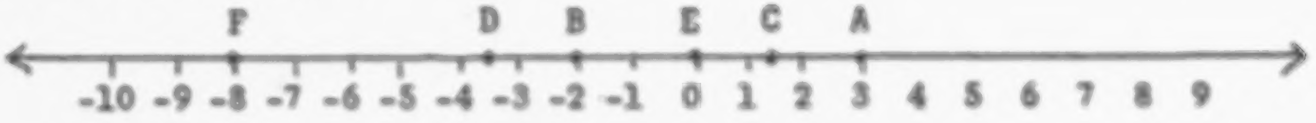
The number line is usually drawn horizontally, but it could be drawn vertically or oblique to the plane. It is an arbitrary decision to have the positive direction to the right and the negative direction to the left, but for convenience we all do it the same way.



Point A is 4 units from the origin and in a positive direction. Point B is 3 units from the origin and in a negative direction.

1. Written exercises

Have the students draw a line of indefinite length, use the midpoint of the line they draw as the origin, and mark off points $\frac{1}{4}$" in a positive and negative direction from the origin.



Locate the following points on the line: $A = 3$, $B = -2$, $C = 3/2$, $D = -7/2$,
$E = 0$, $F = -8$

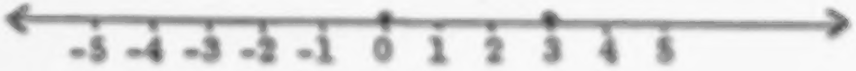
After graphing, answer questions of the type:

How far is A from the origin? How far is it from D to C? etc.

B. Graphs of solution or truth sets

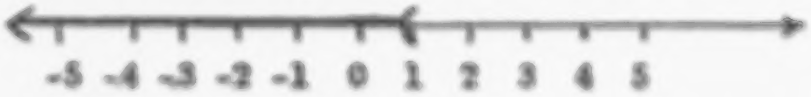
The graph of the truth set of an open sentence containing one variable is the set of all points on the number line whose coordinates are the numbers which make the sentence true.

Example: $x(x - 3) = 0$ The truth set is $\{0, 3\}$
$x = 0$ and $x = 3$



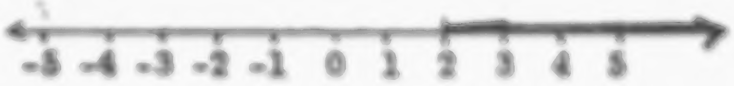
Example: $x < 1$ (an inequality)

The truth set is all those numbers less than 1.



Note the half circle (to show less than 1. Some books use the whole circle.

Example: $x \geq 2$



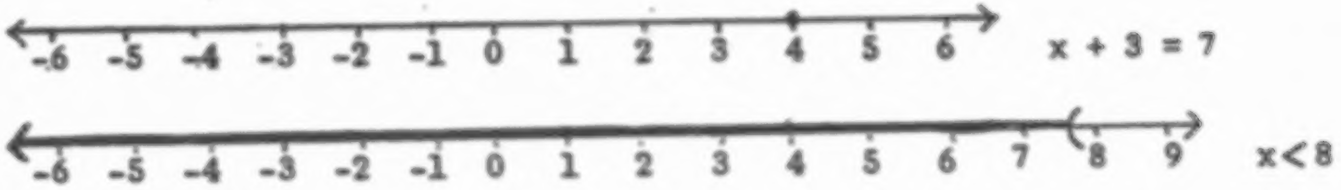
The truth set is all those numbers greater than 2, and including 2.

C. Compound sentences

Explain to the student that a compound sentence in mathematics consists of two open sentences connected by the word and or or.

Example: $x + 3 = 7$ and $x < 8$

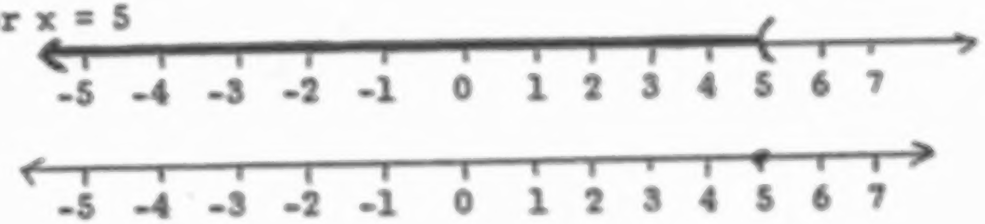
The truth set of $x + 3 = 7$ is $\{4\}$. The truth set of $x < 8$ is the set of numbers less than 8.



Notice that the only point the two graphs have in common is 4. The solution set of $x + 3 = 7$ and $x < 8$ is $\{4\}$. All points which the two graphs have in common will be the truth set of the compound sentence.

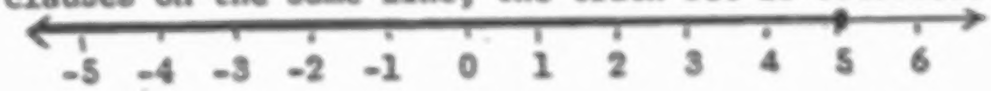
The students should understand that a second type of compound sentence uses the word or rather than the word and.

Example: $x < 5$ or $x = 5$



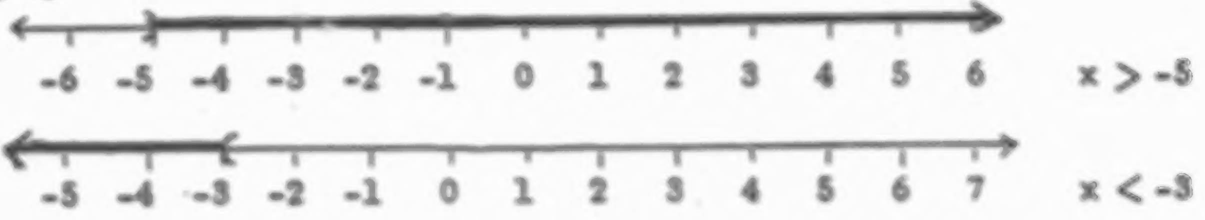
It should be agreed that a sentence of this type is true if either the first clause is true, the second clause is true, or if both clauses are true.

By placing both clauses on the same line, the truth set is obtained.



In this compound sentence, the truth set is 5 and all numbers less than 5. ∴ $x \leq 5$

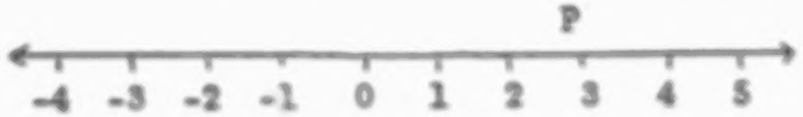
Example: $x > -5$



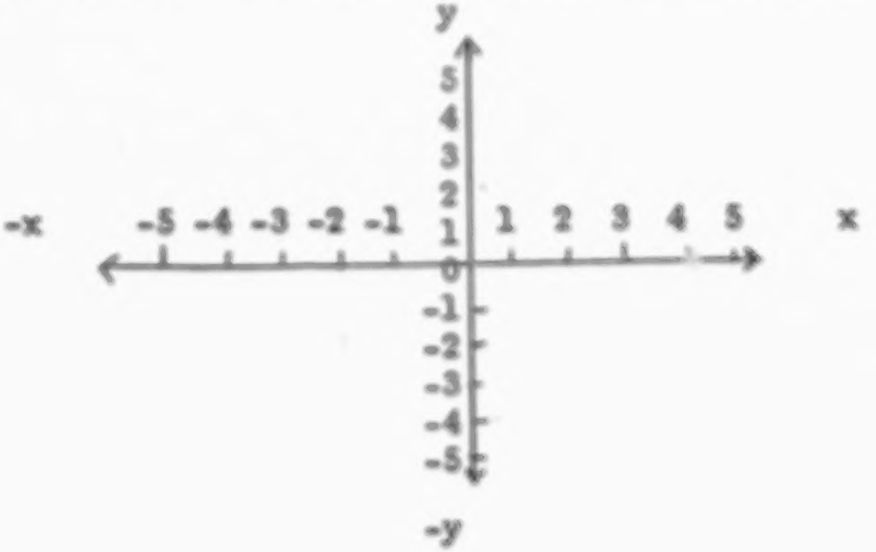
The replacement set is the set of all real numbers.

D. The coordinate plane

Explain to the students that a point may be plotted on a plane as well as on a line. Where only one number was needed to plot a number on a line, something more will be needed to describe a point which may be above the line or below the line.



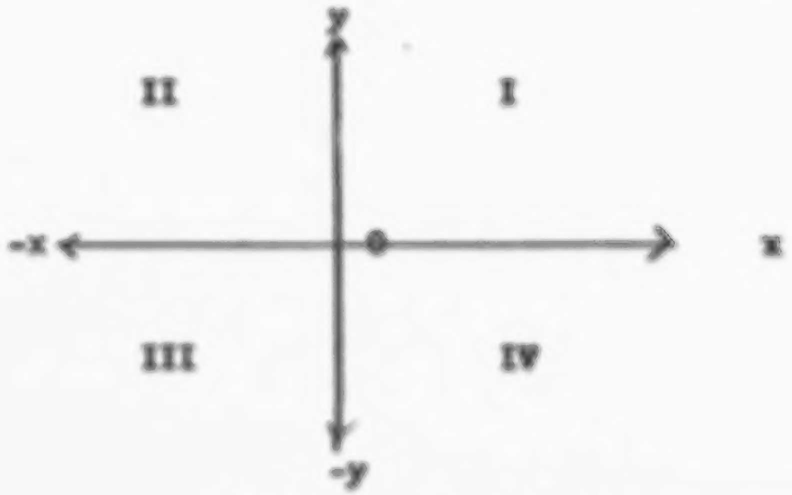
How could thig point be described? This is a good introduction to the coordinate plane.



One line is drawn horizontally and the other line is a vertical line ⊥ to and intersecting the horizontal line at 0.

The two lines are called the coordinate axes. The horizontal line is called the x-axis, and the vertical line is called the y-axis.

The point P (the figure on the preceding page) could now be described as 2 units to the right on the x-axis and 3 units up on the y-axis. The 2 and the 3 are called the coordinates of the point P and are written as ordered pairs of numbers (2,3). The distance on the x-axis is always written first and is called the abscissa of P. The distance on the y-axis is called the ordinate of P.



(arbitrary decision)

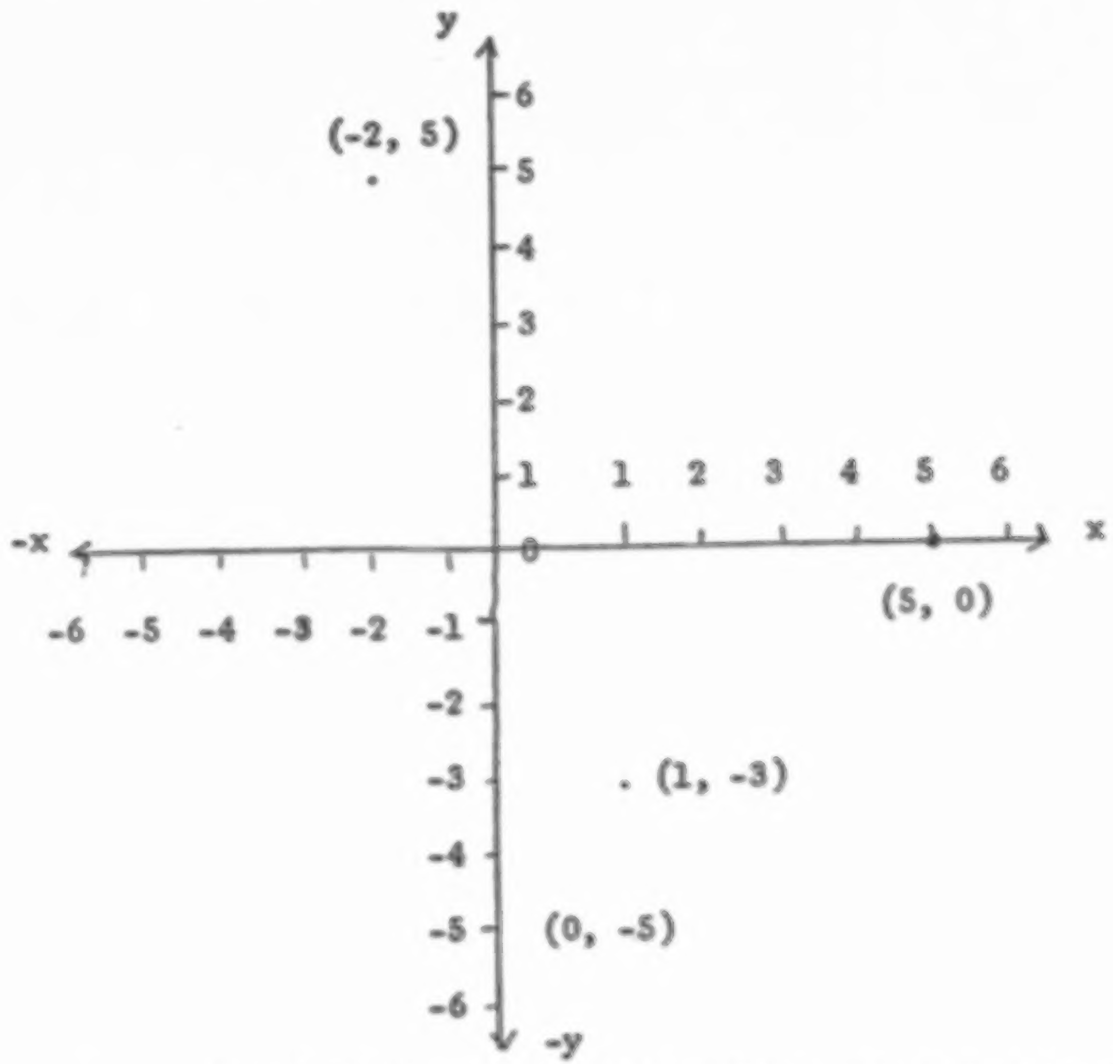
The coordinate plane is separated into four quadrants, numbered in a counter-clockwise direction.

1. Exercises

The students should have many examples of ordered pairs. Stress that graphs must be neat.

Examples: Graph the following ordered pairs. (Graph paper should be used for accuracy and time saving)

1. (1, -3) 2. (-3, 5) 3. (0, -5) 4. (5, 0) etc.



This type of exercise should terminate this unit for the average eighth grade class.

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APPENDIX

GRADE EIGHT

CHAPTER 1

1. The symbol x^b means b number of x's as a product of factored form. T
2. There are K number of symbols used in base K. T
3. A numeration system utilizing five symbols is a quinary system. T
4. Base ten is simple additive system. F
5. The numeral 7 may be written in an infinite number of ways. F
6. If K equals fourteen and (K + 1) = 10, then the base is fourteen. F
7. If 1000 - 1 = 888, then the base is nine.
8. 0 + A = A + 0 = A is true for every numeration system. T
9. A binary point is used in base five as a decimal is used in base ten. F
10. If four times six is equal to 40_a, then base "a" is five. F

CHAPTER 2

1. An empty set is a null set. T
2. If n can be any natural number, then 0 is equal to n. F
3. The grouping symbols in mathematics serve as punctuation. T
4. This sentence is an example of the associative property; $(5 + 8) + 3 = 3 + (5 + 8)$. F
5. Two sets are equal only if they both have the same elements. T
6. Every natural number closed under addition is also closed under multiplication and division. F
7. Subtraction of natural numbers is an example of the cummututive property. F
8. The identity element of multiplication is zero. F
9. N is equal to 88 in the following equation: $N = 43 + (15 \times 3)$. T
10. If the quotient of two numbers is 1, then one of the numbers must be 1. F

CHAPTER 3

1. The set of natural numbers is included in the set of integers. T

2. On the number line, greater than is interpreted to mean "to the right of," and less than is interpreted to mean "to the left of." T

3. Every integer has an opposite. F

4. Addition of integers may be defined as the renaming of two addends as a sum. T

5. The following is a true sentence: $8 = -9 + 17$. T

6. The process of finding an unknown addend is subtraction. T

7. To subtract an integer, its opposite may be added. T

8. The set of integers is closed under the operations of addition and subtraction. T

9. The identity element for addition is zero. T

10. The quotient of two negative integers is a positive number. T

CHAPTER 4

1. A Venn diagram is used to show the relationship between sets. T
2. Disjoint sets are two sets with one common element. F
3. If U is the replacement set and T is the truth set of an equation, then T is a subset of U. T
4. To solve an equation, one must find the solution set. T
5. An equation is a sentence which states that two expressions name the same number. T
6. The addition property of equations states that for every a, every b, and every c, a = b if and only if a + c = b + c. T
7. The multiplication property of equations may be expressed as follows: for all integers n, y, and z, if n = y, then ny = yz. T
8. For all integers a, d, and e, if a = d, then $\frac{a}{e} = \frac{d}{e}$ if $e \neq 0$. T
9. An open sentence is either true or false. F
10. The following is an equivalent expression: 5(2 - 3) and 52 - 15. T

CHAPTER 6

1. Prime numbers have only themselves and one as factors. T

2. The number of primes is infinite. T

3. Goldbach's Conjecture states that every natural number can be named as the sum of two prime numbers. F

4. Every natural number can be factored into any number of prime numbers. F

5. The Sieve of Eratosthenes sorts prime numbers from natural numbers. T

6. All even numbers have another even number to which it is relatively prime. F

7. If a fraction's numerator and denominator are relatively prime, the fraction is in its lowest terms. T

8. The factors of a negative number are the same as a positive number. F

9. The product of two natural numbers is equal to the product of their least common multiple and their highest common factor. F

10. A whole number is even if it has the factor 2. T

CHAPTER 7

1. Rational numbers cannot be replaced by fractions. F
2. Any rational number multiplied by one equals the rational number. T
3. Two numbers whose product is 1 are called reciprocals of each other. T
4. The reciprocal of $-\frac{1}{7}$ is $+\frac{1}{7}$. F
5. A fraction such as $\frac{3}{4}$ over $\frac{2}{5}$ is called a complex fraction. F
6. The set of rational numbers is closed under division. F
7. The denominators of decimal fractions are powers of ten. T
8. The following fraction could be expressed as a terminating decimal: $\frac{7}{20}$. T
9. Every rational number can be named by either a terminating decimal or a repeating decimal. T
10. An example of an irrational number is π (pi). T

CHAPTER 8

1. Between any two points on a line there is another point. T

2. A line segment has a definite length. T

3. To draw a geometric figure means to draw its picture. T

4. The following statements are both true: There is exactly one line through any two given points; There is an unlimited number of lines through a point. T

5. The intersection of two sets is the set of elements common to both sets. T

6. A ray has no endpoint. F

7. A line and a plane that intersect are parallel to each other. F

8. If two sides of a polygon have a common endpoint, they are called adjacent sides. T

9. A rectangle and a square are both quadrilaterals. T

10. If two lines intersect so that right angles are formed, the lines are called perpendicular lines. T

CHAPTER 9

1. All measuring units can be sub-divided into still smaller units. T
2. The greatest possible error is ½ the unit of measure. T
3. Measurement is never exact, but only approximate. T
4. The greatest possible error of the measurement 3½" is ¼". T
5. The accuracy of a measurement is independent of the unit of measure used. T
6. The smaller the unit of measure, the greater is the precision. T
7. The smaller the relative error, the greater is the accuracy of the measurement. T
8. 528.14 feet and 528.14 yards have the same accuracy. T
9. The measurement 6700 feet has two significant digits. T
10. Zeros in decimals used merely to locate the decimal point are never significant. T

CHAPTER 10

1. Two line segments are congruent only if they have the same length. T

2. Two angles are congruent if they have the same measure. T

3. S.A.A., S.S.S., S.S.A., and A.S.A. are all properties in constructing congruent geometric figures. T

4. If two parallel lines are cut by a transversal, then alternate interior angles are congruent. T

5. If two parallel lines are cut by a transversal, then alternate exterior angles are congruent. T

6. If two triangles coincide, then they are not congruent. F

7. The symbol denoting congruency is $\cong$. T

8. If $\overline{CB}$ and $\overline{AE}$ bisect each other at point D, then $\triangle BDA \cong \triangle CDE$. T

9. If two triangles are congruent, then for each angle or side of one triangle, there is a congruent angle or side in the other angle. T

CHAPTER 11

1. The perimeter is the surface region of a polygon. F

2. The area of a geometric figure means the area of the figure and its interior. T

3. Volume is always expressed in squared measure. F

4. Indicate whether the following formulas are true or false.
   a. Triangle: $A = \frac{1}{2}bh$. T
   b. Parallelogram: $A = bh$. T
   c. Circle: $A = \pi r^2$. T
   d. Cylinder: $V = 1/3bh$. T

5. A diameter of a circle is a chord of the circle. T

6. Opposite sides of a trapezoid are congruent. F

7. Opposite sides of a parallelogram are congruent. T

CHAPTER 12

1. A ratio states the relation between two sets of numbers. T
2. A ratio states how many elements there are in a set. F
3. In the case of a number pair 6 feet to 3 seconds, the number pair is called a rate instead of a ratio. T
4. In most problems involving percent, it is convenient to express the percent numeral as a fraction. T
5. The following is referred to as the percentage formula; P = rb. T
6. Forty is $12\frac{1}{2}\%$ of 320. T
7. The following ratio, 42:168, may be expressed as 1/6. F
8. The following ratio may be expressed as shown: 4/25 or 16%. F
9. The answer for n in the following equation is 45. N = (.625)(72). T
10. The following sentence is true: 5/15 = 4/12. T

CHAPTER 13

1. If two angles of a triangle each measure 80°, the third angle would measure less than 20°. F

2. The sum of the measures of four of the angles of a quadrilateral is 360°. T

3. The sum of the measures of 2 consecutive angles of a parallelogram is 180°. T

4. Two line segments are divided proportionally when the measures of the segments of one of them have the same ratio as the measures of the corresponding segments of the other. T

5. If two triangles are congruent, they are also similar. T

6. If corresponding angles of two triangles are congruent, and the corresponding sides are proportional, the two triangles are similar. T

7. If corresponding angles of two polygons are congruent, the polygons are similar. F

8. In any right triangle, the altitude from the vertex of the right angles to the hypotenuse separates the triangles into two right triangles. T

9. The following states the Pythagorean property: The square of the measure of the hypotenuse of a right triangle is equal to the sum of the squares of the measures of the other two sides. T

10. If $\frac{a+n}{n} = \frac{5}{3}$ then $\frac{a}{n} = \frac{2}{3}$. T

CHAPTER 14

1. The measure of the hypotenuse of a right triangle squared is equal to the sum of the remaining sides squared. T

2. The Pythagorean Theorem can best be expressed in mathematical terms as $a^2 + b^2 = c^2$, with "b" expressing the hypotenuse. F

3. The Pythagorean Theorem pertains only to numbers less than 100. F

4. The numeral 27 is a perfect square. F

5. A triangle that has sides of two equal measures is an isosceles triangle. T

6. The square root of 10 is between 4 and 5. F

7. Every positive integer has two square roots--one which is positive and one which is negative. T

8. If a number is the product of two equal factors, then either of the factors is the square root of the number. T

9. The measure of a side of a right triangle can be found if the measure of the remaining sides are known, by using the Pythagorean Theorem and the commutative property. T

10. $\sqrt{}$ (radical sign) denotes positive square roots, therefore: $\sqrt{25}$ is the square root of 625. F

CHAPTER 15

1. If a number is in the power of ten, the first factor, which is one, is usually not indicated. T
2. Ten to the zero power equals zero. F
3. 10^3 shows that 10 is to be multiplied by 3. F
4. The weight of an object is prone to change along with its mass. F
5. The identity element of multiplication can be expressed as an exponent. T
6. 4,250 m.m. equals to 42.5 m. F
7. The units of volume and the units of capacity are one of the same. T
8. The metric system of measure is based upon base ten numeration. T
9. For all positive integers a and b, $10^a \times 10^b = 10^{a+b}$. T
10. If a and b represent integers, then $10^a - 10^b = 10^{a-b}$. T

CHAPTER 16

1. The coordinate of a point on a number line indicates magnitude but not direction. F
2. The coordinate notation B(-3) means the point B whose coordinate is -3. T
3. A horizontal number line is used only for sake of convenience. T
4. An inequality is a compound number sentence in which the two clauses have unequal answers. F
5. Every rational number in existence can be paired with its equal on a number line. T
6. A real number line is so called because it is composed of all real numbers. T
7. There is not a one-to-one correspondence between the real numbers and the points on a line. F
8. The two axes of the coordinate system divide a plane into three parts. F
9. The truth set of an open sentence may be the empty set. T
10. The equation $x + 2 = 5$ is an example of a compound open sentence. T

ED026270

O.C.S.E.I.P. SYLLABUS

General Math

SE006070

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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INTRODUCTION

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

The first draft of this syllabus was written during an 8 week session at University of California, Irvine during the summer of 1966 by:

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GENERAL MATHEMATICS

PREFACE

The usual General Mathematics course is taught in the first year of high school, and the textbook resembles very closely the textbook used in the seventh and eighth grades. The student has a decided feeling of frustration when he sees that he is faced with exactly the same material that has defeated him for eight years.

This material is written for a General Mathematics class which could profitably be taught in the third or fourth year of high school, when added maturity has made the student understand that he needs some practical mathematics.

Sufficient problems under each topic will give the student a facility with the fundamentals of arithmetic in such a way that he will see the practical applications, and will impress upon him the need for this type of material in everyday living.

It is hoped that the comments on the topics will be of special help to those who, although not mathematics teachers, are assigned to teach a course in general mathematics.

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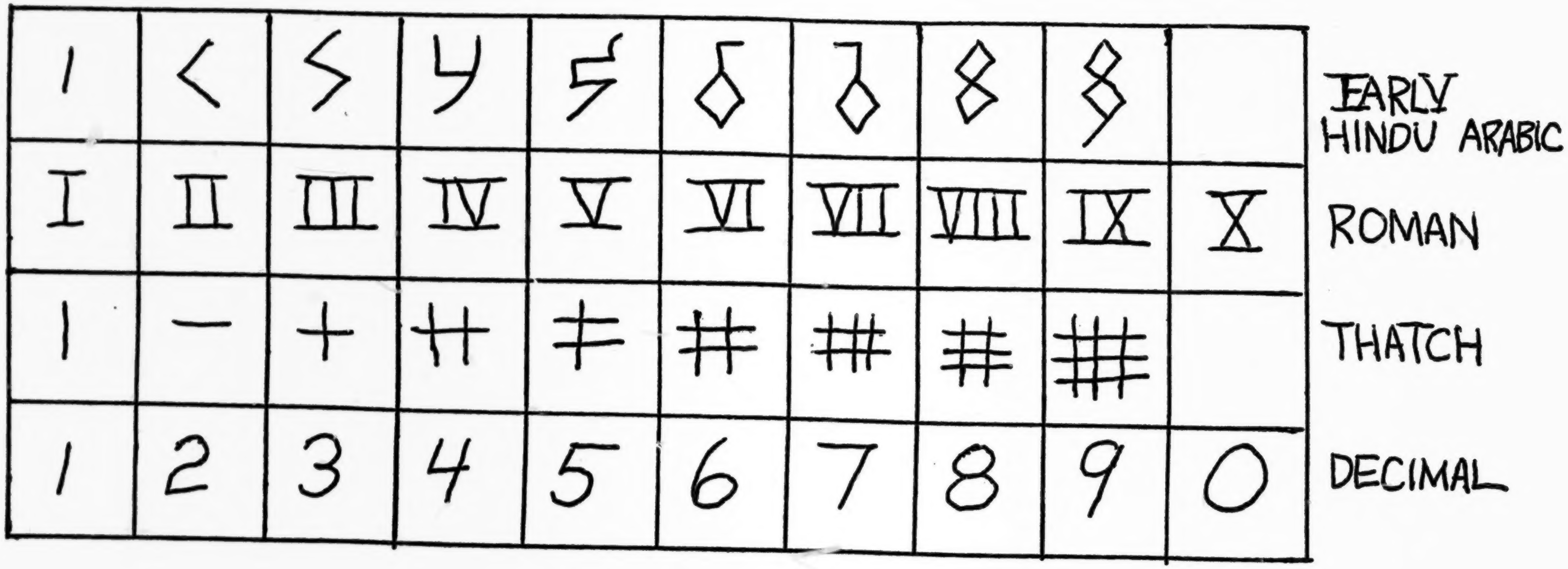
I. How to Read and Write Large Numbers

A. Decimal system of numeration

1. Place value

This system of numbers is adopted from the Hindu-Arabic numbers.

SOME NUMBER SYSTEMS


EARLY HINDU ARABIC
I II III IV V VI VII VIII IX X
ROMAN
THATCH
1 2 3 4 5 6 7 8 9 0
DECIMAL

The principles which make our number system different from others are place value and convenience in manipulations.

2. History of Measurement

Man first used his body for measurements, probably because it was so handy and he could use hands, fingers, feet, etc. as measuring devices that were always with him. The reason we count by tens is probably due to the fact that most humans have ten digits on their hands. Since all number systems are grouping devices it became relatively easy for man to group by power or groups of ten.

It is interesting to note that many of the"modern" measurements we use today are based on body measurements used and developed centuries ago. Some examples are: The fathom used by ancient Egyptian sailors and still used by our Navy today. The fathom is approximately 6 feet and is the distance between the outstretched arms of an adult from fingertip to fingertip. The inch was decreed by a king to be the measure of three barleycorns placed end to end. Others are the yard, mile, rod, foot etc. (See *How Much and How Many*. The story of weights and measures by Jeanne Bendick, McGraw-Hill Book Co. Inc., 1947)

3. Base 10

It is agreed that the value of a number is increased by ten times as the decimal is moved one place to the right and decreased by 1/10 as the decimal is moved one place to the left.

The use of a place value chart for the decimal system and a comparison of other base numbers is recommended.

B. Writing a whole numeral

A numeral is separated into periods, by commas, every third place left of the decimal. Each period contains a units place, a tens place, and a hundreds place. Each period then has its own name (i.e. units, thousands, millions, billions, trillions, quadrillions, etc.).

C. Reading a numeral

To read a numeral, simply read the digits in each period and say the period name. That is: hundreds, thousands, millions, billions, trillions etc.

Use the word "and" only for the decimal point.

D. Expanded numeral, exponential notation and number bases

1. Show the place value of each digit

$483 = 400 + 80 + 3$ or $(4 \times 10^2) + (8 \times 10) + (3 \times 1)$

$1101_{\text{base } 2} =$

$$1101_2 = \begin{array}{c|c|c|c|c} 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ \hline & 1 & 1 & 0 & 1 \end{array} = (1 \times 2^3) + (1 \times 2^2) + (1 \times 2\ ) = 13$$

2. The pattern of some number bases

10^6	10^5	10^4	10^3	10^2	10^1	10^0	Units base 10 (0,1,2,3,4,5,6,7,8,9) Decimal Numbers
2^6	2^5	2^4	2^3	2^2	2^1	2^0	Units base 2 (0,1) numbers
3^6	3^5	3^4	3^3	3^2	3^1	3^0	Units base 3 (0,1,2) numbers
5^6	5^5	5^4	5^3	5^2	5^1	5^0	Units base 5 (0,1,2,3,4) numbers
12^6	12^5	12^4	12^3	12^2	12^1	12^0	Units base12 (0,1,2,3,4,5,6,7,8,9, T,E) numbers

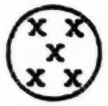
It is suggested that one should find similarities between our decimal system and the other number base systems. Work should be given in translating base ten numbers to other number bases and in converting other number base numbers to base ten.

Ex: 19 (nineteen) in base two would read from left to right as "one, oh, oh, one, one, base two." Therefore, 19 is equivalent to 10011_2

2^5	2^4	2^3	2^2	2^1	2^0
	1	0	0	1	1

Ex: $134_5 = (1 \times 5^2) + (3 \times 5^1) + 4 \times 5^0) = 25 +$

$15 + 4 = 44$

Note: A question that is often asked is: "Why do you say there is no digit 5 in the base five system yet you use fives as the headings of each place value in base five?" The answer is that the referred to fives are grouping symbols used to designate a quantity: as one or more groups of this many  (actually, depending on the textbook used, the teacher may find either type of notation and, in at least one case, both.) Note that in base ten there is no digit for ten.

3. Expanded number, base ten

$483 = (4 \times 100) + (8 \times 10) + (3 \times 1)$

4. Use of exponents with base ten

$483 = 4 \times 10^2 + 8 \times 10^1 + 3 \times 10^0$

In multiplying powers of ten add the exponents.

Ex: $10^3 \times 10^2 = 10^5$

Show that $1,000 \times 100 = 100,000 = 10^5$

5. One and zero as exponents

The exponential notation such as 10^2 means 10 to the second power. The power numbers are the exponents. It should not be confused with 10 x 2.

You might use the following: $1000 = 10^3$ and $100 = 10^2$. In dividing by powers of 10, subtract the exponent of the divisor from the exponent of the dividend.

$1000 \div 100 = 10$ or $\frac{10^3}{10^2} = 10^1$ (subtract exponents)

Therefore $10^1 = 10$

Ten to the zero exponent means one.

$10 \div 10 = 1$ Therefore $\frac{10^1}{10^1} = 10^0 = 1$ (see above)

E. Scientific notation: any number can be written as a number between 1 and 10, multiplied by a power of 10

Scientific notation is used in writing very large numbers. It shows the significant figures and gives practice in rounding off. 5,625,478 is written in scientific notation, to 2 significant figures, 5.6×10^6.

It may be possible to extend scientific notation to negative exponents with a good class.

F. Rounding off

1. How to round off whole numbers

If the figure dropped is 5 or more, increase the preceding figure by 1. Any numbers dropped become zeros.

a. Round off to nearest 10, 100, 1000, etc.

Round off to nearest 10: Example: 225 to 230

b. Practical application in the form of reading problems.	5232 Ford cars were produced this month. In rounding off to the nearest 1000's this could be read as approximately 5000 cars of this type were produced this month."
2. How to round off a mixed number	Add one to the nearest whole number if the fraction is half or more. If the fraction is less than half, drop it and don't change the whole number e.g. 5 3/4 rounds off to 6 5¼ rounds off to 5
3. How to round off a decimal number Round off to nearest .1, .01, .001, etc.	Rewrite as many figures as are needed and drop the rest. If the first figure dropped is 5 or more, increase the preceding figure by 1. In rounding off a decimal number, look one digit to the right of the number to be rounded off. e.g. 9.6236 - to nearest .001 is 9.624 to nearest .1 is 9.6 If the number immediately to the right of the one to be rounded off is less than five, drop it and all numbers to the right. If the number is five or greater, add one to the number to be rounded off and drop the remaining numbers to the right. There are other methods of rounding off, but the above method is preferred at this level due to its relative simplicity. (However, if the teacher wishes, the "odds-evens" method of rounding may be discussed with a somewhat advanced group.)

G. Base Two number games

1. First game

Have each student make four separate cards similar to the ones shown below.

A	
1	5
3	13
7	15
9	11

B	
2	6
10	11
14	7
15	3

C	
4	7
12	5
6	13
14	15

D	
9	10
15	12
11	14
13	8

Using the above cards one can always "guess" the number a person is thinking of under the following conditions:

1. That the person thinking of the number point to the cards his number is on.
2. That the number being "guessed" is from 1 to 15.

Solution

Give each card a numerical value as one would in base two.

Example: A = 1
B = 2
C = 4
D = 8

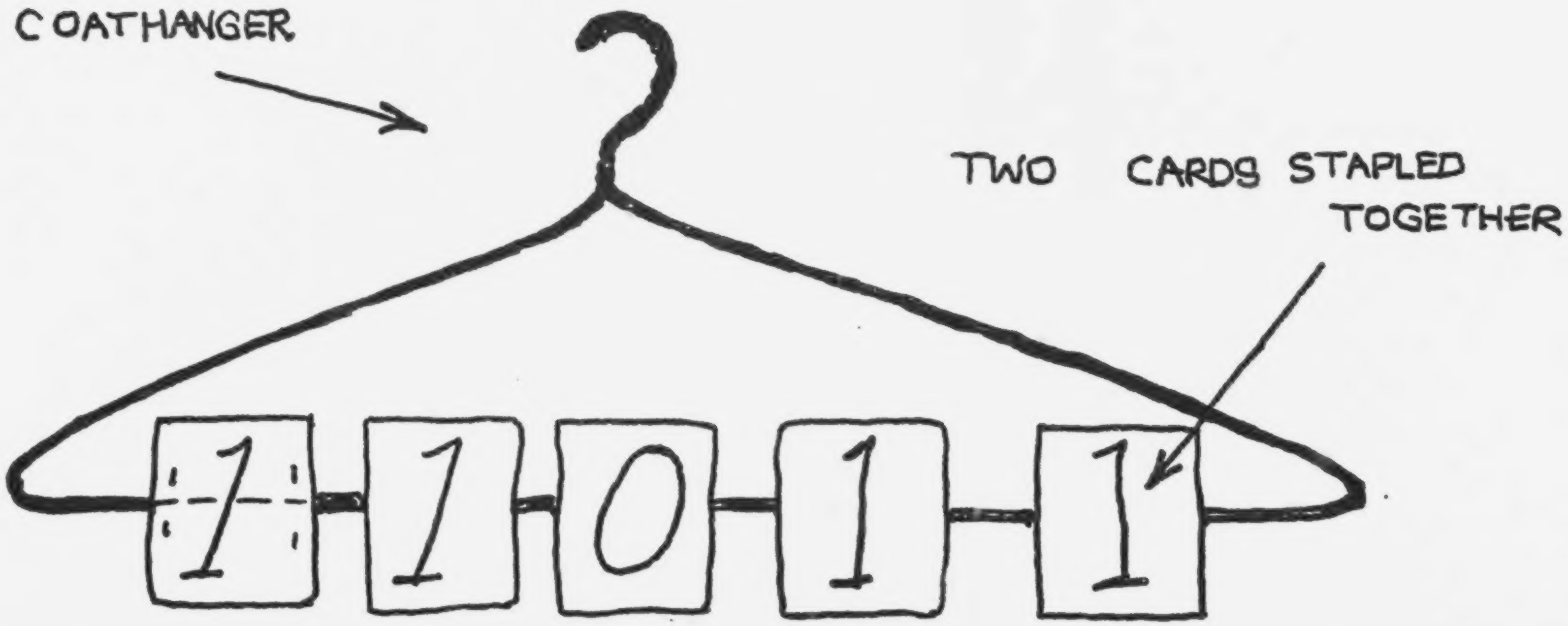
If a person points to cards A, C, and D then his "number" would be the sum of these values or 13.

A + C + D = 1 + 4 + 8 = 13

3. More cards may be added by giving them values of 16,32,64, etc.
4. The numbers can be placed on the cards by the following process.

32	16	8	4	2	1	Value	
F	E	D	C	B	A	Letter	
	1	0	0	0	0	= 16	showing that 16 would appear only on card E
	1	0	0	0	1	= 17	showing that 17 would appear only on cards E and A.

2. Second game



Two 3 x 5 index cards with one (1) printed on one side and zero (0) on the other. The cards can turn about the coat hanger so that a teacher may show all ones, or all zeros or any combination of both numbers. The student is then to read this number in base two and translate this number to base ten.

SAMPLE TESTS

Test #1

I. Write the following numbers

a. one hundred five 105

b. one thousand one hundred five 1,105

c. ten thousand one hundred five 10,105

d. one hundred thousand one hundred five 100, 105

e. one million one hundred thousand one hundred five 1,100,105

f. seven million sixty thousand sixty 7,060,060

g. sixty thousand sixty 60,060

h. four million one hundred thousand two 4,100,002

i. ten thousand two 10,002

j. one thousand two 1,002

II. Find the product

$$\text{a.}\ \begin{array}{r} 67 \\ \underline{24} \\ 1608 \end{array} \qquad \text{b.}\ \begin{array}{r} 234 \\ \underline{129} \\ 30186 \end{array} \qquad \text{c.}\ \begin{array}{r} 672.9 \\ \underline{3.4} \\ 2287.86 \end{array} \qquad \text{d.}\ \begin{array}{r} 829 \\ \underline{829} \\ 687241 \end{array} \qquad \text{e.}\ \begin{array}{r} 1872 \\ \underline{7} \\ 13104 \end{array}$$

III. Add the following

a. $$\begin{array}{r} 9 \\ 2 \\ 4 \\ 1 \\ 7 \\ 2 \\ \hline 25 \end{array}$$

b. $$\begin{array}{r} 68 \\ 79 \\ 88 \\ \hline 235 \end{array}$$

c. $$\begin{array}{r} 422 \\ 699 \\ 788 \\ 220 \\ \hline 2129 \end{array}$$

d. $$\begin{array}{r} 46.9 \\ 32.4 \\ 16.2 \\ 8.1 \\ \hline 103.6 \end{array}$$

e. $$\begin{array}{r} .03 \\ 1.04 \\ 10.17 \\ 4.50 \\ \hline 15.74 \end{array}$$

IV. Place commas in the following numbers

a. 99,911,112

b. 10,279

c. 149,000

d. 672,002

e. 6,767,890,987

f. 91,091,091,011

Test #2 Number Bases

1. The 3 in 356_7 stands for 3×49 or 3×7^2 or 147

2. 10^4 has a decimal value of 10,000

3. 14_{12} = 16 in decimal numerals

4. The decimal system uses 10 different symbols

5. The numeral after 37_8 is 40 base eight

6. Find the value:

   a. 5^4 625   b. 2^4 16   c. 7^2 49   d. 3^3 27

7. Write the following numerals in the indicated ways:

   a. 15 in base two 1111_2 b. 16 in base five 31_5

   c. 32 in base eight 40_8 d. 64 in base two $1000\ 000_2$

   e. 256 in base five 2011_5 f. 256 in base eight 400_8

8. What is the decimal name for:

   a. $6E_{12}$ 83   b. $T5_{12}$ 125   c. TE_{12} 131   d. $4T_{12}$ 58

9. Change the following numerals to base 5 numerals

   a. 14 24_5 b. 23 43_5 c. 37 122_5 d. 56 211_5 e. 127 1002_5

10. Find the decimal value of 3420_5 485

Test #3 Power Numbers

Find the value

1. $4^3 = \underline{64}$
2. $2^5 = \underline{32}$
3. $3^0 = \underline{1}$
4. $10^6 = \underline{1,000,000}$
5. $4^5 = \underline{1024}$
6. $10^0 \times 10^2 = \underline{100}$

Write in shorter form

7. $6 \times 6 \times 6 \times 6 = \underline{6^4}$
8. $8 \times 8 = \underline{8^2}$
9. $10 + 10 + 10 = \underline{30}$
10. $2 + 2 + 2 + 2 + 2 = \underline{10}$

Expand the following

11. $2^5 = \underline{2 \times 2 \times 2 \times 2 \times 2}$
12. $33^4 = \underline{33 \times 33 \times 33 \times 33}$
13. $21_{five} = \underline{(2 \times 5) + (1 \times 5^0}$)
14. $30_{five} = \underline{(3 \times 5) + 0}$
15. $44_{five} = (4 \times 5) + (4 \times 5^0)$

Find the value in Base 10

16. $8^3 + 4^2 = \underline{528}$
17. $4_{five} + 3_{five} = \underline{7}$
18. $1101_{five} = \underline{151}$
19. $300_{five} = \underline{75}$
20. $5^4 = \underline{625}$
21. $5^1 + 5^2 + 5^3 = \underline{155}$
22. $5^2 \times 5^2 = \underline{625}$
23. $2022_{five} = \underline{262}$
24. $401_{five} + 304_{five} + 40_{five} = \underline{200}$

II. Changing Common Units of Measurement

A. British-American units of length

Introduction:

For every measure there is a unit. These units of measure can be further changed into smaller units which are identified by different names. These names or divisions are parts which make up a system of measurement.

1. Equivalents

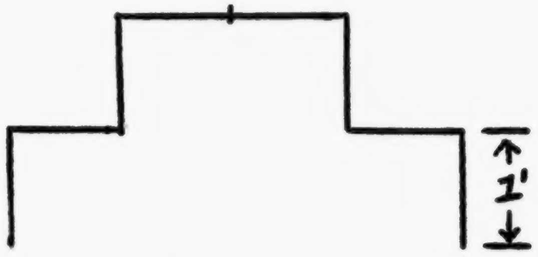
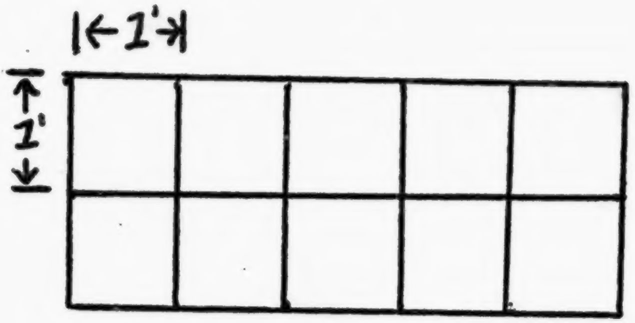
12 in = 1 ft

3 ft = 1 yd

$\left\{\begin{matrix}5\frac{1}{2}\text{ yd}\\16\frac{1}{2}\text{ yd}\end{matrix}\right\} = 1\text{ rd}$

$\left\{\begin{matrix}5280\text{ yd}\\1760\text{ yd}\end{matrix}\right\} = 1\text{ mile}$

The differences between linear square and cubic measure should be carefully pointed out as many students get these three dimensions confused with one another. Although the reasons for not adding linear units to square units or cubic units to square units may be clear to the teacher, many students fail to quickly grasp a working knowledge of these concepts. Models of geometric figures would be very helpful at this point for some audio-visual instruction.

Lines	Plane Geometric Figures
a. One or single dimensional measure	a. Two dimensional measure
b. Linear or line measure	b. Square measure
c. Distance measure	c. Area or surface measure
d.  Length of above figure = 8' (Simply add measurements)	d.  Total area of above figure = 10 sq ft

Lines

e.

f. Lines

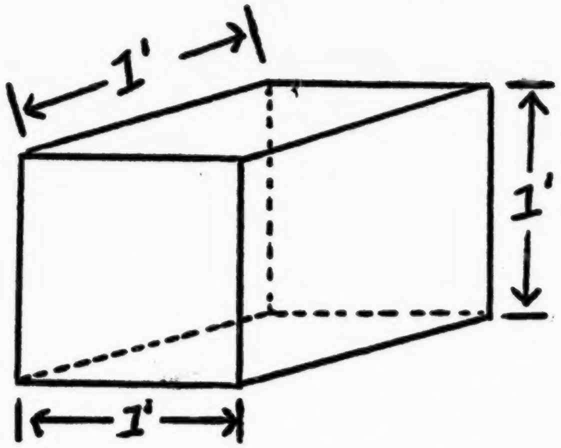
Solid Geometric Figures

a. Three dimensional measure

b. Cubic measure

c. Capacity or volume measure

d.



Total volume of above figure = 1 cu. ft.

e. V = L x W x H (1 = 1 x 1 x 1)

f. Solid geometric figure

Plane Geometric Figures

e. A = L x W (10 = 5 x 2)

f. Plane geometric figure

2. Change to a smaller unit of measurement	Example: 5 yd = ___ ft (1 yd = 3 ft) 3 is a conversion factor; multiply 3 x 5 = 15 ft Example: 5 ft = ___ in 5 x 12 = 60
3. Change to a larger unit of measurement	Example: 72 in = ___ ft (1 ft = 12 in) 12 is a conversion factor, divide Example: 60 in ___ ft $\frac{60}{12} = 5$ ft
4. Fractions	One of the equal parts into which a whole is divided is called a "unit fraction." Example: a disk has been divided into four equal parts. One of these equal parts is a unit fraction. We would indicate this fraction by the symbol 1/4. The 4 is called the denominator because it denominates (or names) the fraction by telling into how many equal parts the whole is divided. The 1 is called the numerator because it tells what number of equal parts is being taken.
a. Proper and improper fractions	Any fraction whose numerator is greater than or equal to its denominator is called an improper fraction. Any fraction whose numerator is less than its denominator is called a proper fraction. (Expressing fractions in other terms) The numerator and denominator are called the terms of a fraction. When two fractions employ different terms to represent the same portion

of the whole, they are called equivalent fractions. (To express any fraction as an equivalent fraction, multiply or divide both the numerator and the denominator by the same number.) This is sometimes called the "principle of multiplying by one" since one times any number does not change the value of that number and if we multiply by $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$, etc., we are really multiplying by one.

b. Addition

To add like fractions add the numerators and write the sum over the common denominator.

Example: $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$

To add unlike fractions: (1) Express the fractions as like fractions with the lowest common denominator, (2) add the resulting like fractions and (3) reduce to lowest terms.

Example: $\frac{1}{8} + \frac{1}{16} = \frac{2}{16} + \frac{1}{16} = \frac{3}{16}$

$\frac{1}{8} + \frac{1}{16} = \frac{2}{16}$

c. Subtraction

Like fractions may be subtracted directly. However, unlike fractions must first be expressed as like fractions.

Example: $\frac{1}{4} - \frac{1}{8} = \frac{2}{8} - \frac{1}{8} = \frac{1}{8}$

d. **Multiplication**

To multiply one fraction by another: (1) Multiply the numerator of each fraction to determine the numerator of the product. (2) Multiply the denominator of each fraction to determine the denominator of the product. (3) Reduce to lowest terms.

e. **Division**

To divide one fraction by another multiply the dividend (the number being divided) by the inverse of the divisor (the number by which you are dividing).

There is another way to divide two fractions whereby you find the common denominator and then divide, but the students find it easier just to invert and multiply.

(See the sample test at the end of this unit. It may prove helpful in the testing of knowledge on understanding of fractions.)

f. **Proportions**

Proportions are simply two fractions connected by an equal sign. Since the two sections must be of equal value, by using the (PMI) principle of multiplying by one, a student could find any one number that happened to be missing from the proportion. Another method would be to multiply the numerator of one fraction by the denominator of the other fraction. These "cross-products" formed must be of equivalent value, thus if one number was missing to could be easily found.

Example: $\frac{3}{4} = \frac{N}{24}$ $\quad$ $3 \times 24 = 4 \times N$

$72 = 4 \times N$ $\quad$ Therefore: $N = 18$

Emphasize that the understanding of proportion will aid the student throughout his mathematics program.

Example: Change 7 feet to inches.

(12 inches = 1 ft) $\frac{12}{x} = \frac{1}{7}$

5. More than one method is used for problem solving

The teacher may demonstrate that more than one method is used to change from one unit of measurement to another. You may also use specific rules.

Example: To change feet to inches, multiply number of feet by 12. To change inches to feet divide number of inches by 12.

Use problems with fractions, decimals and mixed units.

1. Mix problems so that you compute in four fundamental operations, addition, subtraction, multiplication, and division.
2. Offer a choice in answers: decimals or fractions.

Ask questions such as "What fractional part of a unit of measure is a certain larger (or smaller) unit?"

B. Area

Here we are changing from units of one denomination to units of another

1. Equivalents 1 sq ft = 144 sq in 1 sq yd = 9 sq ft 1 sq rd = 30¼ sq yd 1 acre = 160 sq rd 1 sq mi = 640 acres	The specific rules for changing from one denomination to another may also be used. (See earlier examples) 1. Compute problems, using all equivalent units of measure. 2. Use practical applications in problems. 3. Express all measurements in the same denominations. In problems involving area, express the answer as square units.
2. Area of plane figures a. Square b. Rectangle $A = bh$ c. Parallelogram d. Triangle - $A = \frac{1}{2}bh$ e. Trapezoid - $A = \frac{1}{2}h(b_1 + b_2)$ f. Circle - $A = \pi r^2$	A = Area b = base h = height $\pi = 3.1416$ r = radius
3. Area of solid figures a. Total area of cubes, rectangular solids	The total area of a cube is the lateral area + the area of the two bases. Since all of the faces and the two bases are equal squares, the total area is found by the formula, $T = 6s^2$. T = total area S = length of a side

b. Lateral area of a cylinder

c. Area of a sphere

The formula for finding the lateral area of a cylinder is $S = 2\ rh$, and the formula for the area of a sphere is $S = 4\ r^2$.

C. Weight - changing from one unit to another

1. Equivalents

16 oz = 1 lb

2000 lb = 1 T

2240 lb = 1 Long T

Use problems and examples for changing from one unit to another. Ask "what part of one unit is of another?" Use word problems expressing the practical application. Introduce manipulation of visual objects where possible.

D. Dry (capacity) - changing from one unit to another

1. Equivalents

2 pt = 1 qt

8 qt = 1 pk

4 pd = 1 bu

2½ bu = 1 bbl

Develop examples and explain how to solve these problems by using patterns. Use visual objects. Work problems and examples for changing from one unit to another. Ask "what fractional part one volume is of another?" Use word problems.

E. Liquid (capacity) - changing units of liquid measurement

Example: 3 gal = ___ qt (a)

1 gal = 4 qt 4 x 3 = 12 qt

3 gal x 4 qt = 12 qt (b)

1 gal = 4 qt; multiply by 4 (c)

1. Equivalents 1 oz = 8 drams	Develop some of your own measuring devices. (Use of felt board with cutouts can be advantageous.)
1 pt = 16 oz 1 qt = 2 pt 1 gal = 4 qt	Make use of practical applications through word problems.
F. Volume - changing units of measurement	Use previously explained methods - proportion should be emphasized.
1. Equivalents 1 cu ft = 1728 cu in 1 cu yd = 27 cu ft 1 board ft = 144 cu in or 12" x 12" x 1"	Measure volume and express all units in the same denomination. Use rectangular prisms, cubes, cylinders, spheres, cones, pyramids, etc. Problems using different units of measurement should be worked by the students.
G. Metric system of weights and measure 1. A few approximate equivalents 1 M = 39.37 in 1 M = 1.1 yd	Compare units in metric and English system. Establish equivalents. Use formulas. Develop problems and examples for changing from one unit to another. Visual aids are used to develop insight into the metric system. Compare meter stick with yard stick. Measure everyday objects in metric units.
$1\text{ M} = \frac{39.37}{12}\text{ ft}$ 2.54 cm = 1 inch 1 M = 100 cm	As a project, students might give a short history on the metric system and a forecast on its future.

1 M = 1000 mm

1 M = 10 decimeters

1000 meters = 1 kilometer = .62 mile

SAMPLE TESTS

Test #1

Write as a per cent or fraction

1. $\frac{2}{55}$ = ___%
2. 3% =
3. 20% =
4. $\frac{7}{8}$ =
5. $\frac{3}{10}$ =
6. $12\frac{1}{2}\%$ =
7. $\frac{5}{6}$ =
8. $\frac{1}{12}$ =
9. 75% =
10. $\frac{7}{8}$ =
11. 90% =
12. $\frac{1}{3}$ =
13. 99%
14. $\frac{3}{5}$ =
15. $16\frac{2}{3}\%$ =
16. $\frac{3}{8}$ =
17. $\frac{5}{6}$ =
18. 35% =
19. $\frac{3}{25}$ =
20. $\frac{7}{25}$ =
21. $\frac{2}{15}$ =
22. $\frac{7}{12}$ =
23. $2\frac{1}{4}$ =
24. 92%
25. 45% =
26. $33\frac{1}{3}\%$

Divide

27. 7.382 ÷ 2
28. 68436 ÷ .03
29. 0.03 ÷ .02
30. .9740 ÷ .005
31. 4.734 ÷ .0003

Multiply

32. .02 x .05
33. 32.5 x 1.84
34. 78.2 x .014
35. .004 x .05

Add or Subtract

36. $\frac{1}{3} + \frac{1}{4} + \frac{2}{3}$
37. $\frac{7}{15} + \frac{3}{5}$
38. $\frac{1}{5} + \frac{1}{6} - \frac{1}{3}$
39. $3\frac{1}{3} - 1\frac{5}{6}$
40. 3.64 + 5.746

Test #2

Farmer Hill had 100 chickens. From the diagram to the left, tell:

1. ____% of white chickens
2. ____% of red chickens
3. ____% of black chickens
4. ____% of white and black chickens together

Change the following fractions to percentages.

5. $\frac{2}{3}$ = ____%
6. $\frac{1}{6}$ = ____%
7. $\frac{3}{8}$ = ____%
8. $\frac{3}{5}$ = ____%

Change the following percentages to fractions. (reduce to lowest terms)

9. 45% =
10. 16 2/3% =
11. 60%
12. 66 2/3% =

13. 256 ÷ 100 =
14. 256 ÷ .01 =
15. 2.56 ÷ 10 =
16. 25.6 ÷ 100 =
17. .256 ÷ .001 =
18. $10.8\overline{)324}$
19. $1.08\overline{).324}$
20. $.018\overline{).0324}$

Test #3

I. Solve the following problems and reduce answers to lowest terms.

A. Add

1. $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$

2. $\frac{3}{7} + \frac{2}{21} = \frac{11}{21}$

3. $\frac{3}{8} + \frac{5}{16} = \frac{11}{16}$

4. $\frac{5}{6} + \frac{4}{9} = 1\,\frac{5}{18}$

5. $\frac{3}{8} + \frac{3}{4} + \frac{1}{16} = 1\,\frac{3}{16}$

6. $\frac{5}{7} + \frac{4}{9} + \frac{1}{3} = 1\,\frac{31}{63}$

7. $\frac{13}{17} + \frac{5}{8} + \frac{2}{1} = 3\,\frac{53}{136}$

8. $\frac{3}{4} + \frac{7}{8} + \frac{1}{12} + \frac{1}{6} = 1\,\frac{7}{8}$

B. Subtract

9. $\frac{8}{7} - \frac{5}{7} = \frac{3}{7}$

10. $\frac{11}{12} - \frac{5}{6} = \frac{1}{12}$

11. $\frac{3}{1} - \frac{5}{6} = 2\,\frac{1}{6}$

12. $\frac{3}{3} - \frac{1}{6} = \frac{5}{6}$

C. Multiply

13. $\frac{7}{8} \times \frac{5}{6} = \frac{35}{48}$

14. $\frac{6}{5} \times \frac{9}{7} = 1\,\frac{19}{35}$

15. $\frac{1}{2} \times \frac{3}{1} \times \frac{4}{9}$

16. $\frac{5}{8} \times \frac{12}{25} \times \frac{2}{7} = \frac{3}{35}$

17. $\frac{7}{9} \times \frac{4}{5} = \frac{28}{45}$

D. Divide

18. $1\frac{2}{7} \div \frac{3}{5} = 2\frac{1}{7}$

19. $3 \div \frac{1}{2} = 6$

20. $\frac{13}{24} \div 6 = \frac{13}{144}$

21. $\frac{8}{9} \div \frac{7}{5} = \frac{40}{63}$

22. $7\frac{1}{8} \div 3\frac{1}{7} = 3\frac{47}{176}$

III. Graphs

There are many practical considerations to be taken account of in the study of graphs. Graphs can be a useful tool if used correctly, however they are often misused and misread.

Graphs are a picture representation of two or more groups of items. They are usually figures placed in such an order that these figures may be compared at a glance with each other or with other figures.

One has to be careful in interpreting graphs as they can be made to give a false impression. A clever person can distort graphs to the point that they give almost any impression the person wants to leave. For example, in pictographs, suppose if one dairy A is comparing its milk products with a competitor B. Dairy A may show a picture of big healthy cows on its graphs and small scrawny cows on competitor B's graphs insinuating that dairy A has more, bigger, healthier cows than B and as a result better products. Or company A may compare its products with company B using a different scale or comparison ratio; again trying to give a picture that its product is superior to company B in some way or another.

At this writing, there is an auto insurance company that advertises on television by showing an impressive looking graph intending to prove that this company is the fastest and best company in paying out claims for auto accidents. Now this may be true, but there seems to be no real evidence of this deducible from the graph flashed on the screen for a few seconds.

There is an old adage to the effect that "Figures don't lie, but liars do figure" and this can be applied to the construction of graphs. Graphs must be read very carefully in order to be meaningful.

Some of the things one should keep in mind when interpreting graphs are the following:

1. Graphs should be identified or have an explanation and be dated.
2. Graphs should be clearly labeled.
3. Most graphs have two scales, a horizontal and vertical scale. These scales should be clearly printed and each scale should have an even distribution.
4. Any graphs that are to be compared should use the same scales and be in the same proportion.
5. The bars or figures of a graph should be within the bounds of the scale used.

The ability to read and interpret graphs is of extreme importance: a graph can sometimes produce a clearer, faster comprehension of a set of numerical facts than can any other method of presentation. "A picture is often worth a thousand words." Different examples of graphs can be found in most newspapers and magazines. Different types of graphs should be secured and interpreted.

A. Constructing graphs	After practice in reading various types of graphs, have the students construct their own. Students can secure their own data for some problems, either: 1. Directly from life (keep daily temperatures for a week or so, or even hourly temperatures for one day); the family budget; scores on football or other games, etc. 2. From data presented in almanacs and other publications. The latest almanac provides such interesting data as batting averages of baseball players, big league standings, world records in ski jumping, swimming, track, etc.; geographical or historical data such as population of the United States from 1610 to the present, distribution of the population of the U. S. by race (Negro, white, Indian, etc.); the federal income tax for different incomes for recent years; weather information such as rainfall, temperatures, wind.
1. General methods in construction of graphs	In constructing graphs, certain over-all principles should be followed:
a. Titles	1. Have a clear title explaining the meaning of the graph, printed neatly across the top or bottom.
b. Labels for axes	2. For pictographs, bar and line graphs, label both the vertical and horizontal axes clearly.

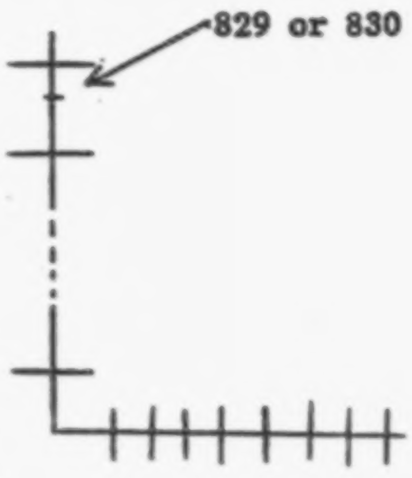
c. Determining the scale

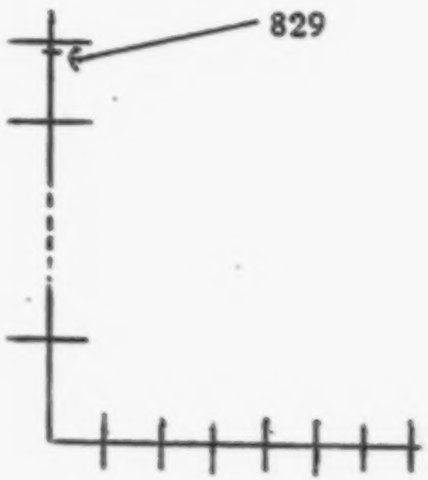
3. To find a suitable scale (or the number assigned to each space), divide the largest number by the number of spaces available and use the next larger suitable number: 1, 2, 4, 5, 10, 50, 100, 500, 1000, etc.

   For example, if 8,256 is the largest number on the list, and there are 20 spaces available, 8,256 divided by 20 equals 400 (approximately). Use either 500 or 1000 for each space. This proceduce will tend to produce a graph large enough to give a clear presentation of the data.

d. Rounding off

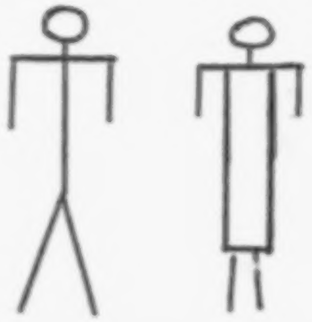
4. The rounding off of quantities is often a necessity when constructing graphs, and provides good experience in this activity. For example, on the scale of 50 units per space at the left below, the number 829 would be rounded to 830 since it could be virtually impossible to place the point any more accurately than this. However, if the scale used had a spread of only 10 units, such at that at the right, then 829 could actually be plotted.



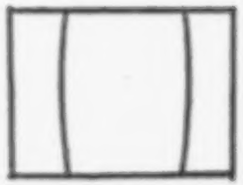


2. Pictographs

Pictographs are the most easily read and counted, but more difficult to construct, for some students. However, for a population graph it is easy to draw stick men (or women) such as those at the left below. It is certainly not necessary to have elaborate figures such as those shown at the right.

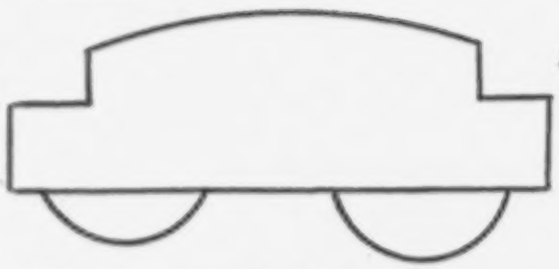


television sets



rather than

automobiles



rather than

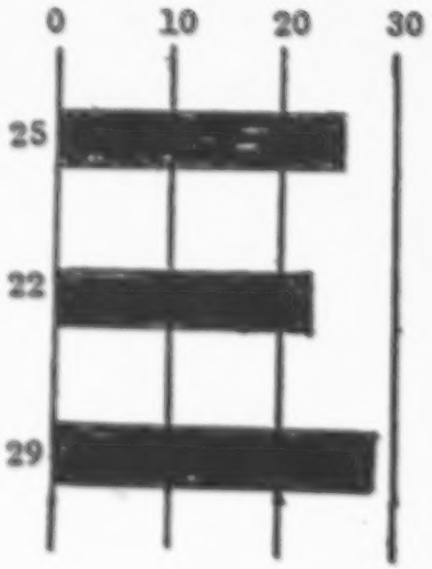
3. Bar graphs

The bar graph is used to compare quantities. For this reason the bars must all start from the same line, marked zero.

The bars should all be the same width, and usually one vacant space is left between them.

The bars can be drawn either vertically or horizontally, although it is usually easier to read the names of the bars if they are horizontal.

4. Line graphs



The line graph is usually used to show changes that take place with the passage of time, rather than to show a comparison of sizes, as do bar graphs. Line graphs usually do not start from zero, since the emphasis is on the amount of change from one period of time to another.

One of the greatest difficulties in constructing either bar or line graphs is in estimating the position of the end point when it lies between the labeled lines.

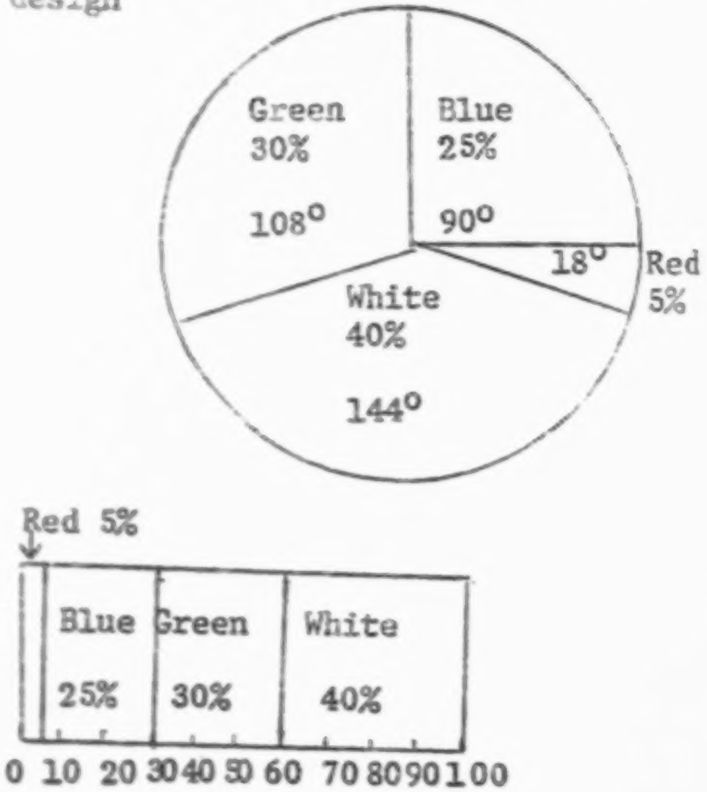
For example, the bar for 25 extends to a point exactly halfway between 20 and 30; 22 is a shade less than one quarter of the distance between 20 and 30; 29 is one-tenth of a division less than 30. Sometimes it helps

5. "Percent" graphs

   a. Circle graphs

   b. Rectangle graphs

B. Distribution of colored tile in a design



to mark the length of the space on scratch paper and divide it off freehand in 10 approximately equal spaces (if each space represents 10).

"Percent" graphs are used to show the parts (or percents) of the whole, and the relationships between them. Each of the parts is reduced to a percent of the whole, then represented as a sector of a circle or a section of a rectangle.

Example: 1 red, 5 blue, 6 green, 8 white

Since the total is 20, the red is 1/20 of the while, or 5%. In the same way, blue is 5/20 or 25%, green is 30% and white is 40%.

Each of these must be made into a sector of the circle (piece of pie). Since the whole circle is 360 degrees, or 18 degrees; the blue is 25% of 360, or 90 degrees, and in like manner the green is 108 degrees and the white is 144 degrees.

Draw one radius of the circle, and lay off on this line an angle of 18°, with its vertex at the center of the circle; this angle forms a sector representing the red. Continue in like manner with the others.

If the percents are decimal amounts, they can be rounded to the nearest whole number, or they can be kept in, say, tenths, and the degrees obtained from them rounded to the nearest whole number if desired.

SAMPLE TEST

Graphs

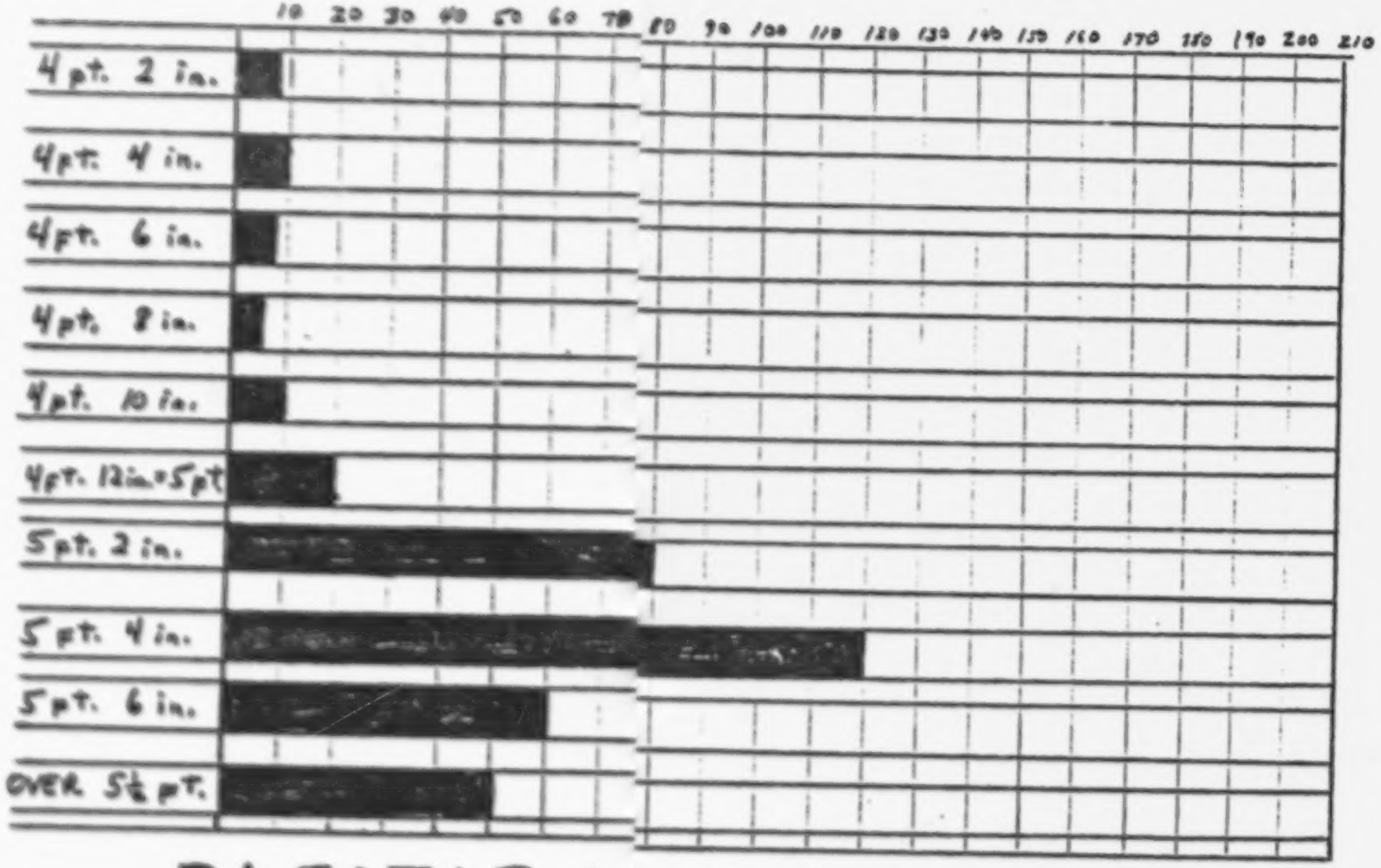
A 1966 census taken of the heights of girls at Pacific High School showed the following:

4 ft. 2 in. - 7 girls
4 ft. 4 in. - 10 girls
4 ft. 6 in. - 8 girls
4 ft. 8 in. - 6 girls
4 ft. 10 in. - 10 girls

5 ft. - 20 girls
5 ft. 2 in. - 80 girls
5 ft. 4 in. - 120 girls
5 ft. 6 in. - 60 girls
Over 5 ft. 6 in. - 50 girls

Make a bar graph of the above information.

HEIGHTS OF GIRLS



PACIFIC HIGH SCHOOL

IV. Meter Reading and Billing

A. How to read a gas meter

Gas is measured in cubic feet. Notice direction of each pointer on meter. Read the number just passed. Four top dials are read in hundreds of cubic feet, read as C. C. F. Gas consumed is the difference from one reading to the next.

B. Rate and billing for consumed gas

Cost of gas consumed is prorated.

1. Prorated would be explained as a method the gas company has established for computing the cost of their product.

Commodity charges

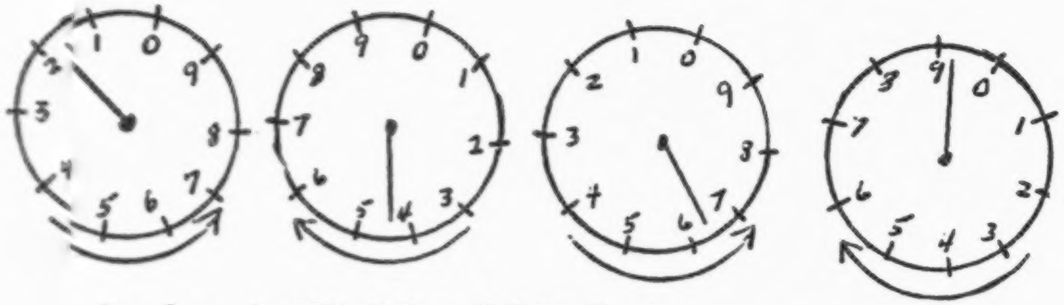
(hundred cubic feet)	
First 10 c.c.f. used per mo.	.22¢ per c.c.f.
Next 10 c.c.f. used per mo.	.19¢ per c.c.f.
Next 30 c.c.f. used per mo.	.12¢ per c.c.f.
Over 50 c.c.f. used per mo.	.10¢ per c.c.f.
Minimum Bill - Fixed charge $1.00	

Discuss fixed charges and sales tax.

2. A fixed charge is a fee paid by the customer for the convenience offered him and for the cost of billing. There is usually an initial fixed charge for connecting or turning on gas service.

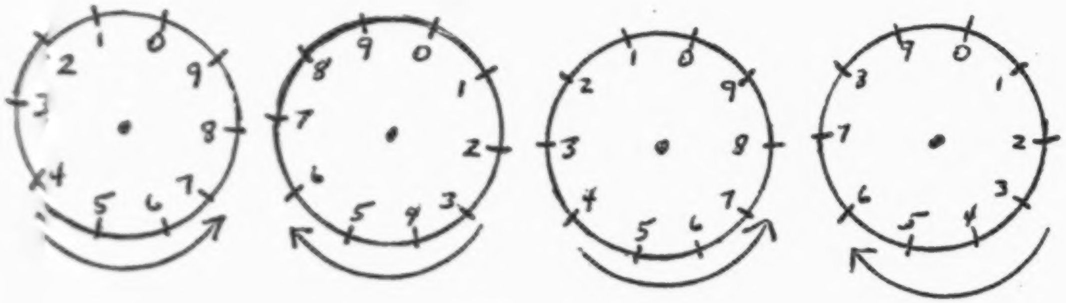
   Sales tax is added to the cost just the same as any other retail commodity.

C. How to read an electric meter



Read as 1,469,000 - Cubic Feet

Use drawings and mock-ups. Develop problems. Read the number just passed by the pointer. Four dials are read, from left to right, in KWH. The difference between two monthly readings is the amount used.



Kilowatt Hours

D. Define:

1. Watt	A unit of electric energy represented by a current of one ampere under a potential difference of one volt
2. Kilowatt	1000 watts of electric energy
3. Watt-hour (WH)	Watts used per hour

V. The Mathematics of Business

A. The Checking Account

1. Why a checking account?	1. The stub is a record of expenditures and a cancelled check is a receipt of payment made.
2. Opening the checking account (Signature card)	2. The signature on the signature card should be in the habitual handwriting of the depositor. This signature should always be used in doing business with the bank for protection of the depositor.
3. The deposit slip	3. Deposit slips bear the date of deposit, total sum of currency and total sum of coins. Checks are listed separately by bank numbers. (Bring in cancelled checks and stubs.)
4. Endorsement of checks	4. Checks are endorsed only when ready to be cashed or deposited. Checks are endorsed across the stub end of the reverse side. Three types of endorsement are: a. In blank - Payable to any person b. In full - Payable to a certain person or company c. Restricted endorsement is usually used in making deposits by mail. The endorsement is "for deposit only" followed by the name of the payee's bank and "to the account of" followed by the payee's endorsement and his number.

5. Check writing	5. The stub is filled in first. On the check, all writing should start at the extreme left of the space provided. The amount of the check should be first written as a decimal; then written again in words, with the cents written as fractions of 100.
6. Precautions	6. Use ink only--never pencil. Be sure the check amounts in decimals and in writing are the same. Avoid overdrawal by keeping accurate account on the stubs. Never correct or change a check; void the old check and write another one. 7. Keep cancelled checks as proof of payment, for income tax purposes and to reconcile the account.
B. Sales slips and receipts	
1. What is a sales slip?	1. A sales slip is the customer's record of a transaction; its duplicate is the company's record.
2. Transaction entries	2. The quantity, description, unit price, and total amount of each item of merchandise is listed. If a charge account, the customer's signature is added. Securing a receipt for each purchase is good business practice.
3. What is a receipt?	3. "Paid," with date and clerk's initials on customer's sales slip is the simpliest receipt. A receipt is a written statement of some sort showing that a sum of money or some other item has been accepted. A money receipt should name the purpose of

	payment. A rent receipt must include the address of the rental, time covered by the payment, and correct dates for the payment.
C. Counting change	When a person's money is accepted it should be kept separate from all other money until the change is counted. State the amount of the purchase, then, starting with the smallest denominations of coins or bills, "count back" to the customer.
D. Percentage	
1. Why is the knowledge of percentage useful?	1. Discounts, mark-ups, profit and loss, interest, commissions, taxes, and budgeting are only a few of the practical uses of percentage. Facility in using percentage simplifies the complicated use of fractions.
2. Meaning of percent	2. "Percent" and "%" both mean "hundredths." To change any number to a percent, multiply by 100 and add the % sign.
3. Relationship of fractions, decimals, and percents	3. Stress methods of obtaining equivalents for percents, decimal and common fractions. Every number can be written as a decimal, fraction, or percent. Example: $4.5 = \frac{45}{10} = \frac{450}{100} = 450\%$
4. Problems using percent; percent of increase or decrease	4. The three things to consider in percentage problems are the whole, part, and percent. Two of the three must be given directly or indirectly in order to find the third.

a. Rules to learn and use

   1. Given the whole and percent, to find the part:

      Example: John had \$10. If he spent 5% of it, how much did he spend?

      Use $W \times \% = P$

   2. Given the whole and part, to find the percent:

      Example: John had \$10. If he spent \$2, what percent of his money did he spend?

      Use $\frac{P}{\%} = W$

   3. Given part and percent, to find the whole:

      Example: Find how much money John had before he spent \$4 which was 25% of his money.

      Use $\frac{P}{\%} = W$

b. To change percent to fractions, drop the percent sign and write the number as a fraction with 100 as the denominator.

c. To change percent to a decimal, drop the percent sign and move the decimal point two places to the left.

d. To change a fraction to percent, change the fraction to a decimal fraction, then move the decimal point two places to the right and annex the percent sign.

e. The three basic types of percentage may be treated as one case through the use of proportion.

1. Find 25% of 12

   Example: $\frac{N}{12} = \frac{25}{100}$

   $100\ N = 300$

   $N = 3$

2. What percent of 12 is 3?

   Example: $\frac{N}{100} = \frac{3}{12}$

   $12\ N = 300$

   $N = 25$

3. 25% of what number is 3?

   Example: $\frac{3}{N} = \frac{25}{100}$

   $25\ N = 300$

   $N = 12$

E. Interest

1. Introduction	1. Interest is the money paid for the use of money. Money borrowed or invested is called the principal. Interest is usually expressed as a percent of the principal. This percent is called the rate of interest expressed as %.
2. Explanation of terms in simple interest formula and value of formula in interest problems	2. By substitution of three known terms in the formula for simple interest (i = prt), the fourth term may be found. Stress the use of this formula. i = interest in money t = time in years p = principal r = rate in percent
3. Difference between simple and compound interest; compound interest tables	3. Simple interest is a certain % of the principal; compound interest is a certain % of the amount. (Amount is the interest added to the principal.) A graph showing the difference between simple and compound interest over a period of 20 years shows students the rapid increase of compound interest. A few compound interest problems worked out by students will demonstrate the value of compound interest tables.

F. Budgets	
1. Plan for setting up a budget; operation of budget account	1. In planning a budget: A. Keep a record of income and expenditures B. Classify the expenditures C. Adjust budget to individual case
2. Usual categories included in a family budget	2. A budget should fit the family and will change with income, price levels, and needs
G. Installment Buying	
1. Meaning of installment buying, and its advantages and disadvantages	1. In installment buying a down payment is usually required and a carrying charge (interest) is added. For supplementary work have students bring in examples of installment buying and have them determine the interest rates on each loan. The student should be aware that the rates of interest vary from lending institution to lending institution.
2. Finding the rate of interest charged in installment plan buying	2. To find the rate of interest, use the interest formula ($i = prt$). Stress "hidden charges" as well as high interest rates of installment buying.
H. Taxes	
1. Various types of taxes	1. Funds for local, state, and federal governments are raised by taxes

2. Assessed valuation and tax rate

2. For computing real estate and personal property rates, the assessor sets the assessed valuation, which is a percentage of the worth of each person's real estate. The tax may change from time to time.

a. Property tax

1. Local taxes

The money needed for the cost of operating local governments is obtained mainly by the property tax. A person's property tax may be used for any or all of the following: Maintain police and fire protection, costs of operating the government including payment of salaries, maintaining roads schools and sanitation, etc.

To determine the amount of taxes needed each tax district makes a budget of expenses for the coming year. To determine the tax rate, the total tax needed is divided by the total assessed valuation.

Formula: $\text{Tax Rate} = \dfrac{\text{Total Tax}}{\text{Total Assessed Valuation}}$

Example: In East Japola the total property tax is $180,000, and the total assessed valuation is $6,000,000. The tax rate is, therefore, 3 cents on the dollar.

$$\text{Tax Rate} = \frac{180,000}{6,000,000} = \frac{3}{100} \text{ or } 3\%$$

This means each property owner will pay a tax equal to 3% of the assessed valuation of his property. For example, if Jack Payplenty owned a home assessed at $40,000 he would have to pay 3% of $40,000 or $1,200 per year as his tax bill. The tax rates may go up or down depending on how each parcel of real estate is re-appraised every few years by the assessor.

SAMPLE TEST #1

Find the entire number if,

1. 12 1/2 % of the number is 5. 40
2. 60% of the number 18. 30
3. 10% of the number is 4. 40
4. 20% of the number is 20. 100
5. 33 1/3 % of the number is 15. 45
6. 16 2/3% of the number is 6. 36
7. 12% of the number is 24. 200
8. 1% of the number is 5. 500
9. 2% of the number is 9. 450
10. 9% of the number is 9. 100

Change the following common fractions to percentages:

11. $\frac{5}{6}$ = 83 1/3 %

12. $\frac{3}{8}$ = 37 1/2 %

13. $\frac{4}{5}$ = 80%

14. $\frac{1}{3}$ = 33 1/3 %

15. $$\begin{array}{r} 4 \\ 12.22\overline{)48.88} \end{array}$$

16. $$\begin{array}{r} .4 \\ 1.222\overline{).4888} \end{array}$$

17. $$\begin{array}{r} 4 \\ 122.2\overline{)488.8} \end{array}$$

18. $$\begin{array}{r} .4 \\ 1222\overline{)488.8} \end{array}$$

19. In a herd of cattle, there were 14 black cows and 7 white cows. What percentage were white cows? 33 1/3 %

20. $\frac{3}{5}$ of a certain number of marbles were green, the remainder were white. If there were 120 marbles, how many marbles were white? 48%

SAMPLE TEST #2

Change these fractions to per cents:

1. 14 1/4 25%
2. 1/8 12 1/2 %
3. 1/6 16 2/3 %
4. 1/7 14 2/7 %
5. 1/12 8 1/3 %
6. 7/8 87 1/2 %
7. 9/10 90%
8. 5/6 83 1/3 %
9. 3/8 37 1/2 %
10. 5/8 62 1/2 %
11. 5/6 83 1/3 %
12. 9/16 56 1/4 %
13. 8/32 25%
14 1/50 2%
15. 3/50 6%
16. 75/80 93 3/4 %

17. Cedar River has a population of 30,000. Ten years ago it was 87½% as large. What was the population then? 26,250

18. In December Mr. Jones sold 60% more goods than he sold in November. What were his sales in December if his November sales were $75,000?

19. A master sergeant in the army is paid $165 per month plus this he gets 50% for overseas duty, and 20% for flying time. How much is his total pay? $280.50

20. A store is having a sale on coats. There is a 12½% off the regular price which is $27.80 before the discount. How much is the discount? $3.47 Price of coat? $24.33

21. We have 30 students in the class and 7 are absent. What % are absent? 28 1/3 %

22. In the class 14 are boys; what % are girls? 53 1/3 %

23. Find the interest on $250 for 1 month at .5% 10¢

24. Find the interest on $4 for 1½ years at 2% 12¢

25. Find the interest on $4,000 for 5 years at 3½% $700

26. In this class of 30 students 5 fail the course, what % fails? 16 2/3 %

SAMPLE TEST #2 (cont.)

27. In 1910 we paid 5¢ a quart for milk, and today we pay 375% of that amount. How much do we pay? 18 3/4 ¢

28. If a record player sells for $45.75, and you make a deal to buy it by paying 1/6 down. How much is your payment? $7.62

29. You have an allowance of $8 per week. You spend 35% for lunch, 5% for paper, 20% for clothes, 15% for entertainment, and the rest is saved. How much is spent for each item? Lunch $2.80
paper $ .40 clothes $1.60 entertainment $1.20 save $2.00

30. Ed made 16 out of 24 free throws. What % did he make? 66 2/3 %

SAMPLE TEST #3

Find the amount of discount on

1. A dress marked $25, but sold at a discount of 12½%. ______

2. A sofa marked $150 with a discount of 5% for cash. ______

3. A $40 radio with a 12½% discount. ______

4. Find the price paid after a 16 $\frac{2}{3}$ % discount on a $50 suit. discount______ price______

5. Find the price on a hand tractor marked $295 with a 15% discount. discount______ price______

6. Joe sold $450 worth of goods and received 2½% commission. How much commission______

7. John works at a store that pays him a commission of 3% plus his salary of $75. He sole $4,500 worth of goods. His commission was______ total salary______

8. Mary had a house that she turned to a real-estate agent to sell. The house sold for $9,560. The commission was 5%. How much commission did the agent receive? How much money did Mary receive? Commission______ Money Mary received______

9. You have a weekly allowance of $9. You spend 25% for lunch, 4% for paper etc., 41% for clothing, 15% for entertainment, 9% misc., 6% savings. How much is spent for lunch______ paper______ clothing______ entertainment______ misc.______ savings______?

Find the entire number if, n stands for number, the number

10. ½ of the n is 24 ______

11. $\frac{2}{3}$ of the n is 28 ______

12. $\frac{7}{8}$ of the n is 28 ______

13. $\frac{5}{6}$ of the n is 45 _____

14. $\frac{5}{16}$ of the n is 15 _____

15. $\frac{3}{10}$ of the n is 21 _____

16. 16 $\frac{2}{3}$% of the n is 18 _____

17. 83 $\frac{1}{3}$ % of the n is 45 _____

18. 62½% of the n is 25 _____

19. 37½% of the n is 57 _____

20. 7% of the money is $21 _____ amount of money in bank

21. 6% of the n is 28 _____

22. 17% of the n is 374 _____

23. 80% of the n is 28 _____

24. 19% of the n is 418 _____

25. 13% of the n is 39 _____

26. 33 $\frac{1}{3}$ % of the n is $26.50 _____

27. 62 $\frac{1}{2}$ % of the n $80 _____

28. 87½% of the n is $560 ________

29. 35% of the n is $105 ________

30. 9% of the n is $927 ________

VI. Investments	
A. Introduction	
"It takes money to make more of it."	People who have savings in excess of their emergency needs may wish to place these funds in investments
How to get most from investments with type of risk we want	Bring out-high risk, high return (or high loss) as an axiom of money investments.
B. Some common means of investing	
1. Banking and savings	Interest is added to the amount on deposit at stated intervals, usually semiannually or quarterly. As a rule, interest is figured on whole dollars only. (Example: If $25.89 is on deposit, the interest would be computed on $25.) Interest is figured on the new balance from the preceding period. When interest is added at regular intervals, it is said to be compounded. Banks use compound interest tables for figuring interest on accounts. Interest on savings accounts vary from 3%-6%. How to bank by mail. This can also be demonstrated and practiced with forms. Compound interest--differences caused by compounding daily, monthly, etc.
2. Bonds	If possible, have a broker come in to discuss investments in bonds, stocks, and mutual funds. An alternative is to get information from a broker--he has an abundance of material along these lines.

	Discuss terms such as: 1. Trustee--bank or corporation which represents the bondsholders as a group. 2. Par value--face value of the bond. Amount that borrower promises to pay. 3. Market value--price at which bond is currently being sold. 4. Premium--difference between market value and par value when market value is greater than par value. 5. Discount--difference between par value and market value when market value is less than par value.
	Interest on bonds is usually paid semiannually. Bonds usually have a low rate of interest, but they have the advantage of having high security value. Formula for bond interest = par value x rate x time
Savings bonds	The government's role in selling them to acquire a loan can be brought out. Percent return related to risk involved can be pointed out.
Municipal and corporate bonds	Percent return and risk involved determines quality of the bond.
3. Stocks	Use the Wall Street Journal and newspapers in the study of stocks.

6

Discuss these classes of stocks:

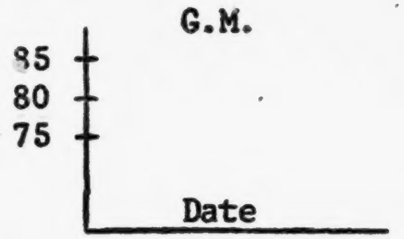
1. Common--ordinary stock of a corporation. Does not provide for the payment of any specified rate of dividend.

2. Preferred--does bear a specific rate of dividend such as 6% or $6.00 a share. There is no guarantee the dividend will be paid.

Discuss the role of the broker and the nominal fee which he charges.

Mutual funds--Share holders buy shares in a company. The company in turn invests this money in stocks and bonds and passes the dividend to the shareholder

Blue chip vs. speculative



Plot high, low, closing prices obtained from newspapers--can see trends of the stock over a period of time.

Bring in a stock certificate to show students what they are. Have students become acquainted with a few of business periodicals These are available through a broker or at a newsstand.

C. Credit Unions

Investment group for members only whereby members may borrow money at lower rates and easier terms than at a commercial institution.

Bring out how the amount of loans is usually limited. Also bring out that the rate of interest is different for the person who owns

	shares in the credit union and the person paying off a loan to the union.
D. Real Estate	Local realtors have pass-out materials available for student use.
1. Buying your own home	Point out that if a person buys a $15,000 house at 6% interest for 30 years that in 25 years $30,000 will have been paid out.
	How much down payment is usually asked for? Point out that the tightness of money determines much of the cost in this respect.
	How much should be spent for a home on how much salary? Some rules of thumb are available from realtors. A family with an income below $7,000/year should borrow 3 times that amount for a home. Above $15,000/year, this ratio is decreased.
Principle and interest	How much is actually being invested in principle? Work out examples.
Cost of upkeep	Is this an investment or is it taking more for upkeep than payment on principle?
2. Buying other property	Undeveloped property and income property
E. Reference:	1. Lankford, Ulrich, Clark "Essential Mathematics," World Book Co.
	2. Piper, Gardner, and Gruber "Applied General Mathematics," Southwestern Publishing Co. 1960.

VII. Insurance

A. Introduction - Probability as it relates to insurance

1. Risks - Define, compare by using actuary and mortality tables. Explain why risks that one individual has is higher than risks of another individual.

2. Profits - From standpoint of the company, show how insurance makes money. Explain the importance of statisticians and their use of probability. Show experiments in probability. Toss coins and let the class record heads and tails. Show how the law of large numbers works in getting closer to the expected probability. Another good experiment is the use of ten thumb tacks. In ten throws one can have 100 expected outcomes.

3. Rates - Explain how premiums are calculated. Again bring out the importance of an up-to-date mortality table. Enumerate the expenses incurred by the company.

B. Types of Insurance

1. Life - Bring in an insurance agent or expert to talk to the class. The Life Institute of America booklet is set out free in quantities on request. (Institute of Life Insurance, 488 Madison Avenue, New York 22, New York).

a. Point out two purposes of life insurance.
(1) Protection against financial loss
(2) As an investment

b. Name and explain the four basic types of life insurance
(1) Term - for a short time period
(2) Straight life - paid as long as insured lives
(3) Limited payment - insured pays premium for a fixed period but is insured for life.
(4) Endowment - Insured for a fixed period. At the end of the period the insured receives cash for the face amount of the policy.

c. List common terms in life insurance.
(1) Insurer - the company
(2) Insured - the person whose life is insured
(3) Policy - the contract between insured and insurer

(4) Beneficiary - person to whom the benefits are payable
(5) Premium - the amount paid regularly to the insurer.

2. Health and Accident

This is a type of insurance which protects the individual, family, or group against unforeseen expenses due to injury or sickness. However, like any other type of insurance it is only as good as the company issuing the policy. Medical insurance policies may vary greatly in their premiums and coverages. A person therefore should know his needs and shop around until he gets the type of policy he desires.

Accident and health insurance is designed to help pay for the losses and defray some of the incurred expenses which accidents and illnesses can bring. If some member of the family requires prolonged medical and/or surgical treatment, the expenses can easily wipe out the family's lifetime savings in a short period of time. If a family has no savings or insurance, it may be unable to secure the best medical treatment and may suffer many inconveniences.

3. Automobile Insurance

There are four major varieties of auto insurance coverage that an automobile owner might consider purchasing for his car.

a. Bodily Injury Liability - insurance protects the driver of an automobile against the cost of injuries that he might inflict on other people through the use of his automobile. Basic bodily injury coverage would be a "5-and-10" policy, which means the company would pay a maximum of $5,000 to any one person in case of injury by the insured and a maximum of $10,000 if more than one person is involved. If the damage exceeds these amounts the insured person is required to pay the difference.

Practical Considerations: Most automobile authorities agree that minimum coverage today should be 50-and-100 (thousand) or greater.

b. Property Damage Liability - insurance coverage for the insured against claims for the damage to other people's property, not including his own, while using an automobile.

c. Collision Insurance - the insurance company will pay the cost of any damage to the insured's care caused by any type of collision or upset. The policy always contains a deductible clause whereby the owner or insured must pay the first $50 or $100 of damage and the company will then pay the rest.

d. Comprehensive Insurance - includes the bulk of all other types of damage that can occur to one's car, including fire and theft. Some other things included in this type of insurance are damages caused by: lightning, transportation, windstorms, hail, earthquake, explosion, riot, falling parts, flood, malicious mischief, and vandalism.

e. Below is listed the costs for a sample insurance policy for a 1966 Olds and its coverage.

Coverage	Premiums
Bodily-Injury Liability 50-100 thousand:	$ 44.00
Property damage 25 thousand:	$ 20.00
Auto medical payments for passengers 2 thousand	$ 15.00
Comprehensive	$ 41.00
Collision, $100 Deductible	$ 42.00
Uninsured Motorist	$ 6.00
Total Yearly	$168.00

4. Fire Insurance

a. In issuing a fire insurance policy, most companies feel the risk they take in insuring property depends on two factors:

(1) The material that was used in the construction
(2) The degree of the fire protection immediately available

b. Insurance rates will vary according to the above two factors. All mortgage companies require adequate insurance on any dwellings they are financing. The face of a policy is the amount of insurance the owner carries. The indemnity is the amount of loss that the company will pay to the insured. This amount can never be more than the worth of the building or the face of the policy and in some policies the insurance company only agrees to pay 80% of the actual loss. Of course, the premiums are adjusted accordingly:

Sample:

Coverage	3-year Premium
Frame, stucco home, dwelling	
Face of policy $15,000	$135.00
Unscheduled personal property $6,000 . (contents)	
Additional living expense $3,000 . . .	

C. References

1. "24th Yearbook," NCTM
2. Johnson, Glen, "Probability and Chance," Webster
3. Johnson, Glen, "The World of Statistics"
4. Haag, Dudley, "Introduction to Statistics"
5. Wilcox, Yarnell, "Mathematics: A Modern Approach"
6. "Mathematics in Action," Institute of Life Insurance
7. "The Mathematics of Life Insurance," Institute of Life Insurance.

VIII. An Introduction to Algebra

A. What is Algebra?

In algebra letters are used to represent numbers. These are sometimes called variables or literal numbers. All the fundamental operations of numbers can be performed with letters. However, in order to add and subtract literal numbers, they must be of the same family or kind.

Example 1: $3a + 4a = 7a$

$a + 2b + 3a = 4a + 2b$

Example 2: $7 \times 7 = 7^2$ or 49

$b \times b = b^2$

Example 3: $\frac{4^3}{4^2} = 4^{3-2} = 4'$

$\frac{b^3}{b^2} = b^{3-2} = b'$

When two or more quantities are connected by an equal sign the terms are said to be equivalent to each other and the whole unit is called an equation. There are certain principles of equations that always hold true; namely that a quantity on one side of the equation can be added, subtracted, multiplied or divided (non-zero division) by any number as long as both sides of the equation are treated in the same way. This is the key to solving equations involving literal numbers. "Do unto one side of the equation as you would have done to the other."

Example:

$$\begin{array}{r} 3x + 4 = 16 \\ \underline{-4 \quad -4} \\ \frac{3x}{3} = \frac{12}{3} \\ x = 4 \end{array}$$

B. Formulas and Algebraic Expressions

Discuss formulas with which the student is familiar--mechanics, lever, electricity, etc. Formulas are equations using literal numbers. Example: $A = lw$. In order to get a numerical solution for a formula, all of the letter values have to be known except the one which is the one the formula will be solved for.

Writing algebraic equations from verbal problems is very important. It will help the students to become familiar with the form of an equation if they start with formulas with which they are acquainted and translate them into verbal statements.

Example 1: $V = lwh$ $L = 3$
$V = 3 \times 6 \times 2$ $W = 6$
$V = 36$ cu. units $H = 2$

Be sure that the student understands that any symbol or group of symbols representing a number is called an "algebraic expression." Algebraic expressions are composed of terms and factors.

Example of a term: a, $2x$, $5xy$, $\frac{3x}{y}$, 5

Each number or expression in a term is called a factor. In the term $5xy$ there are three factors; 5, x, and y.

Examples of an algebraic expressions: $3x + 5y$, $7x^2 + 5$.

These terms are combined by addition or subtraction to form the algebraic expression.

Example 2: Find the value of $2x + 3 = 7$
$x = 2$

Example 3: Find the value of $\frac{X}{2} - \frac{X}{5}$ if $X = 20$

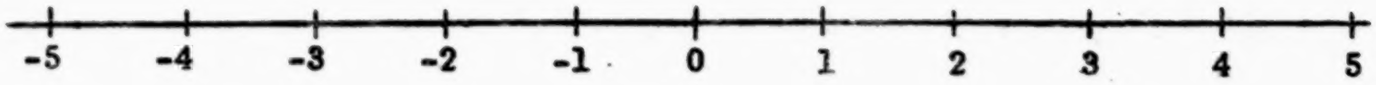
C. Positive and negative numbers

1. Reading a thermometer

The student probably knew of negative numbers for the first time when he saw his first weather thermometer. He heard news broadcasters say that the weather in Montana was "20°below zero."

2. Use of the number line.

It is easy to go from reading a thermometer to the use of the number line to show gain and loss, and to add numbers with unlike signs.



Of two numbers on a number line, the one on the right is greater.

3. Gain and loss on the number line.

A football team gains 5 yards on the first play and is thrown for a loss of 3 yards on the next play. This can be compared to moving 5 units to the right and then 3 units to the left. The students can suggest many other examples.

D. Adding positive and negative numbers.

1. On the number line
2. Rules for signed numbers

After the student has had oral practice adding on the number line by counting either to the right or left, he should see that an algebraic axiom becomes evident. "In addition, if the signs of numbers are the same, the sum of these numbers will have the same sign."

Example: $-5-4-3 = -12$; $+5 +4 +3 = -12$
Alternate way: $(-5) + (-4) + (-3) = -12$

E. Subtracting positive and negative numbers

A. On the number line

Oral practice in subtracting positive and negative numbers by the use of the number line is more difficult for students than is addition.

B. Rules for subtracting signed numbers

If you buy 65 cents worth of candy and give the clerk a dollar, she will count your change, saying 65, 75 (hands you a dime), one dollar (handing you a quarter). The clerk did a subtraction problem by adding. After discussing examples of this type, the student can then learn "To perform a subtraction, replace the subtrahend by its opposite, and add." Show on a number line.

F. Multiplying and dividing positive and negative numbers

A. A positive number times a positive number
B. A negative number times a positive number
C. A positive number times a negative number
D. A negative number times a negative number
E. Law of signs for multiplication
F. Law of signs for division

It is easy to explain that 3(-5) means (-5) + (-5) + (-5), but it is difficult to explain how to take 5 a (-3) number of times. A good example to explain multiplication of positive and negative numbers is a discussion of water flowing into a tank at the rate of 5 gallons per minute. Three minutes from now (+3) there will be 15 gallons more (15) in the tank.

$3(5) = 15$ (positive x positive = positive)

Three minutes ago (-3) there were 15 gallons less (-15) in the tank.

$-3(5) = -15$ (negative x positive = negative)

Suppose that water is flowing out of the tank at the rate of five gallons per minute (-5). Three minutes from now there will be 15 gallons less (+15) in the tank.

$3(-5) = -15$ (positive x negative = negative)

Three minutes ago (-3 there were 15 gallons more (+15) in the tank.

$-3(-5) = 15$ (negative x negative = positive)

For further discussion See University of Illinois Comm. of School Math Materials.

G. Equations - Examples to use

Introduce the students to axioms. An axiom is a statement accepted without proof.
Start with simple equations:

1. $x + 5 = 6$

$$\begin{array}{r} x + 5 = 6 \\ - 5 \quad -5 \\ \hline x + 0 = 1 \end{array} \qquad x = 1$$

2. $x - 5 = 6$

$$\begin{array}{r} x - 5 = 6 \\ +5 \quad +5 \\ \hline x + 0 = 11 \\ x = 11 \end{array}$$

Add 5 to both sides

3. $3x - 5 = 6 - 2x$

Subtract 5 + 2x on both sides

$$\begin{array}{r} 3x - 2x = 6 - 5 \\ x = 1 \end{array}$$

4. $\frac{x}{3} = 6$

Multiply each side by 3.

$$\begin{array}{r} 3(\frac{x}{3}) = (6)(3) \\ x = 18 \end{array}$$

H. Verbal Problems

1. Translating sentences into algebraic language

Students should work many problems of the type: (1) Find the number of degrees in each of two complementary angles if one is 17 times as large as the other.

x = one angle
$17x$ = second angle

$$\begin{array}{r} x + 17x = 90 \\ 18x = 90 \\ x = 5 \end{array}$$

one angle = 5
second angle = 85

(2) What number increased by 24 equals 75?

Let x = unknown number

$$\begin{array}{r} x + 24 = 75 \\ x = 75 - 24 \\ x = 51 \end{array}$$

(3) If I multiply my weight in pounds by 4 and add 60 pounds, I get 400 pounds. How much do I weigh?

x = weight in pounds

$$\begin{array}{r} 4x + 60 = 400 \\ 4x = 400 - 60 \\ 4x = 340 \\ x = 85 \end{array}$$

IX. Length and Angles - Use of Ruler and Protractor

A. Introduction on Length

1. Non-standard units of measure
    a. Measure width of desk
    b. Measure width of book

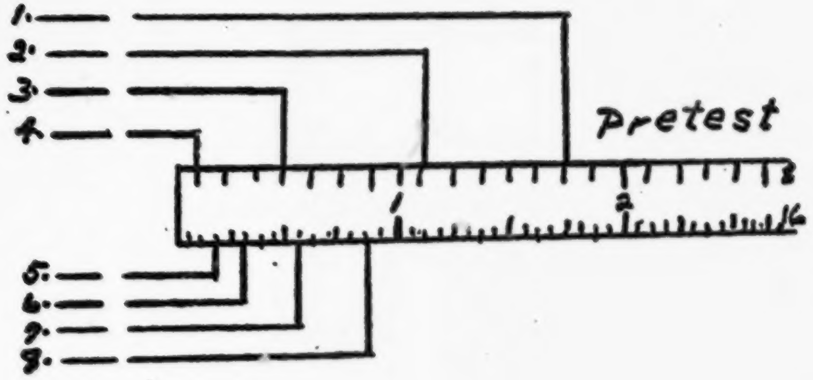
Have students measure by pencil lengths and record the range of the class. Use thumb widths for this. Find examples for students to measure with non-standard units pointing out the necessity for standardization.

2. Standard units of length--inch, foot, yard, etc.

B. Use of lengths

1. Estimations (Rulers needed)
    a. Draw estimates of 1", 1', 1 yd., 5", etc.
    b. Estimate lengths, widths of various objects - small and large.

2. Use of ruler
    a. Pretest by use of suggested activity or other means on how well students can use ruler.



b. Instruct in reading ruler (may not need this much depending upon outcome of pretest).

An overhead projector and a clear plastic ruler can be used for this purpose quite effectively. Ditto sheets may also be used for this purpose.

c. Make some of their own rulers with 1", ½", ¼" divisions on them as well as 1/16".

These can be made of cut up strips of manila folders. They will be used for rounded off readings to be made.

d. Measure various objects in room to nearest 1", ½", 1', etc.

You can use pieces of tape strips placed around the room. This should be a project running for more than one day--give more practice from day to day. The strips of tape can be numbered.

e. Bring in the idea of precision, tolerance, greatest possible error, and accuracy.

Precision can be related to the measurements that have been made. Precision depends upon the smallest subdivision of the measuring device--"it is precise to the nearest ½" " when ½" subdivisions were the smallest. Tolerance is the amount a measurement can vary from some standard to meet specifications.

Example: Measurement must be 3 3/8" ± 1/16" means it can be 3 5/16 - 3 7/16".
This can be related to shop specifications or pattern making.

Greatest possible error is one-half the smallest unit. The relative error is found by dividing the greatest possible error by the measurement itself. The smaller the relative error, the more accurate the measurement.

f. Graph from day to day the % accuracy attained on estimating a certain length.

g. Use tape measures.

Teachers can use tape measures for longer distances as well as for distances around. Students can measure distance with string and also by "stepping off distances." Compare the results of these three types of measurement.

h. Practice some addition and subtraction of lengths.

C. Introduction of Angle Measurement

1. Origins
2. Measurement of direction of rotation

   Angle measurements are used in construction, carpentry, surveying, navigation and fire spotting. From the use of the circle came the need to measure angles. This should be discussed.

D. Use of Angles

Introduce names such as vertex, sides (rays), acute, right, oblique.

1. Names of parts of angles and kinds of angles.
2. Uses of angle measurement.
   a. Need for standard units.
   b. Compass used in navigation.

      This is being discussed in the unit on latitude and longitude.

   c. Compass used in forest observation.

      See next sheet for a sample of a type of dittoed map that could be passed out to students--locating fires and some suggested related questions. Use of protractor is to be taught first. See reference 1, p. 196-199.

E. Use of protractor

1. Pretest by means of dittoed sheets, use of protractor in measurement of angles.
2. Instruct in reading of protractor (not much needed if successful in 1).

   An overhead projector and a plastic protractor can be used. Also a large board protractor may be used. Students have difficulty in understanding the placement of the vertex so a visual demonstration is generally needed.

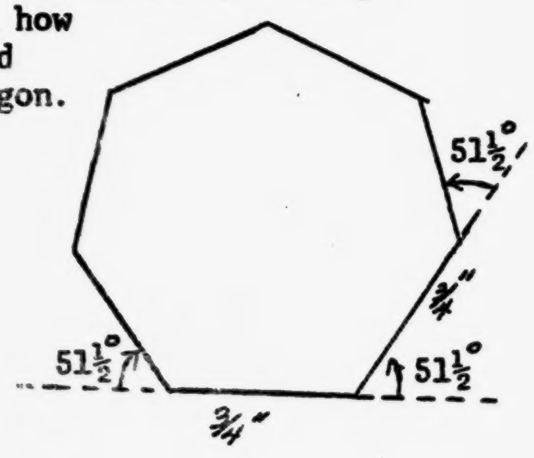
3. Measure various angles--give lots of practice.

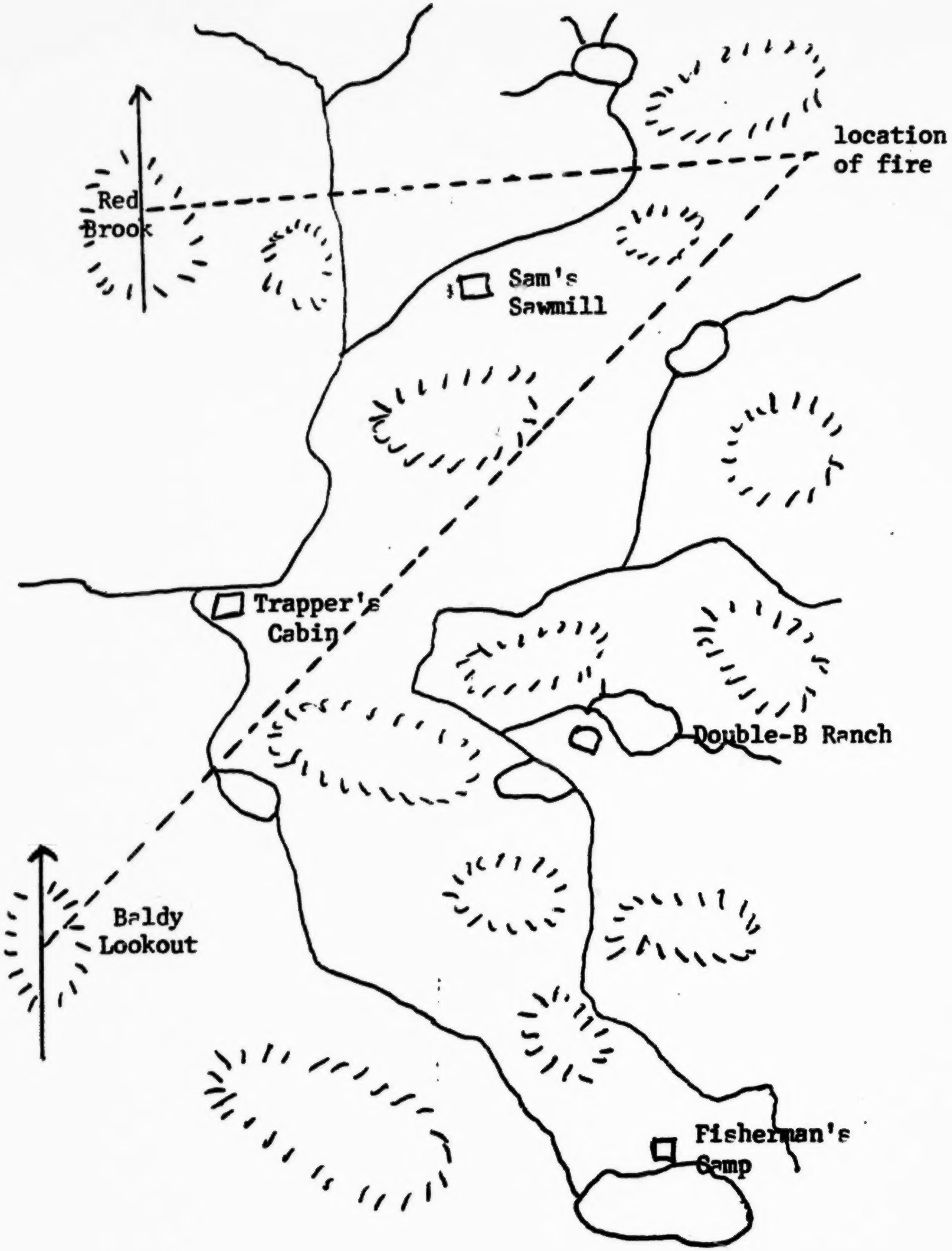
   This is where the map on the next page, or something similar to this, may be used.

4. Give practice in addition and subtraction of angles.
5. Make regular sized polygons. This uses both ruler and protractor.

Example: Construct a regular heptagon with each side 3/4" long. Students will need instruction in how to divide 360 into equal parts and how to use outside angles of polygon.

$$\frac{360}{7} = 51\frac{1}{2}^\circ$$





Naming each of the lake, stream and mountain areas can be done.

You can also incorporate some logic such as trying to analyze whether to send out equipment. (Certain situations sometimes might not warrant it.) For example, a Trapper's Cabin is often used by fishermen who build fire to cook dinner.

Questions such as these may be asked:

1. Since Sam's Sawmill is always burning scraps, what smoke reading should both lookouts ignore in reporting fires?

2. Locate a fire with the following readings: Red Brook 84 and Baldy Mountain 46 . This example is worked out on the map.

These questions can be answered by the use of a protractor, but you can also bring in a magnetic compass to show the students how it is divided into degrees and explain how to convert from direction readings to angle readings.

For the purposes of this project, all should agree on the same directions, such as the example is worked out beginning at North and reading clockwise, and that due East is 90 , etc.

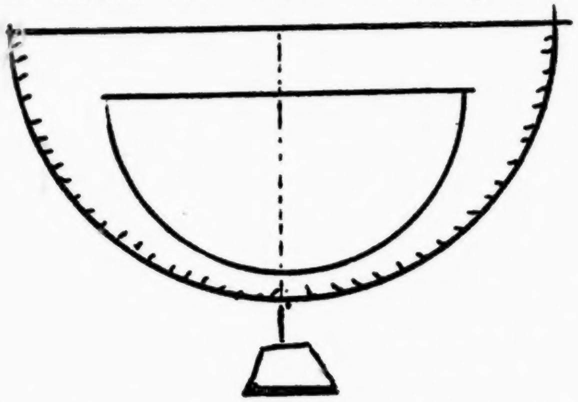
F. Angle measurement for indirect measurement

1. How to find angles of elevation and depression.

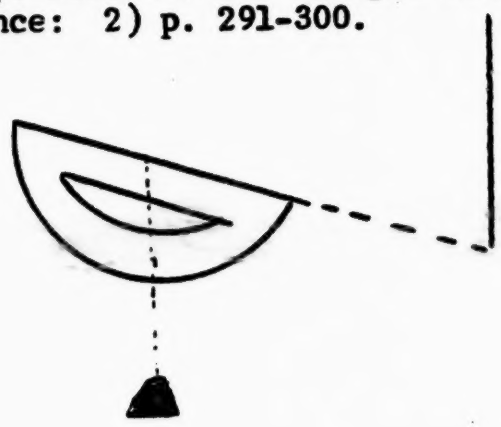
   Bring a surveyor's transit if possible. Alidades and hand levels are less expensive and can also be used.

2. Construct own transit from protractor.

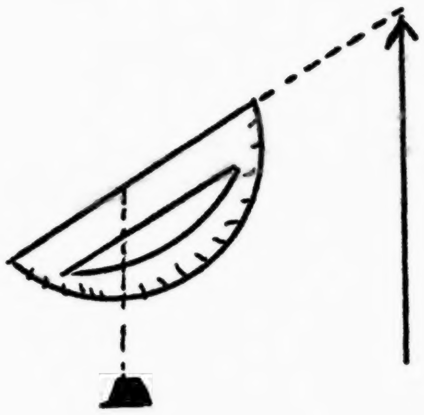
   Take an ordinary protractor and fasten a string to the vertex mark. (Drill with a fine point to make a small hole if there isn't one there.) Tie a weight to the end of the string.



3. Make indirect measurements using own transit. This will give use with both protractor and tape measure. An example would be the measureing of the neight of the room. Reference: 2) p. 291-300.

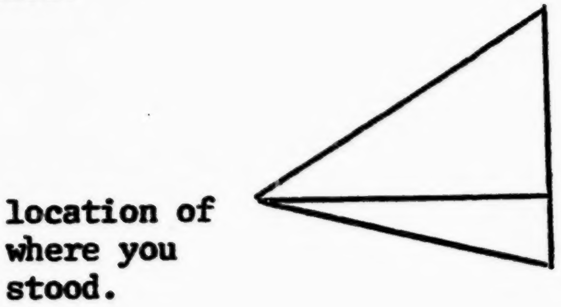


Angle of depression =
90°- 78°= 12°



Angle of elevation =
122°- 90°= 32°

4. Make scale drawing of the measurements made in the previous exercise. *See below for extension ideas.



Measure distance from where you were standing to the object of which you are measuring the height.

G. References:

1. Wiebe, "Foundations of Mathematics"
2. Dodes, "Mathematics; a Liberal Arts Approach"
3. Brown, "General Mathematics"

Ratio, Proportion, Similar Figures, Trigonometric Ratio

I. Ratio and Proportion.

A. Definition of ratio

The quotient of two numbers is called the ratio of the two quantities. When two is compared to 3 the ratio is $\frac{2}{3}$ or is sometimes written 2:3, using the colon instead of a bar.

Point out to students that ratio is simply a fraction and the properties of fractions may be applied when solving ratio problems.

B. Definition of proportion

A proportion is a statement that one ratio is equal to another. The equal ratios $\frac{2}{3}$ and $\frac{4}{6}$ may be written $\frac{2}{3} = \frac{4}{6}$ or using the colon, 2:3 = 4:6. The proportion is read "2 is to 3 as 4 is to 6." It is convenient to name the terms of the proportion. In $\frac{2}{3} = \frac{4}{6}$, 2 is the first, 3 is the second, 4 is the third, and 6 is the fourth term.
In addition, the 3 and the 4 are sometimes called the means and the 2 and the 6 are called the extremes. Notice that if the ratios are equal, the product of the means will always equal the product of the extremes. This fact may be used to check whether two ratios are equal or a true proportion exists. For example: $\frac{5}{7} = \frac{25}{36}$ is not a proportion because their cross products are not equal. ($5 \times 36 \neq 7 \times 25$) In the proportion $\frac{3}{4} = \frac{6}{8}$ the ratios are equal since the cross products are equal. ($3 \times 8 = 4 \times 6$)

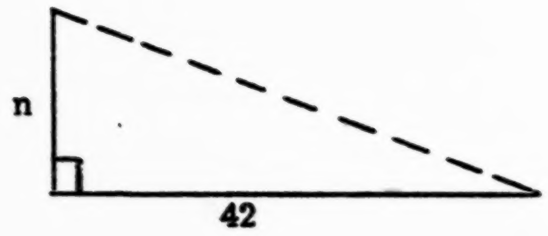
C. Find a fourth term in a proportion

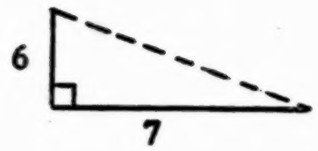
Using the idea of cross product, an unknown term of a proportion may be computed. In the proportion $\frac{5}{7} = \frac{25}{n}$, $5 \cdot n = 7 \cdot 25$. Evaluate the value of n by dividing both sides by 5.

II. Similar figures

A. Definition of similar objects

Similar figures or objects have the same shape but not necessarily the same size. Geometric figures which have the same shape and the same size are called congruent figures. The similarity of two triangles may be studied at this time. Notice that the angles of two similar triangles are equal and the corresponding sides have the same ratio. Given two similar right triangles of sides 3, 4, 5 and 9, 12, 15. The ratio of the corresponding sides is $\frac{3}{9} = \frac{4}{12} = \frac{5}{15} = \frac{1}{3}$ (See Example I). Students who understand proportion may solve problems such as the following: A flagpole casts a shadow 42 feet long. At the same time a 5-foot man casts a shadow of 7 feet. How high is the flagpole? A diagram is essential in solving these problems (See Example II).





Since the triangles are similar, the ratios of the corresponding sides must be equal, $\frac{7}{42} = \frac{6}{n}$ Solve the proportion for n to determine the height of the flagpole.

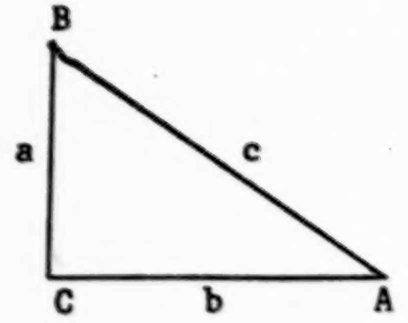
$$7n = 252$$
$$n = 36$$

III. Numerical Trigonometry

A. Trigonometric ratios, sine, cosine, tangent

Supplementary material for the more advanced student:

In trignometry, certain relationships which exist between the sides and angles of a right triangle are used to determine certain parts of a triangle when other parts are known. The ratio of the side opposite an acute angle to the hypotenuse is called the sine (abbreviated sin) of the angle. The ratio of the adjacent side of an acute angle to the hypotenuse is called the cosine (abbreviated cos) of the angle. The ratio of the side opposite an acute angle to the adjacent side is called the tangent (abbreviated tan) of the angle.



$\sin A = \frac{a}{c}$ $\sin B = \frac{b}{c}$

$\cos A = \frac{b}{c}$ $\cos B = \frac{a}{c}$

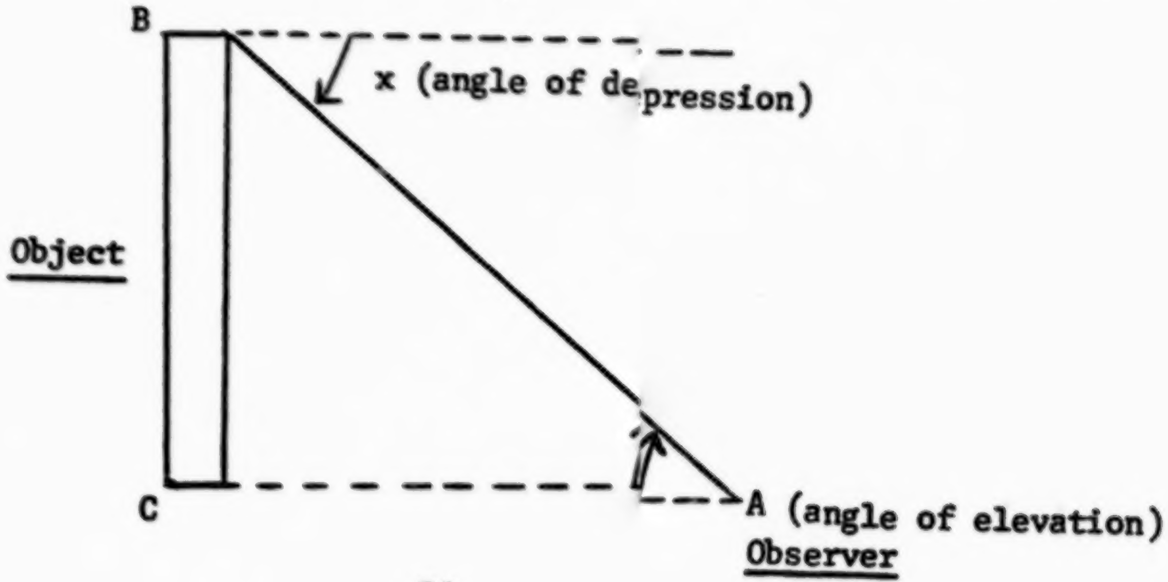
$\tan A = \frac{a}{b}$ $\tan B = \frac{b}{a}$

B. Using trigonometric ratios to solve problems

Mathematicians have evaluated the trigonometric ratios for all the angles of a triangle. The students must become familiar with the use of the Table of Trigonometric Values which may be found in many mathematics textbooks. Using trigonometry, the student may do the following problem: Given a right triangle with $A = 60^o$, $C = 4$, find the adjacent side b. Since the hypotenuse and A are known, the student may use the cosine ratio such that $\cos A = \frac{b}{c}$ and by substituting $\cos 60^o = \frac{b}{4}$ or $.5000 = \frac{b}{4}$ and $b = 2$.

1. Angle of elevation
2. Angle of depression

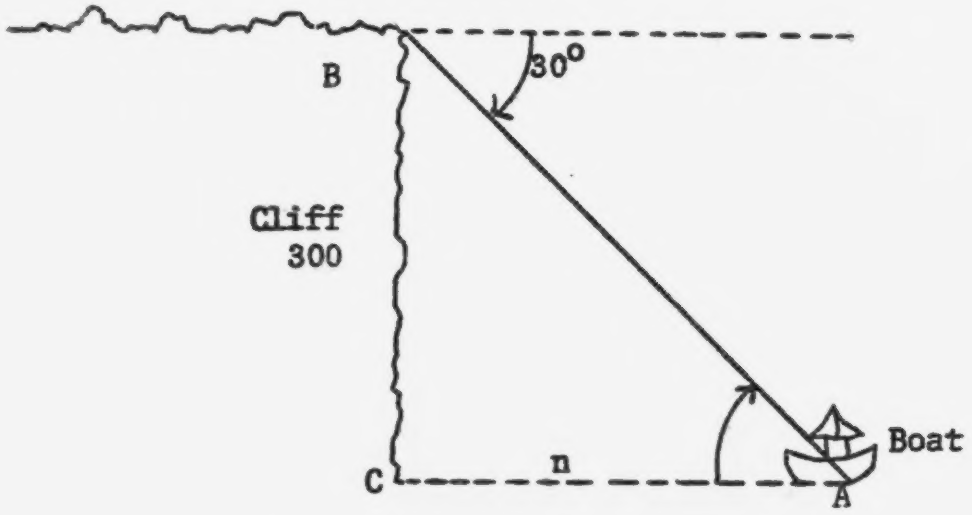
Another type of practical problem using trigonometry requires the use of the angle of elevation which is the angle of rise between the horizontal line and the observer's line of sight when the object is above the observer. When the object is below the observer, then the angle with respect to a horizontal line is called the angle of depression.



Angle A in the diagram is called the angle of elevation where the angle x is known as the angle of depression. Notice that the angle of depression equals the angle of elevation. To solve problems using trigonometric ratios, draw a diagram, locate the right triangle, and select the proper formula and solve the resulting equation.

Problem: From a cliff 300 feet above the sea, the angle of depression of a bost is 30^o. How far is the boat from the foot of the cliff:

Solution:



Since the angle of depression is equal to the angle of elevation, angle A would be equal to 30^o. Using the trigonometric ratio $\tan A = \frac{300}{n}$ or $\tan 30^o = \frac{300}{n}$ or $.5774 = \frac{300}{n}$.

Solve for n to find distance from the cliff to the boat.

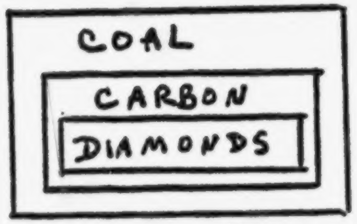
XI. Logical Thinking	
A. Inductive reasoning	
1. Definition of inductive reasoning	General truth or conclusion by investigating a number of particular cases. A scientist makes observations or as a result of many experiments reaches a conclusion. Specific to a general statement.
2. Sources of error	
a. Depend upon measurement	Neither method is accurate due to inaccurate measuring devices, human error, optical illusions. It is suggested that examples be shown to students to demonstrate visual misconceptions.
b. Conclusion reached before all possible cases are studied	An experiment is performed a certain number of times. One cannot prove that it will work on the next trial.
B. Deductive reasoning	
1. Definition of deductive reasoning	Reverse process of inductive reasoning. It proceeds from general to specific.
2. Steps in reasoning deductively	
a. A general statement	Conclusions reached by deductive reasoning are true only when the general statements upon which they are based are true. Example: Every student who intends to graduate from high school must take one year of mathematics.
b. A specific statement	Statement which satisfies all the conditions of the general statement. Example: John intends to graduate from high school.

c. A conclusion	Conclusion or truth deduced from the general statement. Example: Then John must take one year of mathematics. It is suggested that students be given more general statements and a specific statement and asked to deduce a conclusion.
3. If-then statements	
a. Hypothesis	A general statement can be expressed as a complex sentence which has one clause beginning with the word "if" and a second clause beginning with the "then." The hypothesis is the if-clause. In logic the hypothesis is called a premise.
b. Definition of conclusion	The then-clause of an if-then sentence. The conclusion may precede the hypothesis. Example: I live in California if I live in Anaheim. Many statements do not contain an if-clause. In such cases the complete subject is the hypothesis and the complete predicate is the conclusion. Example: An acute angle is an angle less than a right angle. Hypothesis: An acute angle. Conclusion: is an angle less than a right angle.
C. Syllogisms	1. Let us suppose someone wishes to convince you that diamonds are a form of coal. By referring to a chemistry book one can find that diamonds are crystallized carbon. Then one might make the statement that materials composed of carbon are a form of coal. Therefore one may conclude that diamonds are a form of coal. These statements might be arranged in another pattern which is a type of argument called a syllogism, used many years ago by the ancient Greeks. The pattern or syllogism would look something like this: a. Material made of carbon is a kind of coal (major premise).

b. Diamonds are crystallized carbon (minor premise).

c. Diamonds are a form of coal (conclusion).

A diagram of the above sygollism would look something like this:



2. If a syllogism has only one conclusion, made necessary by the premise, it is said to be valid; otherwise it is false. Therefore, a syllogism is true if the minor premise satisfies or follows the hypothesis of the major premise and the conclusion satisfies the inferred proposition of the major premise.

In other words, the major premise and the minor premise must "force" the conclusion. Diagraming a syllogism will often show whether it is valid or not.

3. Tell whether the following are valid or not:

a. All birds have wings
b. Sparrows are birds
c. Sparrows have wings

a. All birds have wings
b. Ducks have wings
c. Ducks are birds

a. All birds can fly
b. Ducks are birds
c. Ducks can fly

a. All trees have leaves
b. Flowers have leaves
c. Flowers are trees

a. John is generous
b. Kind people are generous
c. John is kind

a. If a lotion is used by a movie star, it is the best money can buy
b. The lotion I sell is the best money can buy
c. I sell the lotion used by movie stars

4. Types of syllogisms are used many times to prove a point or to settle a dispute. However, their effectiveness depends on one first assuming the major premise and minor premise to be true. Once the major and minor premises are assumed to be true, they no longer are assumptions but are accepted as facts and can be used as such. When forming statements for an argument, assumptions may be omitted when they seem to be obvious. However, this omission may be a cause of misunderstandings and disagreements. For example, the syllogism above assumes that the lotion referred to is the best one to buy and use for everyone.

a. A boat with a motor on it is a motor boat
b. John put a motor on his canoe
c. The canoe is a motor boat

a. Bill is healthy because he eats vegetables
b. We sell the best vegetables
c. Eat our vegetables and be healthy

a. Numbers that are squared give even numbers
b. One is a number
c. One squared gives an even number

Have students write their own syllogisms and be ready to debate or defend their points of view.

D. Converse statements

Converses are formed by interchanging the hypothesis and the conclusion. Example: Statement: If today is Thursday then tomorrow is Friday. Converse: If tomorrow is Friday, then today is Thursday.

Caution: Teach students that converses may or may not be true.

Example: Statement: If a tree is dead then it has no leaves.

Converse: If a tree has no leaves, then it is dead.

Comment: point out to students that in higher mathematics or in pure mathematics concepts are proven. Experimentation is discouraged since this approach gives results within some percent of error. In a better class students may be exposed to a simple geometric proof to illustrate that mathematics is an exact science and ideas are accepted only after they have been proven.

XII. An Introduction to Geometry

A. Undefined terms

1. Point, line, plane

Most geometry textbooks will be excellent references to use in teaching points, lines and planes.

B. Recognizing lines

1. Parallel lines
2. Perpendicular lines
3. Intersecting lines

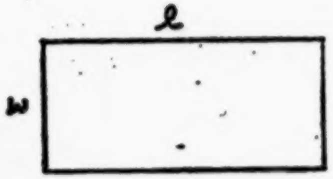
The student should understand thoroughly that one- and two-dimensional figures cannot be picked up and moved. A line has no depth or thickness. Models of the figure may be moved but these models are 3-dimensional.

The physical act of drawing each of these figures as they are discussed will fix the ideas in the minds of the students. Adding just one more dimension, depth (or height) to the two dimensional figures brings the student into geometric solids.

C. Recognizing figures

1. Quadrilaterals

   a. Square, rectangle, parallelogram, trapezoid, rhombus



The area of a rectangle is equal to the length times the width.

w x l = area

2. Triangles

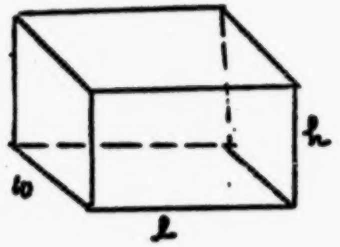
   a. Scalene, isosceles, equilateral, right

All have same formula

$a = \frac{1}{2} b \ H$

3. Other polygons

   a. Pentagon, hexagon, octagon



To find the volume of a rectangular solid find the area of the rectangular base and then multiply by the height.

w x l x h = volume

4. Circles

   a. Radius, diameter, arc, chord, tangent

5. Geometric solids

   a. Prisms, cylinders, cones, pyramids, spheres

D. Constructions

1. Bisection

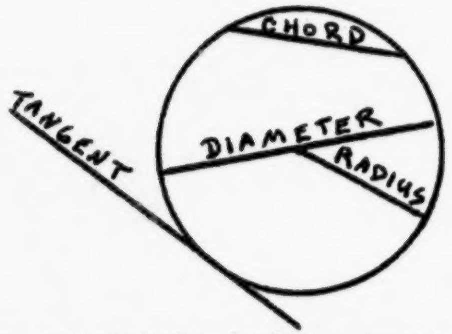
   a. Lines and angles

2. Perpendiculars

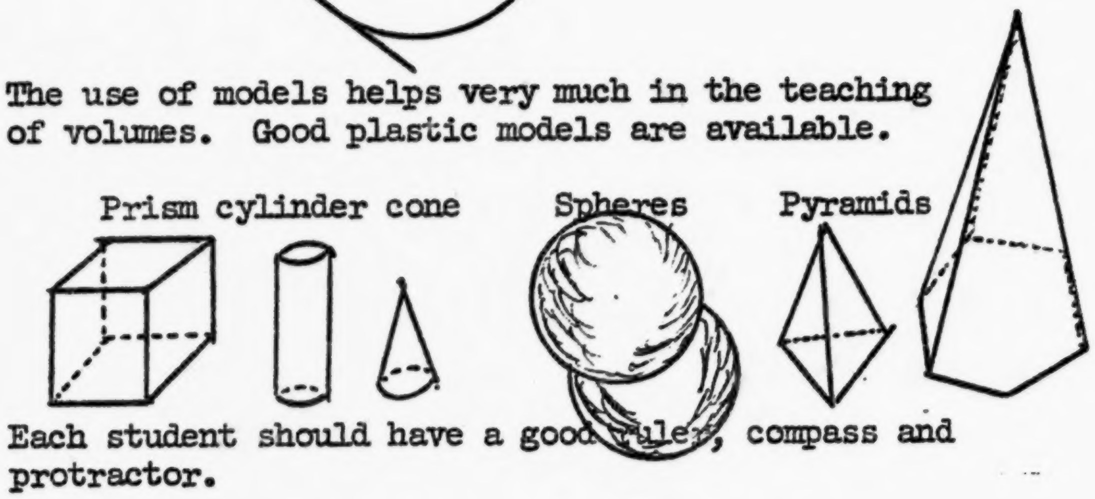
   a. To a line at a given point, from an outside point
   b. Perpendicular bisector

3. Circles

   a. Given the center and radius
   b. How to locate the center of a given circle



The use of models helps very much in the teaching of volumes. Good plastic models are available.



Each student should have a good ruler, compass and protractor.

Bisection: To divide into two equal sections.

It should be explained that constructions are made using only a compass and straightedge, that angles are measured by the use of a protractor, and lines by the ruler.

Perpendicular: A line intersects the line so that the intersections formed are at a 90 °angle.

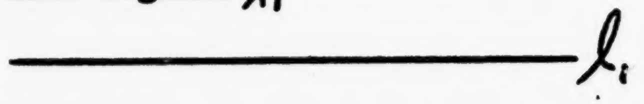
To locate the center of a given circle, draw in any two chords in the circle and construct the perpendicular bisectors of the two chords. These bisectors will intersect at the center of the circle.

4. Parallel lines

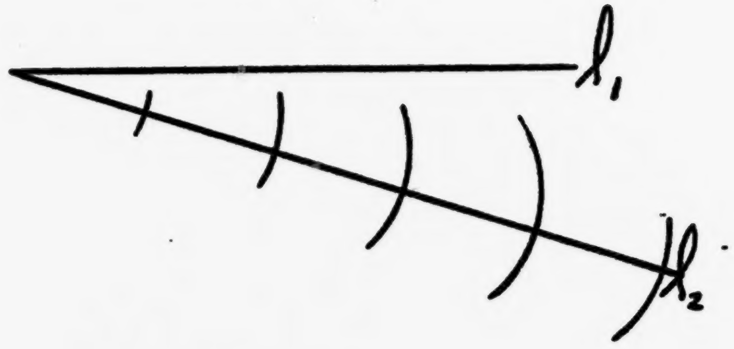
5. Dividing a line into a given number of equal lengths

To divide a line segment into five equal parts:

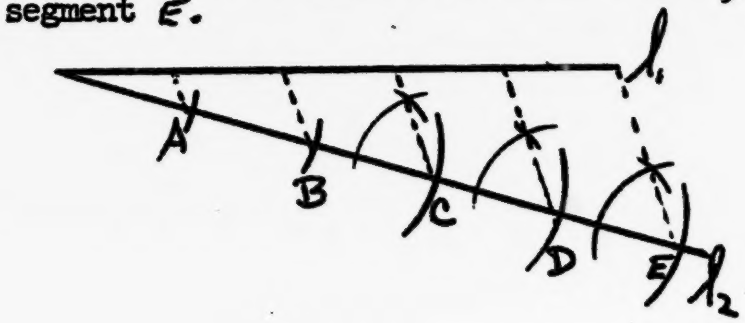
Give line segment ℓ_1



Draw a second line at any angle from either end of line segment ℓ_1



With a compass, beginning at the vertex of the angle, mark off five equal segments and connect the end of ℓ_1 to the fifth segment E.



At D measure the angle, and repeat the measure of this angle at C, B, A.

E. Symmetry 1. Recognizing symmetrical figures 2. Understanding axis of symmetry	Have the students bring in examples of symmetry. Pictures of leaves, butterflies, triangles, balls, etc. Explain point symmetry and line symmetry. If a line can be drawn through a plane figure cutting it so that the two sections are identical, line symmetry is present. The line is called the axis of symmetry (a pea pod and a butterfly have line symmetry). If a point can be found such that all lines drawn through the point and terminating on the figure are bisected by the point, the figure has point symmetry (a spider web and a daisy have point symmetry). Examples from art and architecture can be displayed on a bulletin board
F. Congruent figures 1. Definition of congruency 2. What equal parts make figures congruent	Congruent figures are alike in size and shape and can be made to coincide. (Construct triangles) using 2 sides and the included angle (S.A.S. = S.A.S.), two angles and the included side (A.S.A. = A.S.A.), and three sides (S.S.S. = S.S.S.) Students should cut out triangles, quadrilaterals, etc., to show that one figure can be placed on another to coincide.
G. Scale drawings 1. Defined 2. Reading scale drawings 3. Making scale drawings	A mechanical drawing book as reference will give students practice in reading scale drawings. Have students make scale drawings of a line, a triangle, a quadrilateral, etc.

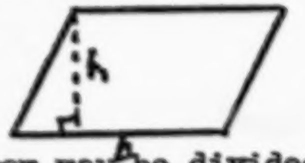
Unit XIII. Areas and Volumes of Geometric Figures

A. Definitions	The practical aspect of this topic makes it very interesting, especially when students consider problems such as those faced by the painter who wishes to determine how much paint to buy to paint a certain room, or the engineer who wishes to compute the amount of liquid contained in a cylindrically shaped tank.
1. Polygon a. Triangle b. Quadrilateral c. Pentagon	
2. Various types of quadrilaterals	It is suggested that students master the computation of the area of two-dimensional objects before proceeding on to three-dimensional geometric solids.
a. Square	A rhombus with 4 right angles
b. Rectangle	A parallelogram with four right angles
c. Parallelogram	A quadrilateral having both pairs of opposite sides parallel
d. Trapezoid	A quadrilateral with one, and only one, pair of parallel sides
e. Rhombus	A parallelogram with 4 sides of equal length
B. Deriving formulas and calculating areas of polygons	
1. Area of a rectangle	
a. Definition of base altitude	Any side may be used as a base. The altitude is perpendicular to the base.
b. Developing formula for area of square as unique type of rectangle	$A = LW$

2. Area of parallelogram
    a. Derive formula in terms of rectangles
    b. Area of rhombus as a parallelogram

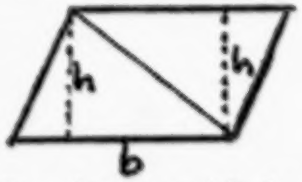
Define the altitude not as a side of the parallelogram, but as a line drawn from a vertex perpendicular to a base.

A parallelogram may be divided into a rectangle and two congruent triangles. Area of rectangle equals the area of a parallelogram of the same base and altitude.



3. Area of a triangle
    a. Triangle as half of a parallelogram
    b. Deriving formula in terms of a parallelogram

A given parallelogram may be divided into two congruent triangles. Since the area of a parallelogram = bh, then area of triangle = $\frac{bh}{2}$

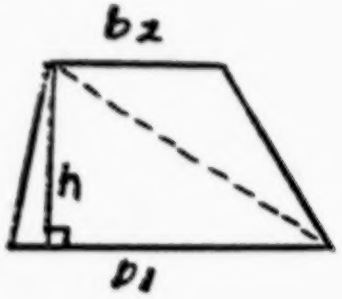


4. Area of a trapezoid
    a. Definition of altitude
    b. Derivation of area formula in terms of triangles

An altitude of a trapezoid is a line drawn from a vertex perpendicular to a base. A trapezoid may be divided into two triangles. The area of the trapezoid may then be determined by adding the areas of the two triangles.

$A = \frac{1}{2}b_1h + \frac{1}{2}b_2h$ $\qquad$ $A = \frac{1}{2}b_1h$

$A = \frac{1}{2}h\ (b_1 + b_2)$ $\qquad$ $A = \frac{1}{2}b_2h$



C. Circles and spheres	
1. Definition of terms	
a. Definition of circle, diameter, radius, circumference, area of circle	Stress that pi (π) is a symbol and 3.14 and 22/7 are approximations. π is the most accurate pi. Circle - A closed curve in the plane, every point of which is the same distance from a fixed point within. Diameter - A line segment joining two points on the circle and passing through the center of the circle. Radius - A line segment extending from the center of the circle to any point on the circle. Circumference - The distance around the circle, $c = 2\pi r$
2. Circumference of a circle from definition of $\pi = C/D$	
3. Area of a circle	The formula for the area of a circle is $A = \pi r^2$
D. Areas and volumes of geometric solids	
1. Area of sphere	Area of sphere = $4\pi r^2$
2. Volume of sphere	Volume of sphere = $\frac{4}{3}\pi r^3$ Students generally have difficulty remembering the r^2 in area formulas and the r^3 in volume formulas. A suggestion may be to think of square units in computing area and cubic units in computing volume.
3. Area and volume of prisms and cylinders	Use models to show shapes of geometric solids. Define bases of prisms as congruent polygons.

a. Definitions	
1. cylinder	Imagine a round shaped object like a tin can. (this is not a mathematical definition, but helps students to visualize)
2. altitude of prism	A length of the perpendicular to the bases.
3. altitude of cylinder	A length of the perpendicular to the bases.
4. faces	The sides of a prism.
5. prism	A solid whose bases are parallel congruent polygons and sides of parallelograms
b. Definitions of lateral area, total area, volume	Develop idea of lateral area as sum of all lateral faces, and total area as lateral area added to the sum of the 2 bases. Inexpensive models of unit cubes are available. Develop the idea of the volume of a prism and cylinder as Volume = area of the base x altitude.
4. Area and volume of pyramids	A pyramid is a solid whose base is outlined by a polygon, i.e., quadrilateral, pentagon, etc. and whose sides are outlined by triangles.
a. Definitions of pyramids, cone, altitude of pyramid, altitude of cone, faces	Stress the differences between slant heigh and altitude. Discuss the similarity between a prism and a cone. A cone has a circular base where a prism may have any polygon as a base. Define faces of a pyramid as congruent triangles. Base of pyramid as a polygon.
	Develop idea of lateral area as the sum of the areas of the faces. By physical or visual demonstration, compare the volume or content of a cone and cylinder of the same base and the same height. Discover and stress the fact that the volume of a cone is exactly one-third of the volume of a cylinder of same height and same base. Develop formula for cone volume as equal to one-third the area of base times height. Repeat demonstration for a pyramid and prism having

the same base and same height. It is highly recommended that the classroom be equipped with geometric solids and a plastic pyramid showing the difference between altitude of the pyramid and altitude of the face.

XIV. Number Processes

A. Prime and composite numbers

Whole numbers (other than 0 or 1) may be separated into prime and composite numbers.

1. Prime numbers

A prime number is a whole number which is divisible only by itself and by one.

a. Sieve of Eratosthenes

A composite number can be expressed as a product of two or more smaller whole numbers.

An example of the Sieve of Eratosthenes may be used to find prime numbers less than 30.

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ~~21~~
~~22~~ 23 ~~24~~ 25 ~~26~~ ~~27~~ ~~28~~ 29

1. 2 is a prime number. Retain the 2 and cross out every second numeral after 2. (These numerals are multiples of 2)
2. 3 is a prime number. Retain the 3 and cross out every third numeral after 3.
3. 4 is already crossed out.
4. 5 is a prime number. Retain 5 and cross out every fifth numeral after 5.
5. Continue the process until all except prime numbers are crossed out.

2. Composite numbers

Composite numbers may be expressed as products of two or more smaller numbers.

a. As a product of two or more numbers
b. Completely factored (prime factors)

6 x 3 for 18, or 9 x 2 for 18, or 18 x 1 for 18, etc.

1. factors must all be prime numbers	To be completely factored, the numbers should be prime factors: $3 \times 3 \times 2 = 18$
B. Factors and Factoring 1. Factor - definition a. Common factors b. Greatest common factor	When 45 is written as 5(9), it is represented in factored form. The numbers 5 and 9 are factors of 45. Factors tell the student that 45 is divisible by both 5 and 9, and that either number divides into 45 an even number of times. There is no remainder when a number is factored. Ask the question, "What are the factors of 18 and 24?" Factors of 18 are 1, 2, 3, 6, 9, 18 Factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24 The common factors of 18 and 24 are 1, 2, 3, 6 The greatest common factor of 18 and 24 is 6. That is, 6 is the largest integer which can be divided into 18 and 24 exactly.
c. Least common multiple	By expressing two or more integers in prime-factor form it is possible to determine their least common multiple. (Lowest common denominator). For example, an integral multiple that has 18 and 24 as factors must have the prime factors that make up 18 and 24 among its own factors. The prime factors of 18 are $2, 3^2$. The prime factors of 24 are $2^3, 3$. Therefore, the factors of the least common multiple must include 3^2 and 2^3. The product of these factors ($3^2 \cdot 2^3 = 72$) is the least common multiple.

C. Expanded notation	
1. Exponents	The student should understand thoroughly that an exponent tells how many times to take the base as a factor (5^2): 5 is the base and the raised 2 (superscript) is the exponent. $5^2 \rightarrow 5 \times 5$. Five is taken twice as a factor. 5^2 can also be read as "5 to the second power."
a. Superscript (2^5)	
b. Base	
c. Power	
2. Writing numerals as an indicated sum or polynomial	Understanding numbers is more thorough if students have experience in writing them as an indicated sum or polynomial. 5,784 written as 5,000 + 700 + 80 + 4, followed with $5(10^3) + 7(10^2) + 8(10) + 4(1)$
D. Inequalities	To solve an inequality is to find the set of all numbers which make the inequality true:
1. Comparing numbers by subtraction	$n + 2 < 6$ $y - 2 > 3$ $3x > 9$ $n + 2 - 2 < 6 - 2$ $y - 2 + 2 > 3 + 2$ $\frac{3x}{3} > \frac{9}{3}$ $n < 4$ $y > 5$ $x > 3$
a. = to, greater than, less than	
2. Symbols defined	$\neq$ is not equal to; $>$, is greater than; $<$, is less than; $\not>$, is not greater than; $\not<$, is not less than; $\geq$ is greater than or equal to; $\leq$ is less than or equal to. If ='s are subtracted from $\neq$'s the remainders are $\neq$ in the same order. When both members of an inequality are either multiplied or divided by the same negative number, an inequality of reverse order results since for the product of two factors to be > 0, both factors must be positive or both must be negative.

$-5x > 25$ $\qquad$ $-3y < -9$

$\frac{-5x}{-5} < \frac{25}{-5}$ $\qquad$ $\frac{-3y}{-3} > \frac{-9}{-3}$

$x < -5$ $\qquad$ $y > 3$

$x = \ldots.-9, -8, -7, -6$ $\qquad$ $y = 4, 5, 6\ldots.$

Students should recognize, understand, and be able to use the symbols.

E. Squares and square roots

1. Squaring a number

   a. Multiplying a number by itself

   b. Reading a table of squares in math tables

It is important that students be instructed in reading the square and square root tables found in mathematic books.

2. Square root

   a. Short method factoring

   b. Long method

   c. Reading a table of square roots

The factoring method used in square root is satisfactory for numbers containing factors 2, 3, 5, and a perfect square. Students can memorize the $\sqrt{2}$, (1.414) the $\sqrt{3}$, (1.732) the $\sqrt{5}$, (2.236)

$\sqrt{48} \rightarrow \sqrt{16.3} \rightarrow 4\sqrt{3} \rightarrow 4(1.732) = 6.928$

$\sqrt{75} \rightarrow \sqrt{25.3} \rightarrow 5\sqrt{3} \rightarrow 5(1.732) = 8.660$

(See Geometry teacher for help)

3. The Rule of Pythagoras

   a. Finding distance by indirect means

The Rule of Pythagoras expresses the relationship of the sides of a right triangle. The student should draw a right triangle, label it, and build the squares on the three sides, learning the algebraic

b. Finding distance by actual measurement

form of the Rule of Pythagoras.

$$a^2 + b^2 = c^2$$

$$a = \sqrt{c^2 - b^2}$$

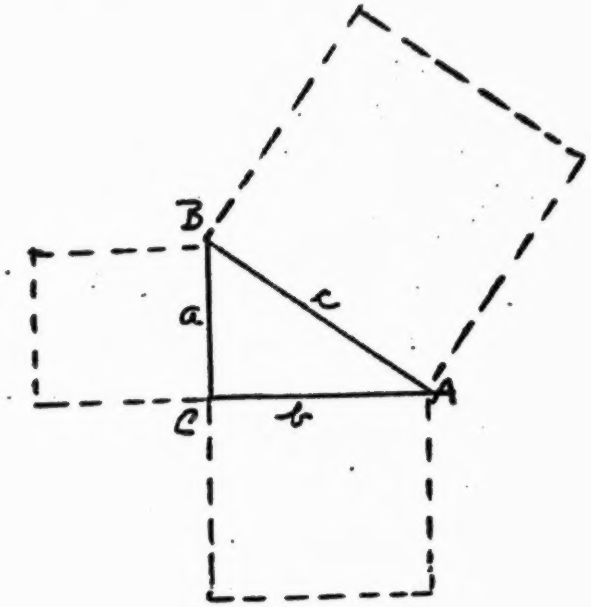
$$b = \sqrt{c^2 - a^2}$$

$$c = \sqrt{a^2 + b^2}$$

Caution: $\sqrt{a^2 + b^2} \neq a + b$

Example: $\sqrt{3^2 + 4^2} \neq 3 + 4$

but $\sqrt{9 + 16} = \sqrt{25} = 5$



F. Sets

1. Defined

Discuss sets with which students are familiar. Sets of dishes, sets of books, set in tennis, etc. Develop from there into sets as a well defined collection of objects.

2. Use of braces to indicate sets

The symbol $\{\ \}$ is used to designate a set, with the objects (called members or elements) listed inside the braces.

3. Subsets

A set of one digit even numbers is expressed $\{2, 4, 6, 8\}$. This is a finite set (you can list all the elements). A set of all even numbers is expressed as $\{2, 4, 6, 8, 10, 12...\}$ The three dots at the end mean "and so on indefinitely." This is an infinite set.

The Greek letter epsilon (distorted) is used to indicate "is a member of the set."

$$T = \{\text{all odd numbers}\} \; ; \; 13 \in T$$

The symbols $\subseteq$, $\supseteq$, are used to indicate that one set is a subset of another. $\subset$, $\supset$ are used with proper subsets (a set that is not the whole set). $A \subset B$, read: "A is a proper subset of B."

$$\{1, 3, 5\} \subset \{1, 2, 3, 4, 5, 6\} \text{ read } \{1, 3, 5\}$$

is a proper subset of $\{1, 2, 3, 4, 5, 6\}$.

4. Commutative and associative properties
    a. Addition
    b. Multiplication

The commutative law of addition (CLA) states that order used in adding two numbers does not affect the sum: $5 + 6 = 6 + 5$. The commutative law of multiplication (CLM) states that order used in multiplication does not affect the product:
$(5)(6) = (6)(5)$
The associative law of addition (ALA) states that in adding 3 numbers, any two of the three may be grouped together and the third number may be added to their sum.

$$(5 + 6) + 4 = 5 + (6 + 4)$$

Have the students practice all possible ways of adding. Only two numbers can be added at any one time, and the grouping $(5 + 4) + 6 = (6 + 4) + 5$ does not affect the sum.

The associative law of multiplication (ALM) states that we may group any two of three numbers together and multiply their product by the third number.

$$4(5 \times 6) = 6(4 \times 5)$$

5. Distributive property of multiplication

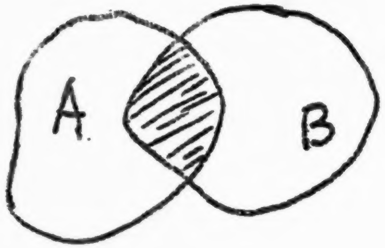
The distributive law of multiplication (DLM) states that when we multiply one number by the sum of two other numbers we get the same product as when we add the product of the first and second numbers to the product of the first and second numbers to the product of the first and third numbers.

$$5(3 + 2) = 5(3) + 5(2)$$

6. Intersection and union

The operation of intersection, indicated by the symbol $\cap$, is used with two sets to find the set composed of the common elements that belong to both sets.

Venn diagrams provide one way of expressing the idea of intersection: Use two streets intersecting as another example.

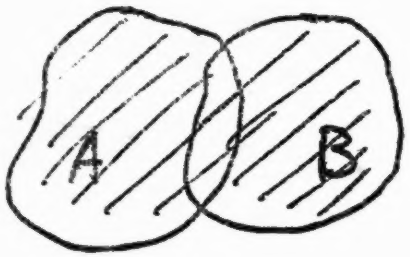


A = all students taking math.

B = all students taking Spanish.

$A \cap B$ = all students taking both math and Spanish.

The operation of union, indicated by $\cup$, is used with two sets of $\overline{\text{A and}}$ B to find those elements which belong to A or B or to A and B.



A = all students taking math.

B = all students taking Spanish.

$A \cup B$ = all students taking math or Spanish.

XV. Longitude and Latitude

A. Great Circles

1. Meridians

   a. Prime or 0° meridian of longitude

   b. Meridians East and West to 180°

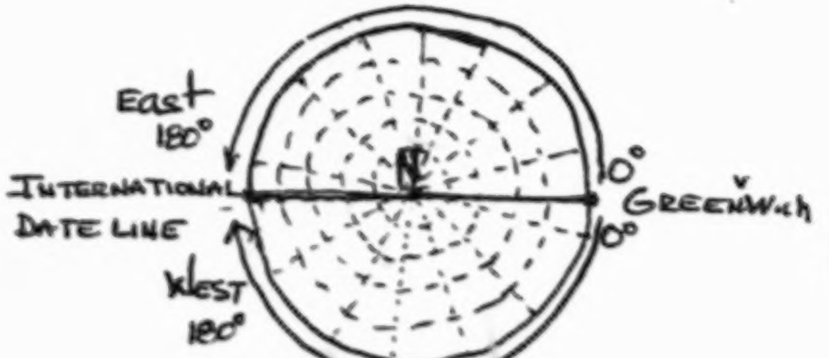


Diagram A (Top view)
Concentric Circles = Latitude
Vertical Circles = Longitude

The position of any point on the surface of the earth is determined by the intersection of the meridian of longitude and its parallel of latitude.

Meridians of longitude are circles (actually half circles)passing through the North Pole and the South Pole. These are called "great circles" of the earth, since they are formed by the intersection of a plane through the earth, passing through the earth's center. Imagine that you are slicing the earth in half, cutting through the center. The intersection of this plane, or cut, with the earth's surface, is a "great circle," and is called a meridian of longitude since it passes through the North Pole and the South Pole. The reason that meridians are actually half circles is that the two halves would have different designations on opposite sides of the earth. One would be a certain number of degrees West, the other would be a different number East.

In Diagram A only the half circle meridians are shown, -- the zero meridian through Greenwich, England (point G), and the meridian of 30° W. Point C is the center of the earth. Note that angle $BCA = 30^\circ$, also arc BA (written $\widehat{BA}$) $= 30^\circ$. If a plane passes through the center of the earth, C, at right angles to the axis of the earth (an imaginary line through the north and south poles), it intersects the surface of the earth to form the equator.

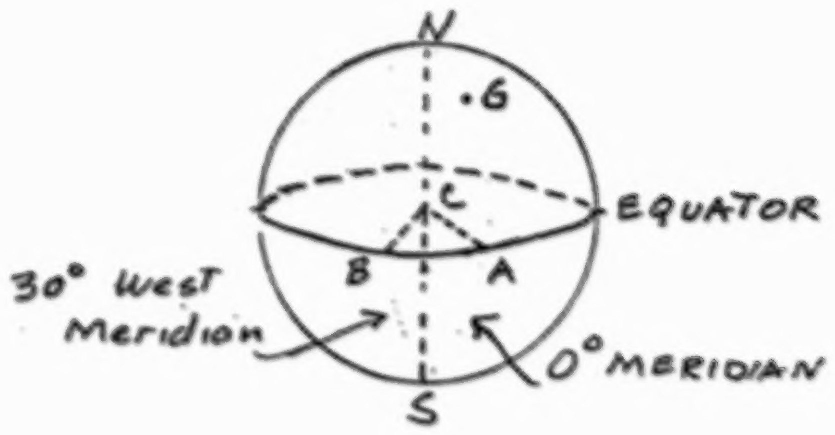


Diagram B (Side View)

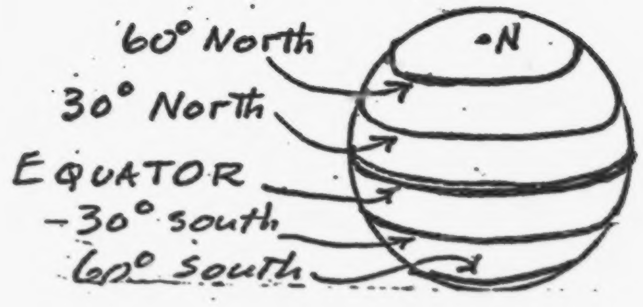
The prime meridian from which longitude is calculated is the meridian that passes through Greenwich near London, England. West longitude extends from this prime meridian (0° longitude) westward halfway around the earth to the international date line (180° longitude). East longitude extends eastward from the prime meridian to the international date line.

2. The equator

The equator is 0° latitude. North latitude is measured north of the equator and south latitude is measured south of the equator. The north pole is 90° north latitude and the south pole is 90° south latitude.

B. Small circles, -- parallels of latitude, north and south

If the plane of cutting does not pass through the earth's center, the intersection is called a "small circle." The parallels of latitude are "small circles," formed by planes cut parallel to the plane of the equator.



C. Latitude and nautical mile: one minute of arc of latitude

Since one minute of arc of latitude equals one nautical mile, to find the distance between two points on the earth's surface located on the same meridian, find the difference in latitudes, convert to minutes of arc, and then to nautical miles.

D. Conversion units

1 statue mi = 5280 ft.
1 naut. mi = 6080.2 ft.
1 statue mi = .8684 naut. mi
1 naut. mi = 1.1515 stat. mi.

Problem: How many nautical miles apart are 2 ships if one is located 35° N, 15° W, and the other is at 26° N, 15° W?

9 degrees = (9 x 60') = 540'
540 minutes = 540 nautical miles

$$\begin{array}{r} 35^\circ \\ \underline{26^\circ} \\ = 9^\circ \end{array}$$

difference in lat.

Answer: 540 nautical miles

Note: If it is desired to convert nautical miles to statue miles, multiply nautical miles by 1.1515 or 1.15.

E. Longitude and time

15° of longitude = 1 hr. time
15' of longitude = 1 min. time
15" of longitude = 1 sec. time

In navigation the longitude is sometimes expressed in units of time. It follows from the table given at the left that one degree of longitude = 4 minutes of time, and one minute of longitude = 4 seconds of time.

1. Changing units of longitude to units of time (divided by 15)

Both world time and longitude are calculated from the meridian of Greenwich, so that the difference between the local sun time and Greenwich time can be used to obtain the longitude of any point on the earth's surface. Ex. Change 47° 14' of arc of longitude to units of time.

$$\begin{array}{r} 3\text{ hrs.} \\ 15\,\overline{)\,47^\circ} \\ \underline{45} \\ 2^\circ + 14' = 134' \end{array} \qquad \frac{134'}{15} = 8.93 \text{ or } 9 \text{ mins.}$$

Answer: 3 hours 9 minutes

2. Changing units of time to units of longitude (multiply by 15)

Note that hours change to degrees, but minutes to minutes and seconds to seconds.

Ex.: Change 2 hours 35 minutes 25 seconds to units measuring longitude.

$15\ (2\text{ hr.} + 35' + 25") =$

$30\ \underline{\text{degrees}} + 525' + 375" =$

$30^\circ + (480' + 45') + (360" + 15") =$

$30^\circ + (8^\circ + 45') + (6' + 15") =$

$38^\circ + 51' + 15"$

Answer: 38° 51' 15"

3. Find the time at point B when the longitude and time at point A and the longitude at point B are known

If point B is east of point A, add the difference in time to the time at point A (remembering that the time increases as we move eastward). If point B is west of point A, subtract the difference in time from the time at point A.

Ex.: When the time is 1326 at point A, 45° 36' W, what is it at point B, 23° 30' W?

Since point B is east of point A, we add the difference in time to "A" time.

$$\begin{array}{r} 45^\circ\ 36' \\ \underline{23^\circ\ 30'} \\ 22^\circ\ \ 6' \end{array} \text{ difference in longitude}$$

$$15\overline{)22^\circ\ \ 6'} \qquad\qquad 15\overline{)426}^{\,28.4'}$$

$$\begin{array}{l} \underline{15} \\ \ \ 7 + 6' = 426 \end{array}$$

$$\begin{array}{l} 13 \text{ hr. } 26' \\ \underline{\ \ 1 \text{ hr. } 28.4'} \\ 14 \text{ hr. } 54.4' \text{ or } \underline{1454} \end{array}$$

Answer: It is 1454 at point B, 23 °30' W

F. The magnetic compass; single navigation

1. Discrepancies in the reading of the compass

   a. Variation

   b. Deviation

When navigating a boat or an airplane by chart and magnetic compass, the navigator must contend with certain discrepancies in the compass readings, called magnetic variation and magnetic deviation.

Since magnetic compass points to the magnetic north pole instead of the true north pole, a correction for variation has to be made. The amount of this correction is different at different localities as indicated on a navigational chart. Note: The term "variation" used in navigation has the same meaning as "declination" in surveying.

	Another descrepancy, caused by the metal parts of a ship or a plane pulling on the compass, is called deviation. The deviation varies as the ship or plane changes the direction in which it is heading, so that a card giving the amounts of deviation for different headings must be kept up to date and always be posted in sight of the navigator.
2. The course of a boat or plane a. True course b. Magnetic course c. Compass course	The direction in which an airplane flies or boat sails over the earth's surface is called its course. The course is expressed as an angle. It is the true course when the angle is measured clockwise from the true north (North Pole). However, since the compass needle points to the magnetic north, the navigator must correct his true course reading. This correction is called magnetic variation or variation and the corrected course reading is called the magnetic course. Since the metal parts of the airplane or boat also affect the compass, the magnetic course reading must be corrected. The correction is called deviation and the corrected course reading is called the compass course.
3. Changing from true course to magnetic course to compass course	To find magnetic course from true course, add west variation but subtract east variation. To find compass course from magnetic course add west deviation but subtract east deviation.
4. Changing from compass course to magnetic course to true course	To find the magnetic course from the compass course, subtract west deviation but add east deviation. To find true course from magnetic course, subtract west variation but add east variation.

5. Sample Exercise

Find the missing numbers:

	True Course	Variation	Magnetic Course	Deviation	Compass Course
a.	28°	5° W.	?	4° E	?
b.	?	3° E.	?	6° W	192°
c.	202°	?	199°	?	203°
d.	315°	12° E.	?	5° E.	?
e.	88°	6° E.	?	6° W.	?

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APPENDIX

GENERAL MATHEMATICS

Tables of Measure

Linear (Line) Measure

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
5½ yards or 16½ feet = 1 rod (rd.)
320 rods or 5280 feet = 1 mile (mi.)

Square Measure

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
30¼ square yards = 1 square rd (sq. rd.)
160 square rods } = 1 acre (A.)
43,560 square feet }

Cubic Measure

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)
27 cubic feet = 1 cubic yard (cu. yd.)
231 cubic inches = 1 gallon (gal.)
1 cubic foot = 7½ gallons

Liquid Measure

3 teaspoons (t.) = 1 tablesppon (T.)
16 tablespoons = 1 cup (C.)
2 cups = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)

Dry Measure

2 pints (pt) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

Avoirdupois Weight

16 ounces (oz.) = 1 pound (lb.)
2000 pounds = 1 ton (T.) short ton

Counting

12 units = 1 dozen (doz.)
12 dozen = 1 gross

Circular Measure

60 seconds (") = 1 minute (')
60 minutes = 1 degree (°)
360 degrees = 1 circumference

Time Measure

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (da.)
7 days = 1 week (wk.)
calendar days = 1 month (mo.)

30 days hath September
April, June and November,
All the rest have 31
Except February which has 28
(on leap year it has 29)

365 days = 1 year (yr.)
366 days = 1 leap year
52 weeks = 1 year
12 months = 1 year
10 years = 1 decade
100 years = 1 century

Apothecaries Weights

60 grains = 1 dram
8 drams = 1 ounce
12 ounces = 1 pound

Liquid Measure (Apothecaries)

60 minims = 1 fluid dram
8 fluid drams = 1 fluid ounce
16 fluid ounces = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon

FORMULAS

Distance. $D = rt$, $r \frac{D}{t}$ $t = \frac{D}{r}$

D = distance, r = rate t = time

Electricity. $E = IR$, $W = IE$, $H.P. = \frac{W}{746}$

E = number of volts, I = number of amperes, R = number of ahms, W = number of watts, H.P. = horsepower.

Inclined Plane. $WH = PL$

W = weight, H = perpendicular distance
P = effort, L = length of incline

Interest. $i = Prt$, $A = p + prt$

i = interest, p = principal, r = rate, t = time in years, a = amount.

Levers. $Wl = wL$.

W = first weight, w = second weight, l = length of first arm, L = length of second arm.

Lumber Measure. Board feet $= \frac{Lwt}{12}$

L = length in feet, w = width in inches, t = thickness in inches.

Pulleys. $\frac{D}{d} = \frac{r}{R}$

D = diameter of large pulley, d = diameter of small pulley, R = number of revolutions per minute of large pulley, r = number of revolutions per minute of small pulley.

Rim speed of pulleys (in feet) R. R.S. = $\frac{\pi(d)\,(r.p.m.)}{12}$. R.S. = rim speed per minute, D= diameter of pulley, r.p.m. = revolutions per minute.

Temperature. $F = \frac{9C}{5} + 32°$, $C = \frac{5}{9}(F-32)$. C = temperature in centigrade degrees, F = temperature in fahrenheit degrees.

Wheel and axle. Wr = PR. W = weight, r = radius of axle, P = power applied, R= radius of wheel

Plane Figures

	Figure	Perimeter	Area
1.	Parallelogram	$P = 2a + 2b$	$A = bh$
2.	Rectangle	$P = 2a + 2b$	$A = ba$
3.	Square	$P = 4a$	$A = a^2$
4.	Trapezoid	$P = a + b + c + d$	$A = \frac{h(B + b)}{2}$
5.	Triangle	$P = a + b + c$	$A = \frac{bh}{2}$
6.	Circle	$C = 2 \quad r$ or d	$A = \quad r^2$ or $.7854d2$

In these formulas

A = the area
a = one side of a figure
B = the longer base of a trapezoid
b = the base or one side of a figure
C = the circumference
c = one side of a figure
d = one side of a figure or the diameter of a circle
h = the altitude
P = the perimeter
  = 3.14 or 3 1/7
r = the radius

Solid Figures

	Figure	Volume	Total Surface	Lateral Surface
1.	Rectangular	$V=lwh$	$T=Ph + 2A$	$S = Ph$
2.	Prism	$V=Ah$	$TPh + 2A$	$S = Ph$
3.	Cube	$V=h^3$	$T=6h^2$	$S = h^2$
4.	Pyramid	$V=\frac{Ah}{3}$	$T= \frac{PW}{2} + A$	$S = \frac{PH}{2}$
5.	Cylinder	$V= \quad {}^2h$	$T=2 \quad rh + 2 \quad r^2$	$S = z \quad rh$ or dh
6.	Cone	$V= \frac{r^2h}{3}$	$T= \quad rH + \quad r^2$	$S = \quad rH$
7.	Sphere	$V = 4/3 \quad r^3$	$T = 4 \quad r^2$ or $d2$	

In all these formulas

A = the area of the base of the solid
d = the diameter
H = the slant height(the perpendicular distance from the vertex to the perimeter of the base of a cone or a pyramid)
h = the altitude of the solid
l = the length
P = the perimeter of the base
  = 3.1416 or 3 1/7
r = the radius
S = the lateral area
T = the total area
V = the volume
W = the width

TABLE OF SQUARES AND SQUARE ROOT

No.	Squares	Square Roots	No.	Squares	Square Roots	No.	Squ	Square Roots
1	1	1.000	34	1,156	5.831	67	4,489	8.185
2	4	1.414	35	1,225	5.916	68	4,624	8.246
3	9	1.732	36	1,296	6.000	69	4,761	8.307
4	16	2.000	37	1,369	6.083	70	4,900	8.367
5	25	2.236	38	1,444	6.164	71	5,041	8.426
6	36	2.449	39	1,521	6.245	72	5,184	8.485
7	49	2.646	40	1,600	6.325	73	5,329	8.544
8	64	2.828	41	1,681	6.403	74	5,476	8.602
9	81	3.000	42	1,764	6.481	75	5,625	8.660
10	100	3.162	43	1,849	6.557	76	5,776	8.718
11	121	3.317	44	1,936	6.633	77	5,929	8.775
12	144	3.464	45	2,025	6.708	78	6,084	8.832
13	169	3.606	46	2,116	6.782	79	6,241	8.888
14	196	3.742	47	2,209	6.856	80	6,400	8.944
15	225	3.873	48	2,304	6.928	81	6,561	9.000
16	256	4.000	49	2,401	7.000	82	6,724	9.055
17	289	4.123	50	2,500	7.071	83	6,889	9.110
18	324	4.243	51	2,601	7.141	84	7,056	9.165
19	361	4.359	52	2,704	7.211	85	7,225	9.220
20	400	4.472	53	2,809	7.280	86	7,396	9.274
21	441	4.583	54	2,916	7.348	87	7,569	9.327
22	484	4.690	55	3,025	7.416	88	7,744	9.381
23	529	4.796	56	3,136	7.483	89	7,921	9.434
24	576	4.899	57	3,249	7.550	90	8,100	9.487
25	625	5.000	58	3,364	7.616	91	8,281	9.539
26	676	5.099	59	3,481	7.681	92	8,464	9.592
27	729	5.196	60	3,600	7.746	93	8,649	9.644
28	784	5.292	61	3,721	7.810	94	8,836	9.695
29	841	5.385	62	3,844	7.874	95	9,025	9.747
30	900	5.477	63	3,969	7.937	96	9,216	9.798
31	961	5.568	64	4,096	8.000	97	9,409	9.849
32	1,024	5.657	65	4,225	8.062	98	9,604	9.899
33	1,089	5.745	66	4,356	8.124	99	9,801	9.950

Sample Test

Areas and Volumes of Geometric Figures

(all answers given in parentheses)

Matching

1. $A = S^2$
2. $A = \frac{1}{2}h\,(b_1 + b_2)$
3. $A = bh$
4. $V = \quad r^2 h$
5. $A = \frac{1}{2}\,bh$
6. $V = wlh$
7. $A = \quad r^2$
8. $A = lw$

(5) A. Area of a triangle
(7) B. Area of a circle
(1) C. Area of a square
(4) D. Volume of a cylinder
(3) E. Area of a parallelogram
(8) F. Area of a rectangle
(2) G. Area of a trapezoid
(6) H. Volume of a rectangular solid

Using the proper formula, find what is indicated in each of the following problems. Show all work in an orderly manner.

9. The area of a rectangle having a length of 6 inches and a width of 7 inches. (42 sq. in.)
10. The area of a circle with a radius of 4 inches. (50.24 sq. in.)
11. The area of a circle with a diameter of 12 inches.
12. The area of a triangle with a base of 4.2 feet and a height of 3.4 feet. (7.14 sq. ft.)
13. The area of a parallelogram having a height of 3 1/3 dm., and a base of 7 1/6 dm. (23 8/9 sq. dm.)
14. The area of a trapezoid whose bases are 9.6 feet and 3.4 feet, and the height is 10 feet. (65 sq. ft.)
15. The area of a square having a side of 3½ cm. (12¼ sq. cm.)
16. The volume of a rectangular solid having a length of 5 in., a width of 7 in., and a height of 3 in. (105 cu. in.)
17. The volume of a cylinder having a base area of 21.6 sq. feet and a height of 12 feet. (259.2 cu. ft.)
18. The volume of a cylinder having a radius of 2 in. and a height of 10 inches. (125.6 cu. in.)
19. A = lw, Find A y L = 5 feet, w = 2½ feet (A = 12½ sq. ft.)
20. V = lwh, Find V if L = 2 in., w = 4 in., h = 3 in. (V = 24 cu. in.)

Powers of 10

Write all answers in scientific notation. Simplify as much as possible.

1. $10^7 \times 10^3 \times 10 =$
2. $7 \times 10 \times 10^{-1} =$
3. $10^{-3} \times 10^{-3} =$
4. $.001 \times 100 \times 10^6 =$
5. $.0003 \times 10^{17} =$
6. $1200 \times 25 \times 150\ 000 =$
7. $80 \times 80 \times 80 =$
8. $4 \times 10^3 \times 25 \times 10 =$
9. $300 \times 10^5 =$
10. $9{,}000{,}000 \times 70{,}000 =$
11. $\frac{10^4}{10^3} =$
12. $\frac{1000}{10000} =$
13. $.00062 =$
14. $10^4 \times 10^6 \times 10^{-3} =$
15. $10^{-3} \times 10^2 =$
16. $\frac{10^4}{10^{-3}} =$
17. $2500 \times 40 \times 10^{-3} =$
18. $\frac{1}{10^4} \times 10^{-3} =$
19. $10^{-6} \times 10 =$
20. $8^4 \times 8^6 \times 8^{-2} =$

VOLUME & AREA

1. A room is 30 ft. long, 25 ft. wide, and 14 ft. high. How many cubic feet of air is contained in the room?

2. Find the volumes of cubes whose sides measure: (a) 8 in. (b) 38 ft. (c) 50 yd. (d) 26 ft.

3. How many cartons 2 ft. by 2 ft. 6 in. by 1 ft. can be stored on a freight car with inside dimensions of 40 ft. by 8 ft. by 8 ft.

4. What is the weight of a steel cylinder 16 ft. long and 4 in. in diameter. A cubic foot of steel weights 490 lbs.

5. What is the weight of 15 cu. ft. of sea water if a cubic foot weighs 64 lbs.

6. What is the weight of a cake of ice, 1½ ft. long, wide, and thick. Ice weights 56 lbs per cu. ft.

7. The volume of a 12 inch cube is how many times as large as the volume of a 2 inch cube.

8. How many cubic yards of dirt must be removed in digging (a) a well 5 ft. in diameter and 54 ft. deep. (b) At $8.40 per cu. yd. how much will the digging cost?

9. How many gallons of gas will a tank hold if it's diameter is 35 ft. and is 30 ft. high. One cu. ft. = 7½ gallons.

10. Find the total area of cubes whose sides measure: (a) 9 inches (b) 25 ft. (c) 40 in. (d) 2 ft. 8 in.

Solve for x

1. $6_x - 3_x = 7 + 11 + 3$
2. $12_x - 5x + 2_x = 9$
3. $3 + 5 = 8a + 2a - ba$
4. $\frac{x}{5} = 7$
5. $x = \frac{2}{3}$
6. $z = \frac{7}{x}$
7. $z = \frac{40}{5x}$
8. $3x - x = 8 - 6$
9. $\frac{x}{7} = 2$
10. $\frac{17x}{2} = 17$
11. $\frac{2}{3} \times \frac{3}{4} =$
12. $.007 \times 2 =$
13. Twice a number divided by 3 equals 8. What is the number?
14. Find the answer when a = 1, b = 2, c = 3

    $\frac{2a}{4b} \cdot 3 =$
15. $1.02\,\overline{.204}$
16. $.102\,\overline{20.4}$
17. $10.2\,\overline{204}$
18. $\frac{5}{6} - \frac{3}{5} =$
19. $\frac{5}{6} \div \frac{3}{5} =$
20. $\frac{4}{5} =$ ________%

Show all work. Follow directions and signs carefully. Add or combine like terms:

1. $\begin{array}{r} 6a \\ \underline{3a} \end{array}$

2. $3.2\ m + 1.9\ m$

3. $4\frac{1}{2}\ h + 2\ 3/8\ h$

4. $\begin{array}{r} 6\ x \\ 14\ x \\ \underline{3.5x} \end{array}$

5. $8\ w + 7\ w + 11\ w - w$

6. $19\ x - x + 10\ y - 3\ y$

Subtract

7. $\begin{array}{l} 7\ 3/8\ p \\ \underline{4\ \frac{1}{2}}\ \ p \end{array}$

8. $365\ c - 2.48\ c$

9. $5x - 1.467x$

Find the quotients:

10. $\frac{45\ n}{9}$

11. $\frac{10\ x}{10\ x}$

12. $\frac{300\ i}{50}$

13. $\frac{5.6\ t}{8}$

14. $48\ y \div 6$

15. $\frac{70\ v}{14}$

16. $\frac{3c}{5} \div \frac{9}{15}$

Perform the indicated operations:

17. $3 + 4 \times 5$

18. $9 \times 4 + 5$

19. $4 \times 19\ n$

20. $5.9\ a \times 10$

21. $3 + 4a - a + 2$

22. $c + 5 + c + 5$

23. $17 \times 3 + 17 \times 3$

24. $3\ x + (\ ) = 24x - 8x$

25. $a + b + 1 + 6 + b + 2$
$+ c + 3 - c + 4b + 6a -$
$1 - a$

Indicate

1. four increased by y

2. 10 less h

3. h less 100

4. The square of k

5. The product of x and y, decreased by the quotient obtained by dividing b by c

6. Find the value of 8 when = 3.14

7. Find the value $6\,ab^2$ when $a = 3$, $b = 6$

8. Simplify $4 + 4 \div 4$

9. Simplify $6 \div 6 + 6$

10. Indicate the cost of P pencils at c cents a dozen

11. If $x = \frac{1}{2}$ find the value of $2\,x$

12. Find the value of $2\,x^3$ when $x = 1/3$

13. What does $3\,x^2$ mean?

14. Find the value of $3\,x^2y$ when $x = 5$ and $y = 7$

Evaluate

15. $5\ (\,-4)$ when $y = 7$

16. What is the left member of the equation $3\,x - 1 = 11$. How many terms does the equation have.

17. Solve the following equations $3\,y = 24$

18. $x - 4 = 10$

19. $3/4\ C = 24$

20. $10 = x - 8$

21. $.03h = 21$

22. Four percent of a number is 28. What is it?

23. $y + 4 = 4$

24. $.04C = 1.2$

25. What is algebra?

1. If $a = \frac{22r^2}{7}$, find A when r = 5.

2. Find the area of a rectangle which is 16 inches long and 8 inches wide.

3. Find the area of a square table top 17 inches on one side.

4. If $V = \frac{Ah}{3}$ find V when A = 24 and h = 5.

5. If I = Prt, find the interest on $800 on 3% for 4 years.

6. Jim can average 14 miles an hour on his bicycle. How far can he go in $2\frac{1}{2}$ hours?

7. If a desk is 4 feet wide and 6 feet long, what is the ratio of its width to its length?

8. Find the area of a trapezoid if the two bases are 9 inches and 12 inches and the height is 5 inches. $A = \frac{h(B + b)}{2}$.

9. A circular race track is 420 ft. in diameter. What is the distance around it?

10. Find the principal if the interest on it is $80, the rate is 5%, and the time is 2 years.

11. $F = \frac{90}{5} + 32$. Find F when C = 60.

12. $\frac{x}{8} = \frac{9}{4}$; find x

13. If a bill carried by a vote of 3 to 7, and 30 voted against it, how many voted for it?

14. How many square yards of linoleum will it take to cover a kitchen floor 10 ft. wide and 13 ft. long?

15. Mr. Smith averages 120 miles an hour in his airplane. How long will it take him to fly 720 miles.

16. How many square feet are there in 288 square inches?

17. If a garden is to cover 360 square feet and is 24 feet long, how wide should it be?

18. If one gallon of weed spray will cover 200 square feet of lawn, how many gallons will be needed for a law 35 feet wide and 40 feet long?

19. $\frac{8}{9} = \frac{x}{2\frac{1}{2}}$ find x.

20. If a 360 mile trip is made in 8 hours, what is the average number of miles per hour?

21. How many square yards are there in 288 square feet?

22. Find the cost of varnishing the top of a square stand 8 ft. on a side at 36¢ a square foot.

23. Find the total surface of a 9 inch cube.

24. How long a fence will be needed to enclose a lot 120 feet long and 50 feet wide?

25. A circular piece of tin has a diameter of $1\frac{1}{2}$ feet. What is its area? Use $= \frac{22.}{7}$

26. How many acres are there in a field 36 rods wide and 80 rods long. 160 sq. rds. = 1 A.

27. Find the volume of an egg carton 8 inches long, 6 inches wide, and 2 inches high.

28. Nancy's candy recipe is as follows: 3 cups of sugar, 3/4 cups of syrup, and 3/4 cups of water. She wants to use only 2 cups of sugar. Then how much syrup should she use?

29. In how many years will $700 earn $210 at 5%?

30. Find the perimeter of a triangle which has one side of 8 inches and two sides of 6 inches each.

31. $\frac{16}{x} = \frac{4}{3}$; find x.

32. When a 3 ft. stick casts a shadow 5 ft., a flag pole is casting a shadow 80 ft. How tall is the flag pole?

33. What is the shortest distance a baseball player can run in making a "home run?" The baseball diamond is 90 ft. sq.

34. Find the rate of interest on a $1000 investment (principal) which earns $70 in 1 year.

35. $\frac{5}{7} = \frac{3}{x}$; find x.

36. How much weight must you exert to lift a 400 lb. crate 1 ft. from the fulcrum of a lever if you are at the other end of the lever 5 ft. from the fulcrum?

37. A 16 inch pulley running 120 r.p.m. is belted to a 10 inch pulley. How fast is the 10 in. pulley revolving?

38. If we use 3 oz. of chocolate for 7 people, how many ounces would we need to use for 10 people?

39. How many cubic yards of ready-mix concrete would you need to pave a driveway 60 ft. long, 9 ft. wide, and 4 inches thick?

40. At 20¢ a gallon, what is the value of the oil in a tank 15 feet high and 14 feet in diameter.
1 cu. ft. = 7½

General Math Final 4th quarter

1. In a recent year the total expenditures for public schools in the United States were five billion, eight hundred thirty-seven million, six hundred forty-three thousand dollars. (a) Write this number in figures. (b) Then round it off to the nearest million dollars.

________________ (b)________________

2. Multiply each number by 10, by 100, and by 1000.

Number	10	100	1000
.729			
6.5			
42			

3. Divide each number by 10, by 100, and by 1000

Number	10	100	1000
34			
26.8			
4.36			

4. Write the answers

a. 4.8 + .4 = ________
b. 4.8 - .4 = ________
c. 4.8 x .4 = ________
d. 4.8 ÷ .4 = ________

e. 1.2 + .6 = ________
f. 1.2 - .6 = ________
g. 1.2 x .6 = ________
h. 1.2 ÷ .6 = ________

5. Round to the nearest whole number, tenth, hundredth, and thousandth.

Number	whole numbers	Tenths	Hundredths	Thousandths
.9356				
8.7631				
25.3175				

6. Write the answers

a. $37\frac{1}{2}\%$ of 24 = ________
b. 4 is 5% of ________
c. 30 is what per cent of 36? ________

7. Solve and check:

a. $x + 4 = 10$
b. $y - 2 = 5$
c. $\frac{y}{3} = 6$
d. $4n = 12$
e. $\frac{3}{4}y = 24$
f. $5x + 2 = 17$

8. Find Mr. Kirk's bank balance: balance brought forward, \$298.65; amounts deposited, \$58.28, \$158.11; and checks written, \$8.25, \$2.65, \$59.75, and \$25.00. Answer____________

9. Mr. Barnes borrowed \$2000 at $3\frac{1}{2}\%$ for 2 years 6 months. At the end of that time what amount will be due? Answer________

10. Find the total cost of insuring Mr. Bass's automobile valued at \$2000, against the following risks: fire, theft, etc., at \$1.85 per \$100, public liability \$25.75, collision \$45.00. Answer_________

11. Jerry paid \$15 down and agreed to pay \$3.75 a month for 12 months on a bicycle whose cash price was \$55. Find (a) the installment price and (b) the carrying charge.

12. The croquet court in the park is 57 feet long and 30 feet wide. Find (a) the area and (b) the perimeter of the court. Answer_________

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13. The sand box in the section of the park for small children is 12 feet square. How many cubic yards of sand are needed to fill it to a depth of 9 inches? Answer__________

14. A large tree in the park has a circumference of 14 feet. Find to nearest tenth of a foot. (a) The diameter and (b) the radius of the tree (use 3 1/7 for). (a)________ (b)__________

15. The roller used in the park to pack the soil after planting grass seed is a concrete cylinder 2 feet long with a radius of 1 foot. (a) Find to the nearest tenth the number of cubic feet of concrete in the roller. (b) How much ground does it cover in one revolution? Round this answer to the nearest tenth. (use 3.14 for) (a)________ (b)__________

16. Find the height of a tree that casts a shadow 20 feet long at the same time that a boy 6 feet tall casts a shadow 4 feet long. Sketch the triangles and label the given parts.. Answer__________

17. In a school of 200 pupils, 35% ride school buses. (a) How many ride the buses? (b) How many do not ride the buses. (a)________ (b)________

18. A basketball team was 15 games and lost 5 games. What per cent of the games played did the team (a) win (b) lose? Answer (a)________ (b)________

19. A television set listed at $350 is offered with trade discounts of 10% and 5%. Find the net price. Answer __________

20. If 3/8 pound of candy costs 30¢, find the cost of 2 pounds. Answer__________

21. How many pounds of milk containing 4% butter fat are needed to make 6 pounds of butter fat? Answer________

22. The price of corn increased from $1.40 to $1.54. Find the per cent of increase. Answer________

23. A 17 foot ladder reaches the edge of a roof when the foot of the ladder is placed 8 feet from the building. Sketch the triangle and find the height of the roof. Answer__________

24. $37\overline{)15096}$

25. $86\overline{)16512}$

GLOSSARY OF MATHEMATICAL TERMS

Abacus - A counting frame to aid in arithmetic computation; a primitive form of computing machine.

Acute Angle - An angle smaller than a right angle; an angle with measure less than 90°.

Additive Inverse - One of the two numbers whose sum is zero.

Altitude - (of a triangle) - The perpendicular distance from a vertex to the opposite side or opposite side extended. (of a parallelogram) - the perpendicular distance between opposite sides.

Angle - The inclination to each other of two straight lines; the figure formed by two rays drawn from one point.

Approximate Numbers - Numbers that do not represent an exact value, such as those obtained from measuring.

Area - The number of square units of measure enclosed by a figure.

Arithmetic Mean - A form of average in which the sum of the elements is divided by the number of elements.

Associative Law of Addition - $(a+b) + c = a + (b+c)$ - find the sum of three or more terms any method of grouping adjacent factors in pairs may be used.

Associative Law of Multiplication - $(ab)c = a(bc)$ - the grouping order in which terms are arranged and multiplied does not effect the answer.

Average - A number that is typical of a set of numbers; various averages are used such as arithmetic mean, median and mode.

Bar Graph - A visual comparison of a set of measures in which the length of each bar is determined by the magnitude of the measure represented by it.

Base - The number of units in a given digit's position, which has the value 1 in the next higher position.

Cardinal Number - The number of members in a set with no regard to the order in which they are arranged.

Center of Gravity - The point about which all the weight of a body seems to be concentrated; the point about which the body is in balance.

Centimeter - A measure of length used extensively in science; one hundredth of a meter; .3947 of an inch; 2.54 centimeters equals one inch.

Central Angle - An angle whose sides are radii and whose vertex is the center of the circle.

Chord - A line segment whose end points lie on a circle.

Circle - A plane closed curve consisting of all points at a given distance from a fixed point, called the center of the circle.

Circumference of a Circle - The length around a circle.

Circumscribed Circle - A circle that is drawn around a polygon in such a way that the circle passes through each vertex of the polygon and the polygon lies entirely within the circle.

Clock Arithmetic - A system of arithmetic built upon a finite set of numbers similar to the numbers used on a clock face.

Closure Law - A statement to the effect that a set is closed under a given operation if when any two members of the set are combined. The result is also a member of the set.

Common Denominator - A common multiple of the denominators of two or more fractions.

Common Fraction - A fraction whose numerator and denominator are both integers.

Commutative Law of Addition - $a + b = b + a$. The order in which two numbers are added does not effect the sum.

Commutative Law of Multiplication - $ba = ab$. The order in which two numbers are multiplied does not effect the sum.

Composite Number - A number that is not prime; a number which can be broken into factors other than itself and one.

Concentric Circles - Circles with the same centers but different radii.

Cone - A solid figure with a circular base and a surface consisting of all the lines which can be drawn between the points on the boundary of the base and some point not in the plane of the base.

Corresponding Sides - Those sides in similar figures which are similarly related to the rest of the figure. In two similar triangles, the longest side in one is the side corresponding to the longest side of the other.

Cube - A box-like solid, each of whose six sides is a square. The sides are known as faces.

Cylinder - A solid consisting of a closed curve surface and two congruent and parallel bases. In a right circular cylinder the bases are disks and the curved surface is perpendicular to the plane of the bases.

Decagon - A polygon having ten sides. It is a regular recagon if all the sides are of equal length.

Decimal Fraction - A proper fraction whose denominator is some power of ten, expressed as a decimal number.

Degree - A unit of angular measure; a measure of direction. The angular measure of a circle is 360°.

Denominator - The term below the division line in a fraction; the term that divides the numerator; the term which indicates the kind of equal parts re-presented by a fraction.

Density - The amount of weight or matter per unit volume; the ratio of weight to volume.

Diagonal - A line connecting two non-adjacent vertices of a polygon.

Diameter - The line segment whose end points lie on a circle and which passes through its center.

Digits - The ten symbols used to represent numbers in the decimal numeration system.

Direct Proportion - A relationship between two quantities such that when one increases or or decreases the second increases or decreases, respectively.

Disk - A circle and its interior area.

Distributive Law - $a(b + c) = ab + ac$. The product of a number and a sum of two or more numbers is equal to the sum of the products of the number and the addends.

Division - Average Method - A method for finding the square root of a number.

Dodecahedron - A polyhedron having twelve faces.

Elastic Limit - The length to which a spring may be streched without distorting its ability to return to its original length.

Elastic Measure - A measuring device used to illustrate the meaning of percent. It contains one hundred units, each unit representing one percent.

Elastic Unit - One of the one hundred units in the elastic measure.

Element - The individual member of a set.

Equation - A statement of equality between two quantities; a sentence equating two different names for the quantity.

Extremes - The first and last terms in a proportion.

Factor - One or two or more numbers whose product is a given number.

Factoring - The process of resolving a number into its factors.

Force - That which pushes or pulls in any way; that which changes a state of rest or state of motion of a body.

Formula - An equation expressing a general answer or rule in mathematical language. A statement of equality.

Frequency - The number of times a vlue occurs or is observed to occur.

Frequency Distribution - A tabulation of the values or scores in a given situation; a classification into categories.

Frequency Polygon - A broken line graph representing a set whose members are ordered pairs of the type (midpoint of the class interval, frequency of the class).

Fulcrum - The point about which a lever turns.

Graph - A picture drawing which shows the relation between sets of numbers.

Gravity - The force or pull exerted by the earth; the force that causes objects to "drop" when released.

Greater Than - An expression indicating the inequality of two quantities; written in symbolic form as $>$.

Hexagon - A polygon with six sides. If the sides are all of equal length, the figure is a regular hexagon.

Histogram - A graphic representation of a frequency distribution of measures that form a continuous scale, such as scores on a test. The graph consists of adjacent rectangles whose height and width are determined by the frequency and range respectively, of the class intervals into which the distribution is divided.

Hook's Law - A law of physics which states the relationship between the amount of change in the shape of an elastic body and the amount of force producing the change.

Hypotenuse - The side opposite the right angle in a right triangle. The longest side of a right triangle.

Improper Fraction - A fraction in which the numerator is larger than the denominator.

Inequality - A statement that two or more quanities are unequal ($\neq$).

Integers - The natural numbers, their number opposites and zero; members of the set $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

Intersection - The overlap of two areas in a Venn diagram; the subset of elements contained in each of two sets; the point in which two lines meet.

Inverse Operation - That operation which, when performed after a given operation, undoes the given operation.

Irrational Number - A number which cannot be written as the quotient of two integers.

Isosceles Trapezoid - A quadrilateral with two parallel sides and equal non-parallel sides.

Less Than - An expression indicating the inequality of two quantities; written in symbolic form as $<$.

Lever - A rigid bar free to turn about a fixed point called a fulcrum.

Lowest Common Denominator - The smallest multiple of the denominators of two or more fractions.

Lowest Common Multiple - The smallest natural number that is exactly divisible by each of two or more given fractions.

Magic Square - A square arrangement of numbers such that the sum of the numbers in any column, row or diagonal is the same.

Means - The second and third terms of a proportion.

Median--(in statistics) - The middle item or score when the items are arranged in ascending or descending order.

Meter - A unit of length; 100 centimeters; 39.37 inches.

Metric System - A decimal system of weights and measures in which the meter is the fundamental unit.

Millimeter - One-thousandth of a meter; one-tenth of a centimeter.

Mixed Number - A number written in terms of an integer and a fraction.

Mode - The score or item appearing most frequently in a statistical distribution.

Multiplicative Inverse - One of the two numbers whose product is one; reciprocal.

Natural Numbers - Counting numbers; members of the set $\{1, 2, 3, 4, 5 \ldots\}$.

N-Gon - A polygon with N sides.

Nonagon - A polygon with nine sides.

Number Opposites - Two numbers of the same magnitude but with opposite signs; the two numbers whose sum is zero.

Numeral - A symbol used to represent a number.

Numerator - The term above the division line in a fraction; the term which is to be divided by the denominator; the term which indicates the number of equal parts represented by a fraction.

Obtuse Angle - An angle greater than a right angle but less than a straight angle; an angle whose measure is greater than 90° but less than 180°.

Octagon - A polygon with eight sides.

Octahedron - A polyhedron with eight faces.

Ordered Pair - Two numbers written in a specific order; symbolically (a, b) so that $(a, b) \neq (b, a)$ unless $a = b$.

Ordinal Number - A number used to indicate the position a particular object occupies in a series. Such words as first, second and third identify ordinal numbers.

Parallel Lines - Lines which lie in the same plane and will not intersect, regardless of how far they are extended.

Parallelogram - A quadrilateral whose opposite sides are parallel.

Pendulum - A body and the cord or rod by which it is suspended.

Pentadecagon - A polygon of fifteen sides.

Pentagon - A polygon of five sides.

Percent - Hundredths; a comparison of some number with 100, usually written with the aid of the symbol %.

Perimeter - The distance around a plane figure.

Perpendicular Bisector--(of a line segment) - A line that divides the given line segment into two equal segments and forms right angles with it.

Perpendicular Lines - Two straight lines which intersect to form equal adjacent angles.

Pi - The name of the greek letter π; the ratio of the circumference of the circle to its diameter; the irrational number whose approximate value is 3.14159+.

Polygon - A closed plane figure whose sides are straight line segments.

Prime Number - A natural number whose only factors are the number itself and 1. One is usually no considered a member of the set of prime numbers.

Prism - A solid figure whose bases are congruent and parallel polygons and whose edges are straight lines.

Proportion - A statement of equality of two ratios.

Protractor - An instrument used to determine the measure of an angle in degrees.

Pyramid - A solid figure whose base is a polygon and whose sides are triangles.

Quadrilateral - A polygon of four sides.

Radius - One-half of the diameter of a circle; the distance from the center to any point on the circle.

Range - The difference between the largest and smallest scores in a set of scores.

Rational Number - A number which can be expressed as the quotient of two integers.

Rectangle - A quadrilateral with opposite sides parallel and adjacent sides perpendicular to each other.

Rectangular Solid - A solid figure whose six faces are all rectangles.

Regular Tetrahedron - A solid figure with four faces, each of which is an equilateral triangle.

Rhombus - A quadrilateral with opposite sides equal and parallel.

Right Angle - An angle whose measure is 90°; one of the angles formed at the point of intersection of two perpendicular lines.

Sample--(in statistics) - A small portion of the population being studied.

Sector--(of a circle) - A region in a plane bounded by two radii of a circle and the arc they intercept.

Set - A well-defined collection of objects: Set of dishes ; Set of tools ; Set of numbers

Similar Triangles - Triangles with corresponding angles equal and corresponding sides proportional

Square - A quadrilateral with four equal sides and four equal angles.

Square Root--(of a number) - One of the two equal factors of that number.

Straight Angle - An angle whose measure is 180°.

Tolerance--(in measurement) - The amount by which an object may vary from the prescribed standard.

Universal Set - The set of all elements under consideration in a given situation; sometimes called the universe.

Velocity - The directed speed of an object.

Venn Diagram - A diagram in which areas represent sets (usually circular).

Vertex - The points where the sides of an angle meet.

X-Axis - The horizontal axis or number line in the coordinate plane.

Y-Axis - The vertical axis or number line in the coordinate plane.

GENERAL MATH FINAL TEST

1. 10^4 means 10 x 10 x 10 x 10.

2. In the symbol 6^3, the exponent is 3.

3. We can make a symbol to mean what we wish.

4. The fourth place from the right in the decimal system has the place value 10^5.

5. Every composite number can be factored into prime numbers in exactly one way, except for order.

6. The sum of an odd and an even number is always an even number.

7. Some odd numbers are not prime.

8. The number 51 is a prime.

9. All even numbers have the factor 2.

10. Any multiple of a prime number is a prime.

11. Even though 1 has as factors only itself and 1, it is not considered a prime number.

12. All odd numbers have the factor 3.

13. No even number is a prime.

14. The least common multiple of 3, 4, and 12 is 12.

15. The least common multiple of 2 and 6 is 12.

16. The product $\frac{3}{7} \cdot \frac{3}{3}$ is equal to $\frac{3}{7}$.

17. Whole numbers are not rational numbers.

18. $\frac{5}{7} + \frac{7}{5} = \frac{12}{12}$

19. In adding rational numbers, if the denominators of the fractions are equal, we add the numerators.

20. The following numbers are all examples of rational numbers: $\frac{3}{4}$, 5, $\frac{8}{3}$, $1\frac{1}{2}$.

21. Zero is the identity element for addition of rational numbers.

22. A rational number multiplied by its reciprocal equals 1.

23. The fractions $\frac{0}{a}$ and $\frac{0}{b}$ represent the same rational number if neither a nor b is zero.

24. The symbol $\frac{24}{8}$ stands for a number which is both a whole number and a rational number.

25. The sum of two rational numbers whose fractions have equal numerators may be found by adding their denominators.

26. The product of zero and any rational number is zero.

27. Even if $a = 0$, $a/7$ is a rational number.

28. If two fractions have the same denominator, the numbers they represent are always equal.

29. The reciprocal of the reciprocal of 3 is 1/3.

30. Even if b equals 0, $\frac{a}{b}$ is a rational number.

31. If one fraction has a larger numerator than that of a second fraction, the number represented by the first fraction is always larger than the number represented by the second fraction.

32. A whole number can never have a fraction as a name.

33. If two fractions have the same denominator, the numbers they represent are always equal.

34. A fraction is a numeral indicating the quotient of two numbers, with denominator different from zero.

35. $\frac{\frac{900}{11}}{100} = \frac{9}{11}$

36. $\frac{19}{21} < \frac{21}{22}$

37. 502 written in expanded form is $5(10)^2 \div 1(10) + 2$.

38. $\frac{1}{3} = 0.1333...$

39. $0.3 \times 0.03 = 0.009$.

40. $.2 \div .08 = 2.5$

41. $\frac{8}{25}$ is another name for the number 32%.

42. If a class has a total of 32 pupils 20 of them boys, the number of boys is 60% of the number of pupils in the class.

43. Five percent of $150 is the same amount of money as 7.5% of $100.

44. A centimeter is 100 times as long as a meter.

45. We can associate rational numbers with points on a line.

46. The sum of positive two and negative two is zero.

47. There is no greatest number on the number line.

48. All negative rational numbers are associated with points on the number line to the right of zero.

49. The point on the number line associated with 2 and the point on the number line associated with -2 are the same distance from the point on the number line associated with 0.

50. The rational number -3 is greater than the rational number -2.

51. The quotient of two negative numbers is a negative number.

52. All negative numbers are smaller than zero.

53. If two numbers are unequal, then one must be greater than the other.

54. $4(5 - 2)$ and $\frac{6 \times 2}{1}$ are different names for the same number.

55. $\frac{4}{2} \cdot 0 = 8(\frac{1}{4})$.

56. $x(3 \div 4) = (4 \div 3)x$ is true for all values of x.

57. The point whose coordinates are (3, 2) is the same as the point whose coordinates are (2, 3).

58. The point (-4, -1) is located in the second quadrant.

59. A right triangle may have two right angles.

60. The length of the hypotenuse of a right triangle is less than the sum of the lengths of the other two sides of the triangle.

61. The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.

62. The hypotenuse of a right triangle is the longest side.

63. The Y-coordinate of points lying on the Y-axis is zero.

64. All points lying in the upper half-plane (above the X-axis) have positive Y-coordinates.

65. The Pythagorean relationship is true for all triangles.

66. The point whose coordinates are (-1, 1) lies on the line $y = x$.

67. Every real number can be written as a rational number.

68. The smallest positive integer is one.

69. $\sqrt{2}$ is a number which when squared is equal to 2.

70. Three and one-seventh is a rational number.

71. Every repeating decimal is a rational number.

72. The square root of 7 is approximately equal to 1.645.

73. Every real number can be represented by a point on the number line.

74. The number zero is not a rational number.

75. If a number is a real number, then it is also a rational number.

76. $2\sqrt{2}$ is both a real number and an irrational number.

77. Irrational numbers cannot be located on the number line.

78. Zero is a number that is both rational and irrational.

79. $10^2 \times 10 = 10^2$.

80. $10^5 \times 10^{-5} = 1$.

81. $93,000,000 = 9.3 \times 10^6$.

82. 11×10^7 is written in scientific notation.

83. $10^3 \times 10^3 = 10^9$.

84. $1,000,000 = 10^6$.

85. $10^3 = 10^{-2} + 10^5$.

MULTIPLE CHOICE

86. A decimal numeral which represents an odd number is:
    a. 461,000
    b. 7629
    e. none of these
    c. 5634
    d. 9,000,000

87. Which is correct?
    a. $5^4 = 5 + 5 + 5 + 5$
    B. $4^4 = 4 \times 4 \times 4 \times 4$
    c. $2^3 = 2 \times 3$
    d. $5^4 = 4 + 4 + 4 + 4 + 4$
    e. none of these

88. Every counting number has at least the following factors:
a. zero and one b. zero and itself c. one and itself
d. itself and two e. none of these

89. In the complete factorization of a number
a. all the factors are primes.
b. all the factors are composite.
c. all factors are composite except for the factor 1.
d. all the factors are prime except for the factor 1.
e. none of these.

90. How many different prime factors does the number 12 have?
a. 1 b. 2 c. 3 d. 0 e. none of these

91. Which of the following is not a prime number? There is only one.
a. 271 b. 277 c. 281 d. 282 e. 283

92. Let a represent an odd number, and b represent an even number; then a + b must represent:
a. an even number b. a prime number c. an odd number
d. a composite number e. none of these

93. A counting number is an even number if it has the factor:
a. 5 b. 3 c. 2 d. 1 e. none of these.

94. Which of the following sets contain only even numbers?
a. $\{2, 3, 4, 5, 9, 10\}$ b. $\{2, 5, 10\}$ c. $\{3, 5, 9\}$
d. $\{2, 4, 10\}$ e. $\{3, 9\}$

95. Which of the following is a prime number?
a. 4 b. 7 c. 9 d. 33 e. none of these

96. Which of the following is not a factor of 24?
a. 2 b. 3 c. 4 d. 9 e. 12

97. Which of the following is the complete factorization of 36?
a. 4 x 9 b. 2 x 3 x 6 c. 3 x 12 d. 2 x 18 e. 2 x 2 x 3 x 3

98. The numbers 8, 9, 16, 20, 27, and 72 are all
a. prime numbers b. even numbers c. odd numbers
d. composite numbers e. none of these

99. The product of two factors must be
a. a composite number b. a prime number c. smaller than one of the numbers
d. smaller than both of the numbers e. none of these

100. Which of the following statements describes a prime number?
a. a number which is a factor of a counting number
b. a number which has no factors
c. a number which does not have 2 as a factor
d. a number which has exactly 2 different factors
e. none of these

101. Which of the following pairs of numbers are both divisible by the same number greater than one?
a. 7, 3 b. 8, 9 c. 7, 28 d. 5, 23 e. none of these

102. Which of the following represents the largest number?
a. .276 b. .076 c. .006 d. .206 e. .267

103. Six percent of $350 is
a. $210.00 B. $21.00 c. $2.10 d. $2100 e. none of these.

104. The interest on $650 for one year at 4% is
a. $2600 b. $1625 c. $26 d. $16.25 e. none of these

105. A class consists of 20 girls and 16 boys. What part of the class is composed of boys?
a. $\frac{4}{5}$ b. $\frac{5}{9}$ c. $\frac{4}{9}$ d. $\frac{9}{4}$ e. $\frac{5}{4}$

106. The sum of -3 and -6 is
a. 3 b. -3 c. 9 d. -9 e. none of these

107. The difference -3 - (-6) is
a. 3 b. -3 c. 9 d. -9 e. none of these

108. The product of -3 and -6 is
a. 9 b. -9 c. 18 d. -18 e. none of these

109. The quotient (-3) ÷ (-6) equals
a. 2 b. $\frac{1}{2}$ c. -2 d. $(-\frac{1}{2})$ e. none of these

110. 4 - (1 - 2) equals
a. -4 b. 8 c. -12 d. 4 e. none of these

111. The number that you multiply -3 by to get 1 is
a. 3 b. -3 c. $\frac{1}{3}$ d. $\frac{-1}{3}$ e. none of these.

112. The difference 3 - 2 is the same as:
a. 3 - (-2) b. 2 - 3 c. 2 + (-3) d. 3 + (-2) e. (-2) + (-3)

113. Choose the missing number to make a true statement in 113-115. 4 + (-5) + ( ) = 0
a. 9 b. -9 c. 1 d. -1 e. none of these

114. (-8) x ( ) = (-16)
a. 2 b. -2 c. $\frac{1}{2}$ d. $-\frac{1}{2}$ e. none of these

115. (-15) - ( ) = 8
a. 7 b. -23 c. 23 d. -7 e. none of these

116. The solution set of the sentence $2 + x > 5$ is
a. 3 b. -3 c. all numbers less than 3
d. all numbers greater than 3 e. 5

117. The phrase (2 x 5) + 4 represents which one of the following?
a. 8 b. 10 c. 14 d. 18 e. 40

118. If x is the number of years in my age now, then my age seven years from now will be:
a. 7 + x b. 7 - x c. 7x d. x - 7 e. none of these.

119. The area of a square whose side is s can be expressed as:
a. 2s b. s x s c. 4s d. 2s x s e. s + 4

120. Which of the following is the solution of the equation x + 7 = 3?
a. x = 4 b. x = -4 c. x = 4, -4 d. x = 10 e. none of these

121. The distance between A(6, 0) and B(-4, 0) is
a. 0 b. 2 c. 10 d. 24 e. none of these

122. The point (3, -1) lies
a. in quadrant II b. in quadrant III c. in quadrant IV
d. on the X-axis e. none of these.

123, Given the two numbers 5 and 12, the sum of their squares is
a. 13 b. 17 c. 25 d. 289 e. 169

124. The number $\sqrt{2}$ may be classified as:
a. real and irrational b. real and rational c. rational but not real
d. irrational but not real e. both rational and irrational

125. Which of the following is a rational number?
a. $\sqrt{2}$ b. $\sqrt{3}$ c. $\sqrt{4}$ d. $\sqrt{5}$ e. $\sqrt{6}$

126. Which of the following is NOT a real number?
a. 0 b. 12 c. $7\frac{1}{2}$ d. .0909 e. all of these are real

127. The product of 10^{-5} and 10^{-3} is equal to
a. 10^{-15} b. 10^{15} c. 10^{8} d. 10^{-2} e. none of these

128. Which is the largest number?
a. 0.01 b. 1.4×10^{-2} c. 15×10^{-4} d. 15.5×10^{-4} e. 0.11×10^{-2}

129. 3.14×10^{2} is how many times as large as 3.14?
a. one b. two c. ten d. one hundred e. none of these

130. $10^{4} \times 10^{-4}$ is the same as
a. 10^{0} b. 10^{-16} c. 10^{0} d. 10 e. none of these

131. Which number is *not* in scientific notation?
a. 3.1×10^{0} b. 3×10^{-5} c. 10^{6} d. 31×10^{2} e. 3.1×10

132. Which of the following is equal to 500,000?
a. 5×10^{4} b. 5×10^{5} c. 50×10^{5} d. 10^{5} e. 50^{5}

133. 10^{0} is the same as:
a. $\frac{0}{10}$ b. 10×10 c. $10 - 10$ d. 1 e. 0

Solve each of the following equations (Problems 134 - 150)

134. $x + 10 = 15$
a. 5 b. 25 c. 5 d. -25 e. none of these

135. $x + (-3) = 8$
a. 10 b. 5 c. -11 d. -5 e. none of these

136. $2x + 7 = 13$
a. 3 b. 32 c. 6 d. -6 e. none of these

137. $x + 3 > 5$
a. 2 b. 8 c. all numbers greater than 2
d. all numbers less than 2 e. none of these

138. $7 - x = 5$
a. 12 b. 7/5 c. -2 d. 2 e. none of these

139. $2 - x = 6$
a. 4 b. 8 c. -4 d. -8 e. none of these

140. $13 = 3x + 4$
a. 5 b. -3 c. 3 d. 5 and 1/3 e. none of these

141. $x + 3 < 1$
a. all numbers less than 2 b. 2 c. -2 d. all numbers less than -4
e. none of these

142. $5x + 4 = 14$
a. 3 b. -2 c. 2 d. 3 and 3/5 e. none of these

143. $3x - 13 = 14$
a. 19 b. -9 c. 1/3 d. 9 e. none of these

144. $3x > -9$
a. -3 b. 3 c. all numbers greater than 3
d. all numbers greater than -3 e. none of these

145. $x/4 = 12$
a. 3 b. 8 c. 32 d. 48 e. none of these

146. $3x - 2 < 7$
a. -3 b. 3 c. all numbers less than -3
d. all numbers less than 3 e. none of these

147. $7x/4 = 56$
a. 56 b. 8 c. 2 d. 32 e. none of these

148. $x - (-4) = 5$
a. 0 b. 2 c. -1 d. 9 e. none of these

149. $15 - 2x = 7$
a. 8 b. 4 c. -4 d. -8 e. none of these

150. $x > -1$
a. all numbers greater than 0 b. all numbers greater than 1
c. all numbers less than -1 d. all numbers greater than -1
e. none of these

COMPLETION

151. Insert a symbol which makes a true statement: $8 + 4$ __ $4 + 8$

152. If K is a counting number, then $\frac{0}{K}$ ___.

153. We are using the _______ property when we say that $3a + 5a$ is another way of writing $(3 + 5) \cdot a$.

154. If the product of 5 and a certain number is zero, then that number must be _____.

155. When a counting number is divided by itself, the answer is always _____.

156. $2/3 + 10/15 =$ _____.

157. $1/3 + 2/5 =$ _____.

158. $5/9 \div 1/9 =$ _____.

159. $2/7 \cdot 3/7 =$ _____.

160. $5/7 - 1/7 =$ _____.

161. 7/11 - 1/9 = _____.

162. The decimal system uses ____ different symbols.

In each case below insert one of the symbols; > , <, = , so as to make the statement true:

163. 7/8 ___ 5/8

164. 3/8 ___ 3/9

165. 3/2 ___ 2/3

166. 9/9 ___ 7/7

167. 6/20 ___ 11/35

168. 0/5 ___ 0/3

169. 19/20 ___ 18/19

170. In the decimal numeral 9384.562, the digit 9 occupies the ___ place.

171. The difference between the sum of 1.05 and 0.75 and the sum of 0.5 and 0.125 is _____.

172. Find the decimal numeral for 5/2.

173. Joyce weighs 90 pounds and Barbara weighs 80 pounds. What is the ratio of Joyce's weight to Barbara's weight?

174. A sofa marked $200 is sold at a 30% discount. What is the sale price?

175. Find the interest on $850 for one year at 6%.

176. The sum of 4 and -6 is ____.

177. The point on the number line midway between 4 and -6 is ____.

178. The difference 4 - (-6) is ____.

179. The product of 4 and -6 is ____.

180. The quotient 4 ÷ (-6) is ____.

181. If -5 is subtracted from 5, the difference is ____.

182. The numbers which we associate with points to the left of zero on the number line are called ____.

183. The opposite of -6 is ____.

184. The Y-coordinate of the point (3, -7) is ____.

185. (-4, -3) names a point located in the ____ quadrant.

186. Each point in the coordinate plane has ____ numbers associated with it.

187. The distance between (3, -4) and (-7, -4) is ____.

188. The closest star to us is 24,500,000,000 miles away. Write this distance using scientific notation.

189. $(7 \times 10^{-7}) \times 10^{7}$ ____.

190. 7 meters = ____ centimeters.

191. 400 centimeters = ____ millimeters.

192. 100 meters = ____ kilometers.

193. 2.54 centimeters = ____ millimeters.

194. The identity element for multiplication is ____.

195. The product of a number by its __________ is equal to 1.

196. The sum of 0 and -7 is ____.

197. The sum of -8 and 17 is ____.

198. The sum of -5 and -9 is ____.

199. The sum of -13 and 17 is ____.

200. The sum of 8 and 18 is ____.

201. The sum of 6 and -6 is ____.

202. The difference of 7 and 0 is ____.

203. The difference of -14 and -15 is ____.

204. The difference of -9 and -9 is ____.

205. The difference of -2 and -7 is ____.

206. The difference of -5 and 8 is ____.

207. The difference of 9 and -15 is ____.

208. The difference of 0 and 7 is ____.

209. The product of -2 and -8 is ____.

210. The product of 0 and -4 is ____.

211. The product of 8 and 5 is ____.

212. The product of 6 and -7 is ____.

213. The product of -5 and -4 is ____.

214. The product of -4 and 9 is ____.

215. The quotient of -28 and -7 is ____.

216. The quotient of 2 and 7 is ____.

217. The quotient of -15 and 3 is ____.

218. The quotient of 64 and -4 is ___.

219. The quotient of -36 and -9 is ____.

220. The quotient of 0 and -8 is ____.

ED0 26270

O.C.S.E.I.P. SYLLABUS

Algebra I

SE 006 070

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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INTRODUCTION

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

The first draft of this syllabus was written during an 8 week session at University of California, Irvine during the summer of 1966 by:

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PREFACE

ALGEBRA I

You, as a teacher, will have some opinions as you read through this syllabus looking for helpful suggestions. Some of the topics now considered standard in many Algebra I textbooks are not included. Others that are included may be in a different order than that to which you are accustomed. Hopefully, the suggestions are sufficiently self-contained to allow you to use what can be helpful. Some observations to be made as you browse:

1. The timing suggestions on the units and topics in the table of contents give a minimum of 111 and a maximum 161 days to I - IX. To this is added 25 - 30 test days and a 10 - 15 day review period which is discussed more fully in the next observation. This brings the timing to 146 - 206 days. Most of the Orange County schools meet about 180 days. With the wide range in the timing, this, of course, means that in no class can maximum time be spent on every topic. It is strongly urged that material through the quadratic function be handled in all classes. It will be possible to move more rapidly in the beginning units as this is (hopefully) primarily a review of student's previous mathematics experience—a condition which will become increasingly true as time goes on and the impact of curriculum change is felt in the lower grades.

2. The 10 - 15 day review period has been reserved for reacquainting students with terminology about sets and properites of operations. These topics now occupy considerable time and space in Algebra I textbooks, but the student has generally had considerable exposure to these topics in his training leading up to Algebra. Therefore, the theory has been to use sets and properties of operations. You may find, by a pretest, that the students have good background in these areas and extension into algebra can be made more rapidly than in 10 - 15 days.

3. The unit discussing proof has been left to the end because of reasons outlined in the introductory remarks of the unit. Encouraging the students to read the proofs in the text materials as the year progresses is definitely recommended. In several places in the units prior to the one on proof, suggestions have been made that a certain proof could be presented at that time. It is sincerely hoped that most teachers in most classes can spend some time on this topic.

4. The topic of Numerical Trigonometry is not written in this syllabus. Teachers have seldom found time for it in Algebra I. If you desire some teaching suggestions on the topic, it is written up in the Geometry syllabus.

5. The numbers in parenthesis ( ) are references which are listed at the end of the syllabus.

6. There is no mention of any A. V. materials in the following units. There are many good films and filmstrips available. Be sure to preview films as some might not be of the level desired. Recommended timing for filmstrips is to show them to introduce a unit or use them for review for slower or absent students. Use of the overhead projector is enjoyed by many math teachers. A good screen mounting on the ceiling makes the overhead more enjoyable for the students.

RECOMMENDED MINIMUM REQUIREMENTS FOR STUDENTS ENTERING ALGEBRA I

(and where the topics are used)

1. Facility and understanding of the fundamentals of arithmetic (used throughout algebra).
   a. The four operations with whole and rational numbers.
   b. Calculations.
2. Facility with adding and subtracting positive and negative numbers (reviewed and extended in Unit II).
3. Familiarity with and ability to evaluate various formulas (used in problem solving from Unit I on)..
   a. Distance.
   b. Interest.
   c. Perimeter and area of plane figures.
   d. Volume of solid figures.
4. Facility in changing fractions, percents, ratios, decimals, from one to another (reviewed and rapidly extended in Unit IV).
5. Ability to read, interpret, and construct graphs (some review and then extension in Unit VI to VIII).

6. Ability to read a protractor and do simple constructions with compass and straightedge (not used until geometry--mentioned in Unit V).

7. Familiarity with the language of sets and operations of intersection and union of sets (used throughout algebra).

8. Facility and understanding of the properties of operations (used throughout algebra).

    a. Commutative.

    b. Associative.

    c. Distributive.

    d. Existence of identities.

    e. Existence of inverses.

    f. Closure of a set under an operation.

9. Reading ability is essential for an algebra student. The ability to read and understand definitions, methods of procedures, and problems cannot be overstated.

Use of these ideas is incorporated into the body of these units, assuming the understanding of them on the part of the students. If students need review, refer to lower grade levels for suggestions. The amount of review will, of course, be determined by the specific class.

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ALGEBRA I

I. Numbers, Variables, Expressions, Sentences, Formulas

A. Numerical expressions or phrases

1. Definition - a numerical expression is an arrangement of numerals, symbols, and operations.

   Examples: (8 + 4) 3, 3 + 2, (6 ÷ 4) + 2.

2. Symbols and operations included are: +, -, ., ÷, ( ), { } and [ ]
   We will use various ways of expressing the product of a pair of numbers. Instead of 4 x 3, we will indicate this product by 3 · 4, (3) 4, 3 (4), or (3) (4).

3. Order of operations - create the need for an order by giving an expression such as 2 + 3 · 4. Students will give answers of 14 and 20. Lead students into deciding upon order, or tell them the order that mathematicians have agreed upon for +, · , ÷, - . (another example of this could be 8 - 4 + 1 where possible answers are 5 and 3.)

   The formal rule for order of operations has been written in order that we all arrive at consistent conclusions in expressions such as 8 ÷ 4 + 2. The following procedure should be followed:

   a. All parenthesis should be removed by performing operations within the parenthesis.

   b. All numerals being raised to a power should be raised to that power.

   c. All multiplications and divisions should be done in order, as they appear from left to right.

   d. Additions and subtractions should be done as they appear in order from left to right.

   Example 1 Simplify: $3^2 \cdot 4 \div 2\ (5 + 3) - 7$

   a. Remove parenthesis $3^2 \cdot 4 \div 2 \cdot 8 - 7$
   b. Raise to a power $9 \cdot 4 \div 2 \,.\, 8 - 7$
   c. Multiply and divide $36 \div 2 \cdot 8 - 7$
   $18 \cdot 8 - 7$
   $144 - 7$
   d. Add and subtract 137

Example 2 Simplify: $2 + 4 \cdot 6 - 4^2 \div (2 + 6) + 3$

a. Remove parenthesis $2 + 4 \cdot 6 - 4^2 \div 8 + 3$

b. Raise to a power $2 + 4 \cdot 6 - 16 \div 8 + 3$

c. Multiply and divide $2 + 24 - 16 \div 8 + 3$

$2 + 24 - 2 + 3$

d. Add and subtract $26 - 2 + 3$

$24 + 3$

27

4. Exponent refers always to the particular use of a numeral to indicate how many times a certain number should be used as a factor. The expression 3^4 is read as "3 to the fourth power"; its meaning is $3 \cdot 3 \cdot 3 \cdot 3$. 3 is the base; 4 is the exponent. (Stress the difference between problems like 4^3 and $4 \cdot 3$)

B. Numerical sentences

1. Definition - a numerical sentence is the connection of numerical expressions with relation symbols.

   Examples: $3 + 2 \leq 6 \cdot 8$, $4 + 2 = 18 \div 3$.

2. Relation symbols to be used are: $=$, $\neq$, $<$, $\nless$, $>$, $\ngtr$, $\leq$, $\nleq$, $\geq$, $\ngeq$

3. Some types of numerical sentences are:

   a. True examples: $3 + 2 = 5$, $4 + 8 \cdot 5 = (9 + 2) \cdot 4$, $8 + 1 \neq 10$.

   b. False examples: $3 + 2 = 6$, $6 - (4 - 2) = (6 - 4) - 2$, $3 + 4 \neq 7$.

4. Check for true and false sentences by writing simpler equivalent sentences.

   Example 1 T or F: $(3 + 4 \cdot 5) + 8 = 6^2 + 2 - 7$

   $3 + 20 + 8 = 36 + 2 - 7$

   $28 + 8 = 38 - 7$

   $31 = 31$

   True

Example 2 T or F:

$$\begin{aligned} 6 \div (3 \cdot 2) &= (6 \div 3) \cdot (6 \div 2) \\ 6 \div 6 &= (2) \cdot (3) \\ 1 &= 6 \end{aligned}$$

False

C. Open sentences

1. Definition - in addition to the true and false sentence there is the open sentence, which is a sentence that is neither true nor false. Those sentences which are neither true nor false can be further classified into three types.

   a. Sentences with at least one replacement.

      Examples: $4x - 2 = 14$, $3 + x > 6$

   b. Sentences with no replacement.

      Example: $a \div a = 2$, $a = a + 1$

   c. Sentences with any replacement.

      Example: $4 + x = x + 4$, $0 \div a = 0$

2. Variables are symbols which hold a place in an expression or sentence, and for which any of the members of a specified set may be substituted.

3. Examples of open sentences that are made true and false.

   a. Open: He is president of the United States.
      True: L.B.J. is president of the United States.
      False: Mr. Jones is president of the United States.

   b. Open: $x + 5 = 7$
      True: $2 + 5 = 7$
      False: $3 + 5 = 7$

c. Open: $3 \cdot x \neq 18$
True: $3 \cdot 7 \neq 18$
False: $3 \cdot 6 \neq 18$

D. Solutions of open sentences. (The use of the word "sentence" instead of equations prevents any new language introduction when getting to inequalities.)

1. Hint. To lead into the solution of open sentences with one variable use examples such as:

5 times a number + 3 = 23

a number ÷ 3 = 18

5 plus 3 times a number is 14

Also include problems with mixed numbers and fractions. This allows the teacher to discuss a "plan of attack" via method of transformations based on the math properties covered in the review section.

2. Some mechanical means of solving open sentences now becomes apparent. Here properties of equality should be introduced.

a. Addition property of equality. If $a = b$ then $a + c = b + c$.

b. Subtraction property of equality. If $a = b$ then $a - c = b - c$.

c. Multiplication property of equality. If $a = b$ then $a \cdot c = b \cdot c$.

d. Division property of equality. If $a = b$ then $a \div c = b \div c$ if $c \neq o$.

II. Positive and Negative Numbers

A. Review of positive and negative numbers

Suggestion 1 - Use measurement situations such as temperature and sea level.

Suggestion 2 - Use life situations - profit and loss, stock market, games (monopoly, elevator.

Suggestion 3 - Use the number line as a visual representation of negative numbers.

Suggestion 4 - Present students with an equation where solution is a negative number. Example: $2x + 4 = 3$

Suggestion 5 - Closure of positive (rationals) under subtraction can be used as a formal justification. This suggestion for the presentation of negative numbers can be used with more able classes. For convenience, we will refer to these numbers as signed numbers.

B. Representation of negative and positive numbers

1. With signs

a. Many texts use a raised symbol to distinguish the number direction from the operation.

Example: $3-({}^{-}a)$, the lower - is a subtraction symbol, the raised one is to show a negative 2. Then a switch is made when subtraction is defined to be $a-b = a +({}^{-}b)$.

b. Some texts do not distinguish between a symbol of operation and a symbol of direction. Some explanation that a bar can be used for both operation and direction should be pointed out. There are actually 3 uses of the bar. For negative integers:

(1) -3 is the integer which is the opposite of 3. i.e., -3 is the solution of $3 + x = 0$.
(2) The opposites of any kind of number ${}^{-}(-3) = 3$, ${}^{-}(-\frac{1}{2}) = \frac{1}{2}$

(3) For subtraction

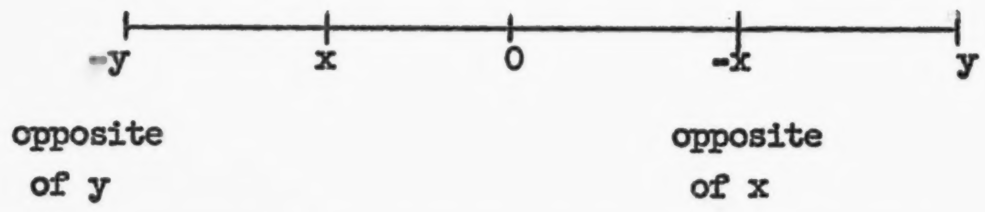
$$2 \overset{\textcircled{c}}{-} \overset{\textcircled{b}}{-} \overset{\textcircled{a}}{-} (3) = -1$$

c. Use of the words "opposite of" (direction) is helpful when referring to -x.

Example: If x = -3 then the opposite of x is 3. It turns out "opposite of" and "additive inverse" are different names for the same number.

2. On number line

a. The last example in the previous section can be shown nicely on the number line. "Opposite of" means "the same distance on the other side of 0".



b. Opposites are additive inverses - that is the sum is zero. Point out that these are synonyms.

c. The students may think of $-(a + b) = (-a) + (-b)$ as a use of the distributive property.

$$-(a + b) = (-1)(a + b) = (-1 \cdot a) + (-1 \cdot b)$$
$$= (-a) + (-b)$$

C. Operations with positive and negative numbers

1. Addition

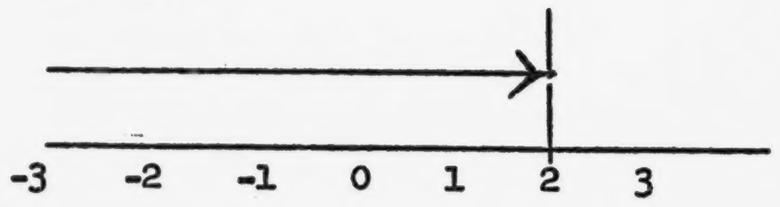
Intuitive approaches

a. Work on the number line is urged.

(1) One way of using the number line is to use only one directed line starting at the location of the first number.

Adding a positive number to the original number moves the sum to the right.

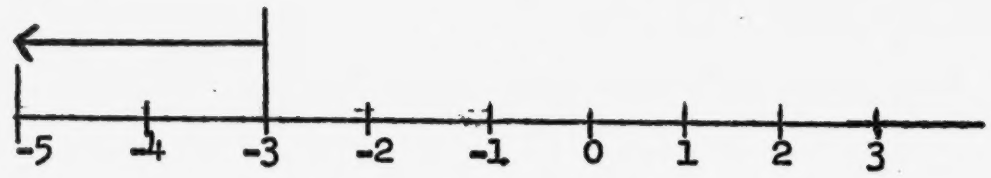
Example: -3 + 5



The sum is positive 2.

Adding a negative to the original number moves the sum to the left.

Example: -3 + (-2)

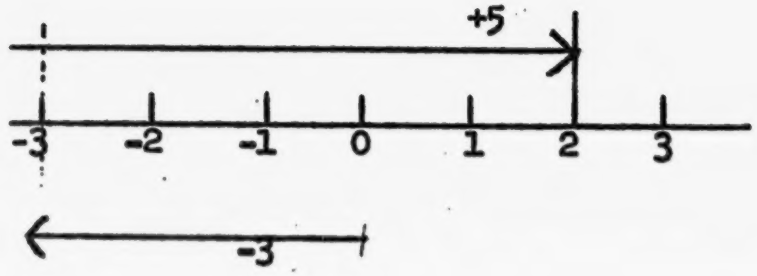


The sum is -5.

The advantage of this method is that the use of only one directed line makes the inverse operation of subtraction more obvious.

(2) Starting point is always the origin. Both numbers are shown by directed lines (vectors). The sum is the arrow head of the second vector.

Example: -3 + 5



(a) Use of the idea of "displacement" in number line operations may clarify the concept of "moving to the left or to the right." The two ideas are equivalent.

(b) Give problems needing addition of signed numbers to be done intuitively.

Example: A submarine has a polaris missile fired when it is 350 feet below sea level. The missile rises 1150 feet from the submarine. How far is the missile above the surface of the water?

(c) Patterns such as a grid could be used to show the rules of addition informally.

2. Absolute value definition and use

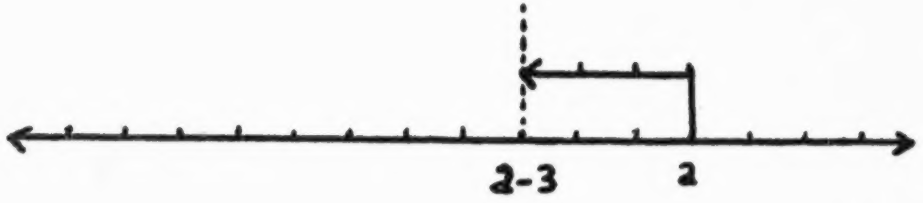
a. Absolute value of a number is the distance between 0 and that number. Distance is only positive or 0.

b. The formal rules of addition of signed numbers are given in terms of absolute value in many texts. Suggestion: it is more difficult to say those rules than to use them. It is, therefore, advised that the students not memorize them rotely at first. It is necessary to understand a definition before memorizing it.

3. Subtraction of signed numbers

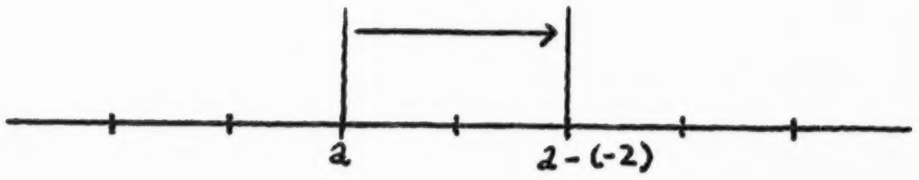
a. Use of only one directed line segment. The number line approach, using the idea that subtraction is the inverse of addition, is one possibility. See C1 above for comparison.

Subtracting a positive number from the original number moves the difference to the left. a - 3.



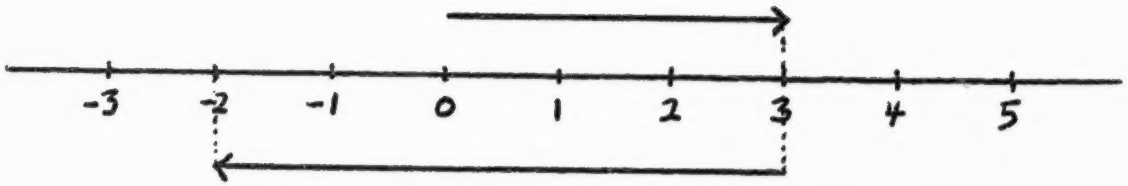
Subtracting a negative number from the original number moves the difference to the right.

a - (-2)



b. Use of two directed lines.

Example: 3 - 5



(1) From examples such as these, students become aware that when they subtract a negative number, they actually are adding a positive number. a - (-2) = a + 2 - also when subtracting a positive, it is addition of a negative a - 3 = a + (-3)

(2) Give problems using subtraction intuitively.

Example: The Greek Mathematician Archimedes was born in 287 B.C. and died in 212 B.C. How long did he live?

(3) Through many examples, as explained above, the student should now generalize to the formal rule of subtraction:

a - b = a + (-b)

(4) Now the student can be shown that because

$$a + (-b) + b = a$$

(addition of inverses = 0)

$$\text{then} \quad a + (-b) + b - b = a - b$$

$$a + (-b) = a - b$$

That is, subtraction is the inverse of addition.

4. Multiplication of signed numbers

Several approaches give some meaning to the formal rules. We suggest holding off these rules until later.

a. Have students see the patterns in the following grid. Something similar can also be worked out for addition and subtraction.

Suggestion is made to have students write out their observations as well as filling in the grid.

·	3	2	1	0	-1	-2	-3
3	9	6	3	0			
2	6	4	2	0			
1	3	2	1	0			
0	0	0	0	0			
-1							
-2							
-3							

b. It is easy to explain 3(-5) means (-5) + (-5) + (-5) but not easy to explain how to take 5 a negative 3 number of times. (The commutative law will help: (-3) x 5 = 5 x (-3)).

A good example to explain multiplication of positive and negative numbers is a discussion of water flowing *into* a tank at the rate of 5 gal./min.

Three minutes from now (+3) there will be 15 gal. more in the tank.

Three minutes ago (-3) there were 15 gal. less (-15) in the tank.

If the water is flowing *out of* the tank at 5 gal./min. (-5) then the following happens:

Three minutes from now there will be 15 gal. less (-15)

Three minutes ago (-3) there were 15 gal. more (+15).

c. Postponement until later in the year of having students doing deductive proofs is recommended.

Example: Prove $(-a)(b) = -ab$ for all a,b.

Many texts show these proofs; others ask students to prove them. With a very advanced class, proofs may be possible. Through formal approaches, the *statement* (not proofs) of these theorems may carry some meaning.

Example: a and b > 0 implies $ab > 0$ -

Example: $(-a)(-b) = ab$ etc.

d. If the student will accept as a property $-a = (-1)(a)$, then all these rules follow using other properties.

Example:

$$\begin{aligned}(-a)(b) &= (-1)(a)(b)\\ &= (-1)\ (a)(b)\\ &= (-1)(ab) = -(ab)\end{aligned}$$

5. Division of signed numbers

a. The numerical value of the division is the same as with positive numbers but the problem for the student is the sign.

$\frac{+}{+}$ the student can see this has always been positive.

$\frac{+}{-} = ?$ can be restated as $(-)(?) = +$ since division is the inverse operation of multiplication. From multiplication, the student can see ? has to be - . Same can be done for other cases:

$$\frac{-}{+} = - \text{ and } \frac{-}{-} = +$$

b. If one accepts that $-a = (-1)\ a$

$$\text{and } \frac{-1}{-1} = 1, \ \frac{-1}{1} = \frac{1}{-1} = -1$$

all the student needs to remember is that the bar (-) can be thought of as a negative one.

c. A difficult concept is that $\frac{+}{-}$, $\frac{-}{+}$ are both the same as $-\frac{+}{+}$. This can be shown by use of the reciprocal (multiplication inverse) and use of multiplication rules of signed numbers and that the product of a number and its reciprocal can only be $= +1$.

Ex: $\frac{a}{-b} = a(-\frac{1}{b}) = -\frac{a}{b}$

Remind students that division by zero is not defined.

d. With the acceptance of ideas previously outlined:

$$\frac{a}{-b} = \frac{1}{-1} \cdot \frac{a}{b} = -1 \cdot \frac{a}{b} = -\frac{a}{b}$$

e. Use of familiar problems will provide some meaning to the students.

Example: A scuba diver dove these depths: -30, -35, -45, -50 ft.
What is his average dive?

D. Equation and inequality solutions

1. Algebraically for equations

The properties of equality used in Chapter I are also applicable to directed numbers.

2. Algebraically - for inequalities

a. Properties for addition and subtraction of signed numbers in inequalities

If $a < b$ then $a + c < b + c$ addition property of inequality

If $a < b$ then $a - c < b - c$ subtraction property of inequality

The order of inequality stays the same (can then handle similar to equations).

b. Properties for multiplication and division

The formal properties.

If $a < b$ and $\underline{c > 0}$ then $ac < bc$ multiplication property of inequality

and $\frac{a}{c} < \frac{b}{c}$ division property of inequality

The order of the inequality remains the same.

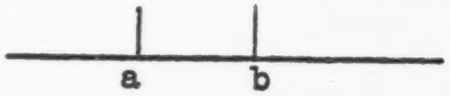
However, when $\underline{c < 0}$

and $a < b$ then $ac > bc$

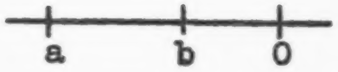
and $\frac{a}{c} > \frac{b}{c}$

The order of inequality reverses.

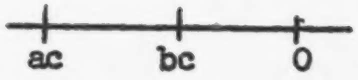
This can be shown intuitively on the number line.



Put zero point any place - or try all possible places.

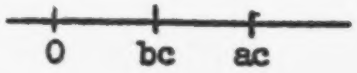


If $c > 0$



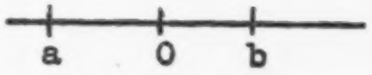
$ac < bc$

If $c < 0$

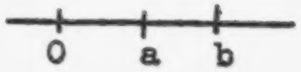


$ac > bc$

Also show the cases



and



In all cases, when $c > 0$, the inequality order stays the same and when $c < 0$, it reverses. Numerical examples may be helpful.

c. For able students the formal proof may be appealing.

$a < b$ Given

$c < 0$ Given

$a - b < 0$ Subtraction property of order.

$(a - b)c > 0$ Product of two negative numbers is positive

$ac - bc > 0$ Distributive property

$ac > bc$ Addition property of order.

At best, this topic is a difficult one to teach! Be careful not to spend an excessive amount of time on it.

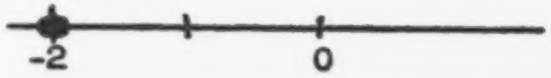
3. Geometrical solutions of open sentences in one variable

This topic is handled nicely in several of the more recent texts.

a. Equations

Students should observe there is only one solution.

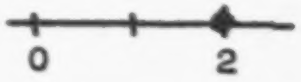
Example: $3x + 5 = -1$



b. Equations containing absolute value

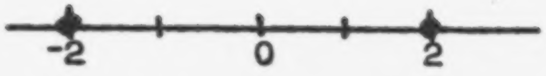
Stress the use of definitions and also give many examples.

Example: 1.) $x = |2|$
one solution



Example: 2.) $|x| = 2$
two solutions

$x = 2$ or
$x = -2$

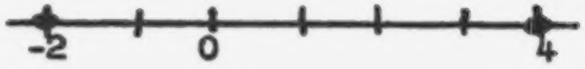


Example: 3.) $|x - 1| = 3$

two solutions

Stress $|\;\;|$ means that that number is a distance of (3 in this case) - from the origin. Distance has no direction. Therefore it can be 3 or - 3.

So $x - 1 = 3$
or
$x - 1 = -3$



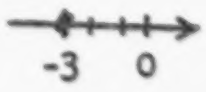
4.) $|x - 1| = -3$

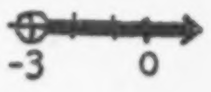
No solutions because distance can have only positive or zero values.

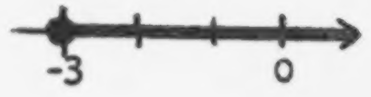
c. Inequalities with $\leq$, $\geq$ symbols

Another form of the or solutions is the type of open sentence $x \geq -3$. This uses union of sets. Simple sentences can be graphed by most students.

$x = -3$ $x > -3$ $x \geq -3$

The union of 

and 

is 

d. Inequalities of form $a < x < b$

This is a form of an and sentence - using interesection of sets.

$a < x < b$ means $x > a$ and $x < b$

E. Solution of problems using positive and negative numbers

Review of more formulas can be done (such as distance formula.)

1. Solutions needing negative numbers

Due to lengthy discussion of this topic in the unit on Numbers, Variables, etc., this place just provides more practice for the student, with negative solutions now possible also.

2. More complicated inequality problems than in unit on Numbers, Variables, etc.

a. Review of some words used for $<$, $>$, $\leq$ $\geq$ such as less than, more than, at most, at least, will aid students in setting up sentences in inequality problems.

b. Some good problems in reference (11) p. 186-188 for example.

3. Formulas

Use negative numbers for evaluation.

Example: $F = \frac{9C}{5} + 32$ when

$C = -20°$

$F = -100°$

III. Powers, Polynomials, Products, Factors

A. Factors and primes

1. Definitions: unique factorization theorem, greatest common factor (G.C.F.), least common multiple (L.C.M.), etc.

   Review according to need. Due to a great deal of work on these topics in 7th and 8th grade, determination of how much the students know about them should be made. Refer to 7th and 8th grade materials for suggestions on approach.

2. Use of factors and primes in operations with rational numbers

   a. Show how G.C.F. is used in reducing fractions.
   b. Show how L.C.M. is used in addition and subtraction of rationals.

      (1) Transition from arithmetic rationals to algebraic rationals can be done at this time. It is assumed that students are familiar with variables being names for numbers. This is a good spot for review of arithmetic operations.

      e.g., the analogy between working the problem $\frac{1}{2} + \frac{1}{3}$ and $\frac{1}{x} + \frac{1}{y}$ can be pointed out.

      Also reducing

      $$\frac{12}{30} = \frac{2\cdot 2\cdot 3}{5\cdot 2\cdot 3} = \frac{2}{5} \quad \text{and} \quad \frac{x^2y}{xyz} = \frac{x}{z}$$

      (2) This is another place to point out that the factors in the denominators $\neq 0$ for then it would be undefined.

      i.e., $x, y, z \neq 0$

B. Exponents

1. Scientific notation

   You can review solution of numerical problems by means of scientific notation.

Example: $\frac{3200}{.04} = \frac{(3\cdot 2)\ (10^3)}{(4)(10^{-2})} = (.8)\ (10^{3-(-2)}) = (.8)(10^5) = 80{,}000$

This can lead into the next section.

2. Laws of exponents

   a. Define power, base factor

      In most instances, this will be review.

   b. Multiplication of two factors of same base means exponents can be added.

      One approach is to continue working with base ten; $(a^m)\ (a^n) = a^{m+n}$:

      Examples: (1) $(10^2)(10^3) = 10^{2+3} = 10^5$

      (2) $(10^2)(10^3) = (10\cdot 10)(10\cdot 10\cdot 10) = 10^5$

      (3) $(8^3)\ (8^4) = 8^{3\cdot}$

   c. Division of two factors of same base means exponents can be subtracted. (If numerator exponent > denominator exponent.)

      The inverse operation of multiplication can be pointed out; $\frac{a^m}{a^n} = a^{m-n}$
      $a \neq 0$:

      Examples: (1) $\frac{10^3}{10^2} = 10^{3-2} = 10^1 = 10$

      (2) $\frac{10^3}{10^2} = \frac{10\cdot 10\cdot 10}{10\cdot 10} = 10^1 = 10$

      (3) $\frac{6^8}{6^3} = 6^{8-3} = 6^5$

d. Define

These definitions are given so that the division property in "c" will still hold even when a = b; $\frac{a^m}{a^n}$ is $a^0 = 1$ if $m = n$; $a \neq 0$:

Examples: (1) $\frac{10^a}{10^a} = 10^0 = 1$ (2) $\frac{6^4}{6^4} = 6^0 = 1$

(3) $\frac{10^a}{10^b} = 10^{a-b} = 10^0$ but when a = b

$\frac{10^a}{10^b} = 1$ so $10^0 = 1$

e. These definitions are given so that the division property in (c) will still hold even when $a < b$;

$$a^{-m} = \frac{1}{a^m} \qquad a \neq 0$$

Examples: (1) $a < b$, for example $a = 3$, $b = 5$

$\frac{10^a}{10^b} = 10^{3-5} = 10^{-2}$

and $\frac{10^3}{10^5} = \frac{1}{10^2}$ so $10^{-2} = \frac{1}{10^2}$

(2) $\frac{6^8}{6^{12}} = 6^{8-12} = 6^{-4}$

and $\frac{6^8}{6^{12}} = \frac{1}{6^4}$

f. Power of a product $(ab)^m = a^m b^m$

This can be developed from definition of a power and commutativity.

g. Power of a power $(a^m)^n = a^{mn}$

This can be shown from definition of power and definition of addition of exponents.

C. Polynomials - Operations

1. Definitions of coefficients, degree, polynomial, terms (like and unlike), factors

Some teachers make use of these words assuming they are communicating with the students when such may not actually be the case. Discussion of the vocabulary as well as mentioning prefixes and roots of words like monomial, binomial, etc. is advised.

2. "Standard" arrangement of polynomials - in descending (or ascending) order of the power of one of the variables.

Necessity for arrangement is in evidence when dividing.

Convenience in arrangement of polynomials is shown in factoring, adding and subtracting.

3. Operations

a. Addition

(1) This can be approached from the idea of adding "like" terms (if you have the same power of the same variables in two terms, the coefficients can be added).

Example: $8a^2b + 5ab + 2ab = 8a^2b + 7ab$

(2) Use of the distributive property is another approach.

Example: $8a^2b + 5ab + 2ab = 8a^2b + (5 + 2)\ ab = 8a^2b + 7ab$

(3) Use analogy of arithmetic

Example: $31 + 52 = 3(10) + 1 + 5(10) + 2$

Units are added to units, tens to tens, etc.

b. Subtraction

This can be taught in terms of addition.

(1) Subtraction of a number (expression) is the same as addition of the opposite of that number.

Example: $3a - (2a - 5)$ means "add the opposite of $2a - 5$ to $3a$"

$3a + (-2a) + 5 = a + 5$

also, by using multiplicative identity and distributive property:

$3a - (2a - 5) = 3a + -1\,(2a - 5) = 3a - 2a + 5 = a + 5$

(2) Remind students that checking of subtraction is done by addition - the inverse operation

$(2a - 5) + (a + 5) = 3a$

c. Multiplication

Product of a monomial and a polynomial

This is the distributive property and the use of exponent properties.

Example: $2ab^2\,(a - 7b + 4) = 6a^2b^2 - 14ab^3 + 8ab^2$

Product of two polynomials

(1) Show analogy of multiplying two two-digit arithmetic numbers and two binomial algebraic expressions

$$\begin{array}{r}
34 = 3(10) + 4 \\
\underline{21 = 2(10) + 1} \\
3(10) + 4 \\
\underline{6(10)(10) + 8(10)} \\
6(10)(10) + 11(10) + 4
\end{array}
\qquad\qquad
\begin{array}{r}
3x + 4y \\
\underline{2x + y} \\
3xy + 4y^2 \\
\underline{6x^2 \quad + 8xy} \\
6x^2 \quad + 11xy + 4y^2
\end{array}$$

(2) The distributive property can again be applied.

$$\begin{aligned}(x + y)(v + w) &= (x + y)v + (x + y)w \\ &= xv + yv + xw + yw\end{aligned}$$

Because of the frequency of the need to find the product of two binomials, a short cut method should be encouraged. See next suggestion.

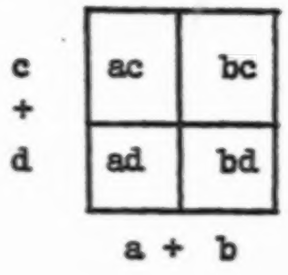
(3) Eventually the students can see the pattern of FOIL for finding the product of two binomials.

Multiply: F(first terms), O(outer terms), I(inner terms) L(last terms).

Then add: F + O + I + L.

(4) A geometric approach by areas may enlighten students about FOIL.

$(a + b)(c + d)$



Area of the rectangle = ac + ad + bc + bd.

(5) By the use of the distributive property the multiplication of polynomials may be done in this manner:

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$$

also $(a + b)(c + d + e) = a(c + d + e) + b(c + d + e) =$

$$(ac + ad + ae) + (bc + bd + be)$$

(6) Applications of multiplication of polynomials often arise in solution of area problems.

Example: The length of a rectangle is 3 feet greater than some number and the width is 2 feet less than the same number. The area of the rectangle is 24 square feet. What are the dimensions?

$$(x + 3)(x - 2) = 24$$

See section 4 on problem solving using factoring.

d. Division of a polynomial:

(1) By a monomial

(a) Use numerical examples to show students the common error of "cancelling."

$$\frac{41}{3} = \frac{35 + \overset{2}{\cancel{6}}}{\cancel{3}} \neq 35 + 2 = 37$$

(b) Use distributive property and definition of division and properties of 1.

Example: $\dfrac{6a^2b + 12ab^2}{3ab} = \dfrac{1}{3ab}(6a^2b + 12ab^2)$

$$= \frac{1}{3ab}(6ab)(a + 2b)$$

$$= 2(a + 2b) = 2a + 4b$$

(2) By a polynomial

(a) Again you can use the analogy with arithmetic division.

$$\frac{342}{14} = \quad \begin{array}{r} \left.\begin{array}{r} 4 \\ 20 \end{array}\right\} 20 + 4 \\ 14\overline{)342} \\ \underline{280} \\ 62 \\ \underline{56} \\ 6 \end{array} \qquad\qquad \begin{array}{r} \left.\begin{array}{r} 2 \\ x \end{array}\right\} x + 2 \\ x + 1\overline{)x^2 + 3x + 3} \\ \underline{x^2 + x\phantom{{}+3}} \\ 2x + 3 \\ \underline{2x + 2} \\ 1 \end{array}$$

$$20 + 4 + \frac{6}{14} = 24\,\frac{3}{7} \qquad\qquad x + 2 + \frac{1}{x + 1}$$

(b) Remind students that checking is done by the inverse operation, multiplication, then add the remainder.

$$(24)\,(14) + 6 = 342 \qquad\qquad (x + 2)\,(x + 1) + 1 = x^2 + 3x + 3$$

(c) Synthetic division is an approach that an able class might understand at this level. The mechanics could certainly be accomplished, but the understanding would require more time. This is explained in the Algebra 2 write-up.

D. Factors and products of polynomials (as reverse operations)

Before factoring, be sure students are well-grounded in the multiplication of polynomials.

Students will be using factoring of algebraic expressions to aid them in work with algebraic fractions--much the same as with arithmetic. Therefore, point out the necessity for complete factoring--into prime factors (leave the door open that a factor may be prime over the rationals but not a prime considered as a polynomial over the reals.) This, then is a unique factorization--as with arithmetic.

1. Common monomial factor

   a. Point out that students should first look for the greatest monomial to aid in complete factorization.

      $5x + 15x^2 + 20xy^2 = 5x(1 + 3x + 4y^2)$

      Have students multiply to check this type of factoring.

   b. This is a use of the distributive property.

2. Difference of two squares-product of sum and difference of two numbers

   a. Have the students look for a pattern from multiplying:

      $(x + 3)(x - 3)$

      $(2 - y)(2 + y)$

      $(3x - 5)(3x + 5)$

      and ultimately induct that

      $(\square + \triangle)(\square - \triangle) = \square^2 - \triangle^2$

      Reverse (inverse) the process to factor $\square^2 - \triangle^2$

   b. Use the above pattern to calculate arithmetic problems:

      $(20 + 2)(20 - 2) = 400 - 40 + 40 - 4 = 400 - 4 = 396$

      $(30 + 1)(30 - 1) = 900 - 30 + 30 - 1 = 900 - 1 = 890$

      Generalization of the product of the sum and difference of the same two numbers may be made from arithmetic examples alone.

c. Perhaps the pattern can be seen more readily in #1 if we:

Let $n = (a - 2)$

Let $m = (x - 3)$

then $(n)^2 - (m)^2 = (n - m)\,(n + m) =$

substituting back in:

$$\left[(a - 2) - (x - 3)\right]\left[(a - 2) + (x - 3)\right]$$

$$(a - 2 - x + 3)\,(a - 2 + x - 3)$$

$$(a - x + 1)\,(a + x - 5)$$

d. Patterns should be helpful to the student in factoring "tricky" problems.

Examples: (1) $(a - 2)^2 - (x - 3)^2 = \left[(a - 2) - (x - 3)\right]\left[(a - 2) + (x - 3)\right]$

(2) $a^4 - 16$ (complete factoring) $= (a^2 - 4)\,(a^2 + 4)$

$$(a - 2)\,(a + 2)\,(a^2 + 4)$$

e. The difference of two squares can also be treated as a quadratic "trinomial." This can be done if $\square^2 - \triangle^2$ is thought of as $\square^2 + 0 \cdot \square - \triangle^2$.

3. Quadratic trinomial

$Ax^2 + Bx + C$,

when $A = 1$

a. Have students look for patterns when multiplying:

$(x + 3)\,(x + 2)$ $(y - 5)\,(y - 7)$ $(z + 6)\,(z - 3)$

Ultimately this can lead to the induction that

$$(x + \square)(x + \triangle) = x^2 + (\square + \triangle)x + \square\triangle$$

Then reverse the process to factor $x^2 + Bx + C$, if possible.

b. Have students practice finding a pair of numbers whose sum is one number and whose product is another.

Example: Sum 5, product -36. The pair is 9 and -4. Apply this to factor $x^2 + 5x - 36$. This is usually done by taking all possible pairs of factors of -36 and finding which pair gives a sum of 5.

c. It is recommended that the factoring of a perfect square trinomial be handled as a special case of the quadratic trinomial. This leaves the student with fewer "cases" to worry about. Speed in squaring a binomial is desirable and the pattern of:

$$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$$

can be developed. Use of arithmetic examples can be made.

$$(37)^2 = (40 - 3)^2 = 40^2 = 2(40)(3) + 9$$

$$= 1600 - 240 + 9 = 1369$$

4. Quadratic trinomial

$Ax^2 + Bx + C$

$A \neq 1$

a. The "look and try" method of factoring a trinomial such as $20x^2 - 9x - 20$ is described in any algebra text. The number of possibilities can be cut down by observing that the original trinomial has no common monomial factor. Therefore, no factor of the trinomial will have a common factor.

Examples: $(4x - 10)(5x + 2)$ isn't possible because

$$4x - 10 = 2(2x - 5)$$

5. Grouping to factor

The use of the distributive property can be pointed out.

Example: $ac + ad + bc + bd =$

$$a(c + d) + b(c + d) = (a + b)(c + d)$$

E. Problem solving using factoring

Use of the principle that if $ab = 0$ then either $a = 0$ or $b = 0$.

1. Have students try to think of a product being 0 and not having one of the factors be 0.
2. This may be a good place to introduce a proof of the principle because many examples will not show the truth for all pairs of numbers. Also, this proof is fairly straight forward.

$ab = 0$

Assume $a \neq 0$ Have students provide

$\frac{1}{a} ab = 0 \cdot \frac{1}{a}$ the reasons.

$1 \cdot b = 0$

$b = 0$

3. Give practical problems showing the use of quadratic factoring. These can also include roots that are not valid in the context of the problem.

Example: Use equation $d = rt + 16t^2$ (number of feet, d, an object falls in t seconds when propelled downward at a starting rate r in ft./ sec.).

An object is thrown from an airplane at 48 ft./ sec. flying at an altitude of 11,200 ft. How soon does it reach the ground?

$$11{,}200 = 48t + 16t^2$$

$$t^2 + 3t - 700 = 0$$

$$(t + 28)\ (t - 25) = 0$$

$$\begin{array}{l|l} t + 28 = 0 & t - 25 = 0 \\ t = -28 & t = 25 \end{array}$$

invalid root

4. Other suggestions for problem solving from unit on positive and negative numbers are still valid here.

IV. Rational Numbers - Operations On Expressions, Sentences, Problem Solving

A. Rational numbers and rational expressions

1. Definitions

a. Rational number

(1) When defining (reviewing) a rational number as any number which may be expressed as the quotient of two integers such as the ratio $\frac{a}{b}$, stress the fact that $b \neq 0$.

This is used throughout the unit (as well as throughout mathematics).

(2) It might be well to point out the same roots to the words ratio and rational at this time. When ratio and proportion are used later in the unit, it may seem more natural to the student.

b. Irrational number

If some student asks "what is not a rational number?" this provides an opportunity to introduce the notion of an irrational number. See introduction to the later unit on Irrational Numbers.

c. Rational expression

Analogy of rational numbers and rational expressions can be utilized throughout the unit.

(1) The definitions follow closely.

A rational expression is an expression that is an indicated quotient of two polynomials, $P, Q : \frac{P}{Q} \quad Q \neq 0$

(2) When $b = 1$, $\frac{a}{b}$ is an integer. The integers are a subset of the rationals. When $Q = 1$, $\frac{P}{Q}$ is a polynomial. The polynomials are a subset of the rational expressions.

Examples: $\frac{3}{1} = 3$ $\frac{x^2 + 3}{1} = x^2 + 3$

(3) A rational expression is also a ratio.

B. Operations on rational expressions

Constant uses of factoring and properties of 1 are made in this section. This may make these concepts more meaningful to the students if the uses are pointed out.

1. Simplifying - use of the greatest common factor (G.C.F.)

a. Arithmetic - rational number versus Algebra - rational expression

Arithmetic - rational number	versus	Algebra - rational expression
$\frac{39}{102} = \frac{3 \cdot 13}{3 \cdot 2 \cdot 17}$	Prime factoring	$\frac{x^2 + x}{x^2 - 1} = \frac{x(x + 1)}{(x - 1)(x + 1)}$ $x \neq 1, -1$
$= \frac{3}{3} \cdot \frac{13}{2 \quad 17}$	Definition of Multiplication	$= \frac{x}{x - 1} \cdot \frac{x + 1}{x + 1}$
$= 1 \cdot \frac{13}{2 \quad 17}$	Division property of 1	$= \frac{x}{x - 1} \cdot 1$
$= \frac{13}{34}$		$= \frac{x}{x - 1}$

b. Point out that a rational expression is simplified when there are no common factors in the numerator and denominator. Use of the underlined words rather than the word "cancelling" may help students avoid a common error such as:

$\frac{\cancel{a} + b}{\cancel{a}} \neq b$ because a is not a factor of the numerator

Another way of pointing out this danger to the students is to remind them that when the multiplication identity property $(a \cdot 1 = a)$ can be used, then simplification is possible.

A further qualification for simplifying used in some texts is that the polynomials in the numerator and denominator must have all integral coefficients. These are called polynomials over the integers. Therefore, when you ask students to simplify, make the directions clear. The word "simplify" means different things to different people.

c. When simplifying expressions such as $\frac{a - c}{c - a}$ $(c \neq a)$

The use of the distributive property can be used to explain the simplification.

$$\frac{a - c}{c - a} = \frac{a - c}{(-1)(-c + a)} = \frac{a - c}{(-1)(a - c)} = \frac{1}{-1} = -1$$

Generalization may be made from several examples of division of opposites:

$$\frac{3}{-3} = \frac{-\frac{1}{2}}{\frac{1}{2}} = \frac{x - 3}{-(x - 3)} = -1$$

2. Multiplication and division of rational expressions

a. Tie the concept of multiplying rational expressions to the familiar notion of multiplying rational numbers.

a, b, c, d integers	A, B, C, D polynomials
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ b, d $\neq 0$	$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$ B, D $\neq 0$

Simplifying should also be stressed.

Remind students that the operations on polynomials in rational expressions should certainly behave like the operations on rational numbers since a polynomial is a name for a number.

b. Division can be explained in the same vein.

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

Multiply both numerator and denominator by the L.C.M. of the denominators

$$\frac{\frac{A}{B}}{\frac{C}{D}}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{(bd)}{(bd)} = \frac{ad}{cb} \qquad \frac{\frac{A}{B}}{\frac{C}{C}} \cdot \frac{(BD)}{(BD)} = \frac{AD}{CB}$$

$b, c, d, \neq 0$ $\qquad$ $B, C, D, \neq 0$

This is to show the student why he "inverts and multiplies" which is actually a short cut to the method shown here or the one in the next suggestion.

Another way of explaining division is:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot 1 = \frac{\frac{a}{b}\left(\frac{d}{c}\right)}{\frac{c}{c}\left(\frac{d}{c}\right)}$$

Use of reciprocals and multiplication property of 1

$$= \frac{\frac{ad}{bc}}{1}$$

Definition of multiplication of rationals

$$= \frac{ad}{bc}$$

Division property of 1

Students can be made aware that a division problem is a ratio of one rational expression to another non-zero rational expression.

3. Least common multiple - L.C.M. - used in addition and subtraction of rational expressions

a. Addition and subtraction of rational expressions generally need a common denominator since the distributive property can then be readily applied. The terms can then be considered like terms:

Example: $\frac{1}{5} + \frac{3}{5} = \frac{1}{5} \quad 1 + \frac{1}{5} \cdot 3$ $\qquad$ $\frac{5}{a} - \frac{3}{a} = \frac{1}{a} \cdot 5 - \frac{1}{a} \cdot 3 \quad a \neq 0$

$= \frac{1}{5}(1 + 3) = \frac{4}{5}$ $\qquad$ $= \frac{1}{a}(5 - 3) = \frac{2}{a}$

b. The parallel use of properties of 1 in adding and subtracting rational numbers and rational expressions can be used in examples:

$\frac{3}{25} - \frac{4}{35}$	Find L.C.M. by factoring denominators	$\frac{m}{m^2 - 2m + 1} - \frac{n}{m^2 - m}$
$= \frac{3}{5\ 5} - \frac{4}{5\ 7}$		$= \frac{m}{(m - 1)\ (m - 1)} - \frac{n}{m(m - 1)}$
$5^2 \quad 7$	L.C.M. is	$(m - 1)^2 \quad m$
$= \frac{3}{5^2}\left(\frac{7}{7}\right) - \frac{4}{5\ 7}\left(\frac{5}{5}\right)$	Multiply by a form of 1 to make each denominator the L.C.M.	$= \frac{m}{(m - 1)^2}\left(\frac{m}{m}\right) - \frac{n}{m(m - 1)}\ \frac{(m - 1)}{(m - 1)}$
$= \frac{21 - 20}{5^2 \cdot 7}$	Use distributive property (or combine numerators)	$= \frac{m^2 - n(m - 1)}{(m - 1)^2 \cdot m}$
$= \frac{1}{5^2 \cdot 7}$	Combine and simplify if possible	$= \frac{m^2 - nm - n}{(m - 1)^2 \cdot m}$

Here is a place to be explicit about "simplify." Some teachers may prefer the multiplication step to be completed.

$\frac{1}{175}$ $\qquad$ $\frac{m^2 - nm - n}{m^3 - 2n^2 + m}$

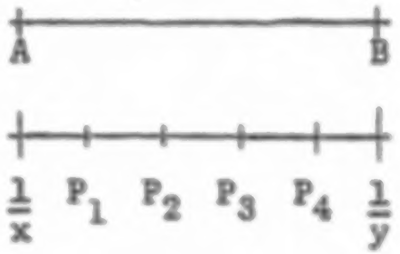
c. Practice in recognizing the opposite of rational expressions (the expression which when added to the original expression, gives a sum of 0) may help the student in avoiding the dangers of subtraction.

Example: Knowing that the opposite of $\frac{x - 2}{x + 1}$ is $\frac{(-x) + 2}{x + 1}$ may help in writing

$$\frac{1}{x} - \frac{x - 2}{x + 1} \text{ as } \frac{1}{x} + \frac{(-x) + 2}{x + 1} \qquad x \neq, -1$$

d. Use of addition and subtraction of rational expressions on a number line is possible.

Example:



Take segment AB and divide it into 5 equal portions.

Call A, $\frac{1}{x}$ and B, $\frac{1}{y}$ $\qquad x, y \neq 0$

A question can be posed to the students:

"What are the names for the points P_1, P_2, P_3, P_4?"

$$\frac{\frac{1}{y} - \frac{1}{x}}{5} = \text{length of each division}$$

Therefore $\frac{x - y}{5xy}$ can be added or subtracted to find names for other points.

$$P_1 \text{ is } \frac{1}{x} + \frac{x - y}{5xy} = \frac{5y + x - y}{5xy} = \frac{4y + x}{5xy}$$

4. Complex fractions

a. Numerical examples can introduce the topic

Example: $\dfrac{\frac{1}{2} - \frac{3}{4} + \frac{2}{3}}{\frac{3}{5} + \frac{5}{6}}$

Point out that this is a complex fraction because it contains fractions in the numerator and/or denominator.

b. Simplification of complex fractions can be accomplished by multiplying the numerator and denominator by the L.C.M. of all the denominators. Point this out as a use of the multiplication property of one.

$$\frac{\left(\frac{1}{2} - \frac{3}{4} + \frac{2}{3}\right)}{\left(\frac{3}{5} + \frac{5}{6}\right)} \; \frac{(2^2 \quad 3 \quad 5)}{(2^2 \quad 3 \quad 5)} = \frac{30 - 3(15) + 2(20)}{3(12) + 5(10)} = \frac{25}{86}$$

$$\frac{\left(a + \frac{1}{x}\right)}{\left(a - \frac{1}{x}\right)} \; \frac{(x)}{(x)} = \frac{ax + 1}{ax - 1} \qquad x \neq 0 \qquad a \neq \frac{1}{x}$$

c. Another method of approach is to carry out the addition and subtraction of fractions in the numerator and denominator and then divide.

$$\frac{a + \frac{1}{x}}{a - \frac{1}{x}} = \frac{\frac{ax + 1}{x}}{\frac{ax - 1}{x}} = \frac{ax + 1}{x} \cdot \frac{x}{ax - 1} = \frac{ax + 1}{ax - 1}$$

C. Sentences containing fractional coefficients (including decimals)

1. Multiplication by the L.C.M. of the denominators is one method of approach. This is a use of the multiplication property of equality (or inequality).

Example #1: $\frac{x}{8} - \frac{x}{12} = \frac{1}{8}$ $\qquad$ $24\left(\frac{x}{8} - \frac{x}{12}\right) = \left(\frac{1}{8}\right) 24$

$3x - 2x = 3$

$x = 3$

Example #2: $.5y - 1.4y = .9$

$10\ (.5y - 1.4y) = 10\ (.9)$

Sentences containing decimal coefficients can also be rewritten as fractions.

Example #1: $\frac{5}{10}\ y - \frac{14}{10}\ y = \frac{9}{10}$

$10\left(\frac{5}{10}\ y - \frac{14}{10}\ y\right) = 10\left(\frac{9}{10}\right)$

Example #2: $\frac{n}{-5} + 2 < -3$

$(-5)\ \left(\frac{n}{-5} + 2\right) > (-5)\ (-3)$

Note:
This is an opportunity to reemphasize when the order of inequality is reversed.

Distributing, then adding and subtracting the coefficients is sometimes preferred by the students.

Example: $\frac{3}{20} = \frac{a}{8} - \frac{a}{10}$

$\frac{3}{20} = \left(\frac{1}{8} - \frac{1}{10}\right) a$

$\frac{3}{20} = \left(\frac{5 - 4}{40}\right) a$

$$\frac{3}{20} = \frac{1}{40} a$$

$$a = 40 \cdot \frac{3}{20} = \frac{20 \cdot 2 \cdot 3}{20} = 6$$

2. Ratio and proportion, define ratio; define proportion

A ratio may be defined as a rational number or a rational expression.

a. Try to have students develop familiarity in recognizing a proportion as two ratios set equal to one another. The "product of means = product of extremes" method of solving a proportion is very useful in solution of some formulas.

Example: Solve for r: $\frac{Ft}{v^2} = \frac{W}{gr}$ None of the variables = 0

$$gr\ Ft = v^2 W$$

$$r = \frac{v^2 W}{g\ Ft}$$

b. Numerical examples using "product of means = product of extremes" can give students practice in recognizing proportions.

Example: Is $\frac{7}{12} = \frac{3}{5}$?

NO because

$$7 \cdot 5 \neq 12 \cdot 3$$

c. Point out that proportions have been used often in previous work. Examples are: Scale drawing, per cent problems, similar triangles problems.

3. Division by 0.

a. "Prove" to students that $2 = 1$. Have them try to pick out the fallacy in the following sequence of steps:

$a = b$	Assume
$a^2 = ab$	Multiply equation by b
$a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides
$(a - b)\ (a + b) = b(a - b)$	Factor
$(a + b) = b$	Divide both sides by a - b
Since $a = b$	
$b + b = b$	Substitute by assumption
$2b = b$	Divide by b
$2 = 1$	

b. This can lead into discussion of extraneous roots of an equation, if the students have the following background:

(1) If $ab = 0$ then $a = 0$ or $b = 0$

(2) Division by 0 is undefined

(3) Solution of rational equations

(4) Some work with finding roots of equations which must satisfy more than one condition (intersection of sets)

(5) Multiplication of both members of an equation by a non-zero number.

Example: $\frac{12}{x^2 - 4} - \frac{3}{x - 2} = -1$

Multiply equation by $(x - 2)\ (x + 2)$. This is defined in this problem only when $x \neq 2$ or -2.

$$(x^2 - 4)\left(\frac{12}{x^2 - 4} - \frac{3}{x - 2}\right) = -1\ (x^2 - 4)$$

$$12 - 3(x + 2) = -x^2 + 4$$

$$x^2 - 3x + 2 = 0 \quad \text{and} \quad x \neq 2 \quad \text{or, } -2$$

$$(x - 2)\ (x - 1) = 0 \quad \text{and} \quad x \neq 2 \quad \text{or, } -2$$

The solution set is only 1 rather than 1 and 2.

A check of all the possibilities will show that 2 gives an undefined expression.

D. Problem solving

1. Applications of equations (and inequalities) using rational numbers

Introduce problems requiring fractional representations. Draw on the use of proportions, percents, ideas from scale drawings, the science class, etc.

Example #1: A plane travels 1,204 miles in $3\frac{1}{2}$ hours. How far will it travel in $4\frac{1}{2}$ hours traveling at the same rate?

Example #2: A boy received 80%, 82%, 74% on three tests. What mark must he receive on the fourth test to have at least an 81% average?

2. Changing subject of a formula

Use of formulas from the sciences can be made which contain rational expressions.

Example: $C = \frac{en}{R + nr}$ is a formula from electrical measurement. Have the students change the subject from C to e, n, R, r.

V. Irrational Numbers

A. Introduction - What is an irrational number?

To answer this question, comparisons can be made to the rational numbers.

1. a. A possible definition of a rational number is that it is an infinite decimal expansion which can be represented by a repeating (or terminating) decimal.

Example: $4.2727\overline{27} \ldots = N$, N is rational.

To show that this agress with the definition $N = \frac{a}{b}$, a, b integers, $b \neq 0$, the following method can be practiced by students.

Example: Multiply by the power of ten coinciding with the num ber of digits being repeated.

$$\begin{array}{rrl} & 100\ N = & 427.27\overline{27}\ldots \\ & N = & 4.27\overline{27}\ldots \\ \cline{2-3} \text{Subtract to get:} & 99\ N = & 423. \end{array}$$

then $N = \frac{423}{99} = \frac{47}{11}$, with 11, 47 integers.

b. Going from the ratio of two integers to the repeating decimal form can be used as practice in division.

Example: $\frac{8}{7} = 1.\overline{42857}\ldots$

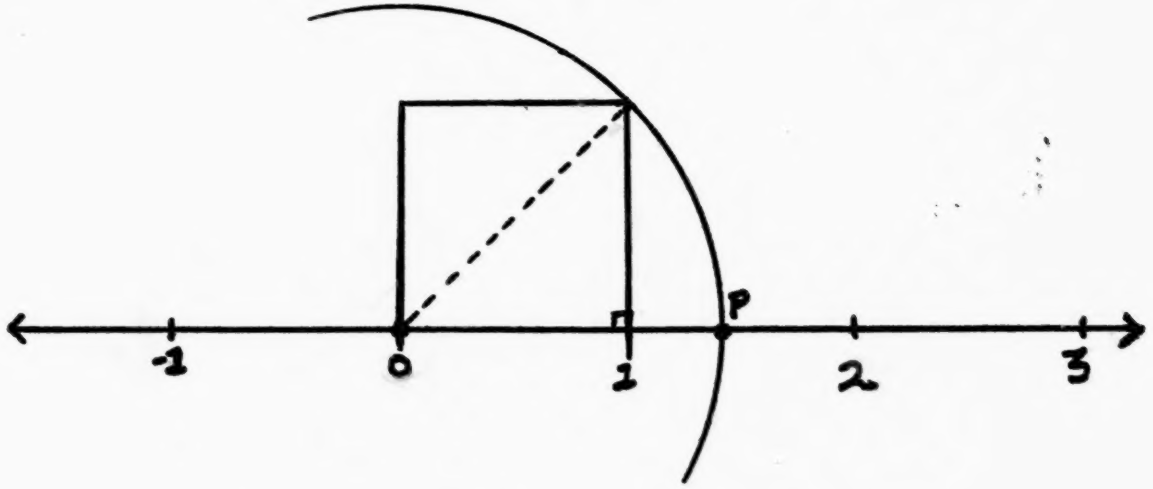
2. Irrational numbers are then numbers that have a non-repeating pattern in an infinite decimal representation. "How is one formed?" may be a question by the students.

a. Give students infinite decimal representations which have a pattern but not a repeating one.

Example: .282882888...

b. Point out that π is irrational even though it is usually approximated by a rational number: $\frac{22}{7}$ or 3.14.

3. Construct (by straight-edge and compass) a one-unit square on a number line. Using the compass, mark off the length of the diagonal on the number line. From the Pythagorean theorem--which most students have in their backgrounds--the point marked P on the diagram is the irrational number $\sqrt{2}$.



It is irrational because no matter how many rational divisions are made, P does not correspond to the representation of any of them.

Mention can be made of the dilemma of the Pythagoreans when they found the incommensurability of $\sqrt{2}$. Secrecy surrounded the discovery. For more information about the Pythagorean cult, see (7) P. 228-232.

4. With a very able class, an algebraic indirect proof that $\sqrt{2}$ is irrational may be possible. Refer to any Algebra 2 text.

5. The completion of the real number system is accomplished with irrational numbers. The students may appreciate the completion at this stage, especially after seeing the relationship to the number line.

6. Listed below are the properties of the system of real numbers which are identical with those of the system of rational numbers:

   a. The set of real numbers is closed under addition.
   b. The set of real numbers is closed under multiplication.
   c. The commutative law of addition holds.
   d. The associative law of addition holds.
   e. Zero is the additive identity.
   f. Every real number x has an additive inverse, -x.
   g. The commutative law of multiplication holds.
   h. The associative law of multiplication holds.
   i. The number 1 is the multiplicative identity.
   j. Every real number x, except zero, has a multiplicative inverse, $\frac{1}{x}$.
   k. The distributive law of multiplication over addition holds.

B. Operations with irrational numbers
   1. Terminology

      a. Radical, radicand, root of a number, principal square root, radical sign, index are vocabulary words with which the student should be familiar.

      b. Students should understand that the symbol $\sqrt{}$ refers to the positive (principal) square root only. Indices other than 2 can be introduced. Point out that $\sqrt[2]{}$ is written as $\sqrt{}$. Any index other than 2 has to be designated: $\sqrt[3]{}$, $\sqrt[4]{}$ etc.

      c. Point out that the square root operation is defined (in this course) for non-negative numbers only. Have the students try to find the $\sqrt{-9}$ after the definition is understood.

      d. In some texts $\sqrt{a^2}$ is defined to be $|a|$ (the absolute value of a) to take care of the case where a may be negative. In others, this difficulty is overcome by allowing only non-negative values for a. It is recommended by these writers that the former approach be used so the replacement set for variables can be all real numbers.

         Example: $-\sqrt{25m^2n^4} = -5|m|n^2$

Note: The n^2 doesn't have to be included in absolute value signs since it is already positive because it is squared.

e. An n^{th} root of a number should not be thought of as the inverse operation of taking the n^{th} power.

Example: $3^2 = 9$ and $\sqrt{9} = 3$ but

$(-3)^2 = 9$ and $\sqrt{9} = 3 \neq -3$

2. Finding a square root
   a. Newton's method

(1) This method could be referred to as the estimate, divide and average method of finding a square root. It is based on the principle that you are looking for a number to be used as a factor twice. This factor is then the square root of the number. Due to its reliance on definitions, it seems to stay with the student longer than the method mentioned in b.

Example: Find the square root of 28. Take an estimate - this is good practice for the student to estimate. Say 5.2.
Then divide into the number.

$$\begin{array}{r} 5.38 \\ 5.2\,\overline{)\,28.00} \\ \underline{26\;0} \\ 2\;00 \\ \underline{1\;56} \\ 440 \\ \underline{416} \end{array}$$

Then average the factors.

5.2 and 5.38

$$\frac{5.2 + 5.38}{2} = 5.29$$

5.29 then provides the next estimate. This can be carried out to as many decimal places as accuracy demands. This provides good arithmetic practice for the students. It's a lengthy procedure, however, so don't assign too many problems.

(2) A thorough discussion of this method can be found in (14) p. 326-332.

(3) Use of a desk calculator to make the computations in class can be shown as an aid in solving problems.

b. Another algorithm for calculating the square root of a number.

(1) This method is based on finding the square root of an algebraic expression. Work a numerical and algebra problem side by side:

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad\qquad (46)^2 = 1600 + 480 + 36$$

$$\begin{array}{r|l}
a & \overline{a^2 + 2ab + b^2} \\
 & \underline{a^2} \\
2a + b & \quad 2ab + b^2 \\
b & \quad \underline{2ab + b^2}
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \quad 40 + 6 \\
40 & \overline{1600 + 480 + 36} \\
 & \underline{1600} \\
2(40) + 6 & 480 + 36 \\
6 & \underline{480 + 36}
\end{array}$$

$$\text{or } 2116 = \begin{array}{r|l}
 & \;4\;6 \\
4 & \overline{\overset{\frown}{21}\overset{\frown}{16}} \\
 & 16 \\
86 & 516 \\
6 & \underline{516}
\end{array}$$

(2) The actual computations are not difficult to do but generally there isn't much understanding on the part of the students on why it works. For this reason the method in a. is being advocated in recent texts.

(3) A discussion on b. method can be found in (15) P. 432-434 and (11) P. 370-373.

c. **Finding square roots by means of a table**

The students generally need no help in doing this! For other uses of the table, see discussion on simplifying radicals Section 4.

d. **Square roots by factoring**

See Section 4 on simplifying radicals.

3. **Multiplication and division of irrational numbers.**

a. Have students try to generalize the rules from examples like the following:

(1) $\sqrt{100} = \sqrt{4.25}$
$= \sqrt{4} \cdot \sqrt{25}$
$= 2 \cdot 5$
$= 10$

(2) $\sqrt{36} = \sqrt{4.9}$
$= \sqrt{4} \cdot \sqrt{9}$
$= 2 \cdot 3$
$= 6$

(3) $\sqrt{\frac{100}{4}} = \sqrt{25} = 5$

and

$\sqrt{\frac{100}{4}} = \frac{\sqrt{100}}{\sqrt{4}} = \frac{10}{2} = 5$

and finally induct to:

$$\sqrt{ab} = \sqrt{a} \ \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

b. With a very able class, you might want to bring in the idea of fractional exponents, and $\sqrt{a} = a^{\frac{1}{2}}$. Then this topic can be tied together with the topic on exponents. When the base is the same in a multiplication problem, the exponents can be added.

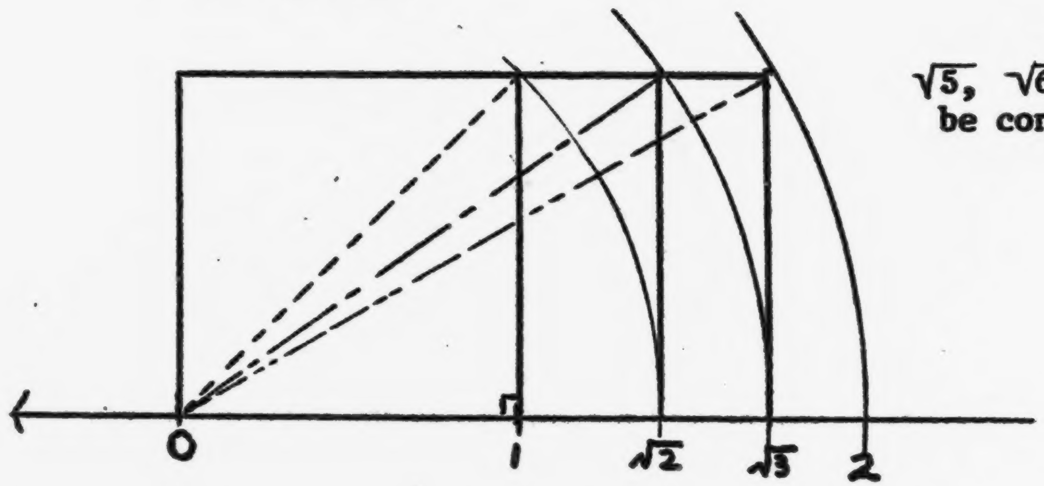
Example: If $a \neq 0, \sqrt{a}\sqrt{a} = a^{\frac{1}{2}} \ a^{\frac{1}{2}} = a^{1} = a$

When the base is the same in a division problem, the exponents can be subtracted.

c. Construct (by straight edge and compass) other irrational numbers using multiplication.

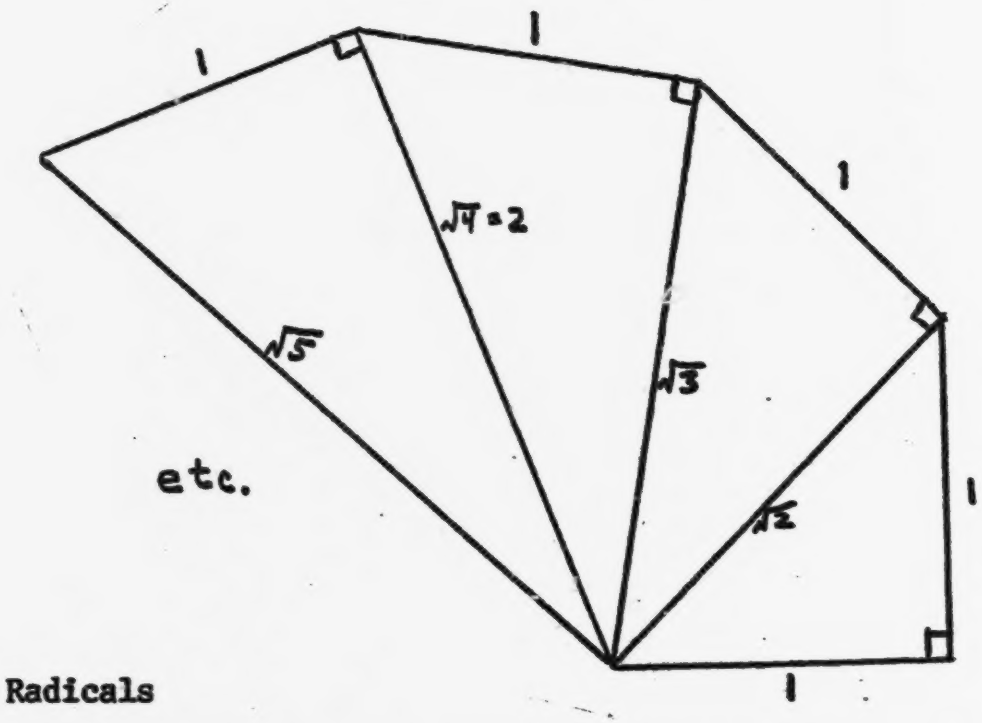
Example: $\left(\sqrt{3}^{2}\right) = 1^{2} + \left(\sqrt{2}^{2}\right)$

1 Geometrically:



$\sqrt{5}$, $\sqrt{6}$, etc. can also be constructed.

This construction is sometimes shown in the "spiral" form.



4. Simplifying Radicals

a. A radicand of a square root is considered simlified if:
   (1) There are no perfect square factors under the radical sign.

(2) The denominator is rational. This also means no fractions.

Discussion of: (1) If the radicand is factored completely, any pair of factors can be extracted using the definition of square root.

Example:

$$\begin{aligned}\sqrt{54x^3} &= \sqrt{2\cdot3\cdot3\cdot3\cdot x\cdot x\cdot x}\\ &= \sqrt{3\cdot3\cdot x\cdot x}\,\sqrt{2\cdot3\cdot x} = \sqrt{(3x)^2}\,\sqrt{6x}\\ &= 3x\sqrt{6x} \quad (x \text{ is non-negative})\end{aligned}$$

Discussion of: (2) To justify rationalizing the denominator, have students work examples such as:

$$\frac{1}{\sqrt{2}} \doteq \frac{1}{1.414} \quad \text{and} \quad \frac{2}{\sqrt{3}} \doteq \frac{1}{1.732}$$

$$\text{and then } \frac{\sqrt{2}}{2} \doteq \frac{1.414}{2} \quad \text{and} \quad \frac{2\sqrt{3}}{3} \doteq \frac{3.464}{3}$$

Example:

$$\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot 1 = \frac{3}{\sqrt{2}}\,\frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

b. Square roots of some numbers can be found now that simplifying is understood:

(1) It can be shown that the square root table is good for finding square roots of numbers other than those listed.

Example: If $\sqrt{5} \doteq 2.24$ and $\sqrt{50} \doteq 7.07$

$$\begin{aligned}\text{then } \sqrt{500} &= \sqrt{5\cdot10^2} = 10\sqrt{5} \doteq 22.4\\ \sqrt{5000} &= \sqrt{50\cdot10^2} = 10\sqrt{50} \doteq 70.7\\ \text{or } \sqrt{.05} &= \sqrt{5\cdot10^{-2}} = 10^{-1}\sqrt{5} \doteq .224\\ \sqrt{.5} &= \sqrt{50\cdot10^{-2}} = 10^{-1}\sqrt{50} \doteq 707\end{aligned}$$

(2) Some common approximate square roots are often known because of repeated use. Other square roots can be calculated from the common ones.

Example: $\sqrt{2} \doteq 1.414 \qquad \sqrt{3} \doteq 1.731$

$\sqrt{75} = \sqrt{3 \cdot 25} = 5\sqrt{3} \doteq 5 \cdot 1.732 = 8.660$

c. Factors are needed to simplify a radicand.

Example: $\sqrt{a^2 + b^2} \neq a + b$

because $(a + b)^2 \neq a^2 + b^2$

5. Addition and Subtraction

a. One approach is to have students think of radicals with the same radicands as like terms, and those with different radicands as unlike terms. They can then apply rules of addition and subtraction of Algebraic terms, by adding or subtracting coefficients.

Example: $3\sqrt{3} + 2\sqrt{2} + 2\sqrt{3} = 5\sqrt{3} + 2\sqrt{2}$

b. Simplification is done to see if terms are alike.

Example: $\sqrt{18} + \sqrt{2} = 3\sqrt{2} + \sqrt{2} = 4\sqrt{2}$

c. The distributive property can be used in showing combining or grouping of like terms:

$5\sqrt{a} + 6\sqrt{a} + 2\sqrt{3} - \sqrt{3}$

$(5 + 6)\sqrt{a} + (2 - 1)\sqrt{3} = 11\sqrt{a} + \sqrt{3}$

d. A common error by students is to think that $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$. Have them work out numerical cases to show this is not true.

Example: $\sqrt{3} + \sqrt{6} \neq \sqrt{3 + 6} = \sqrt{9} = 3$

because $1.732 + 2.449 = 4.181 \neq 3$

6. Rationalizing the denominator when it is a binomial.

Try to have the students see the application of the factors of the difference of two squares in order to rationalize a denominator such as $\sqrt{2} - 3$, by multiplying both numerator and denominator by $\sqrt{2} + 3$.

C. Solving Radical Equations.

1. Definition of a radical equation.

A radical equation is an equation containing a variable as a radicand or part of a radicand.

2. How to solve a radical equation in one variable

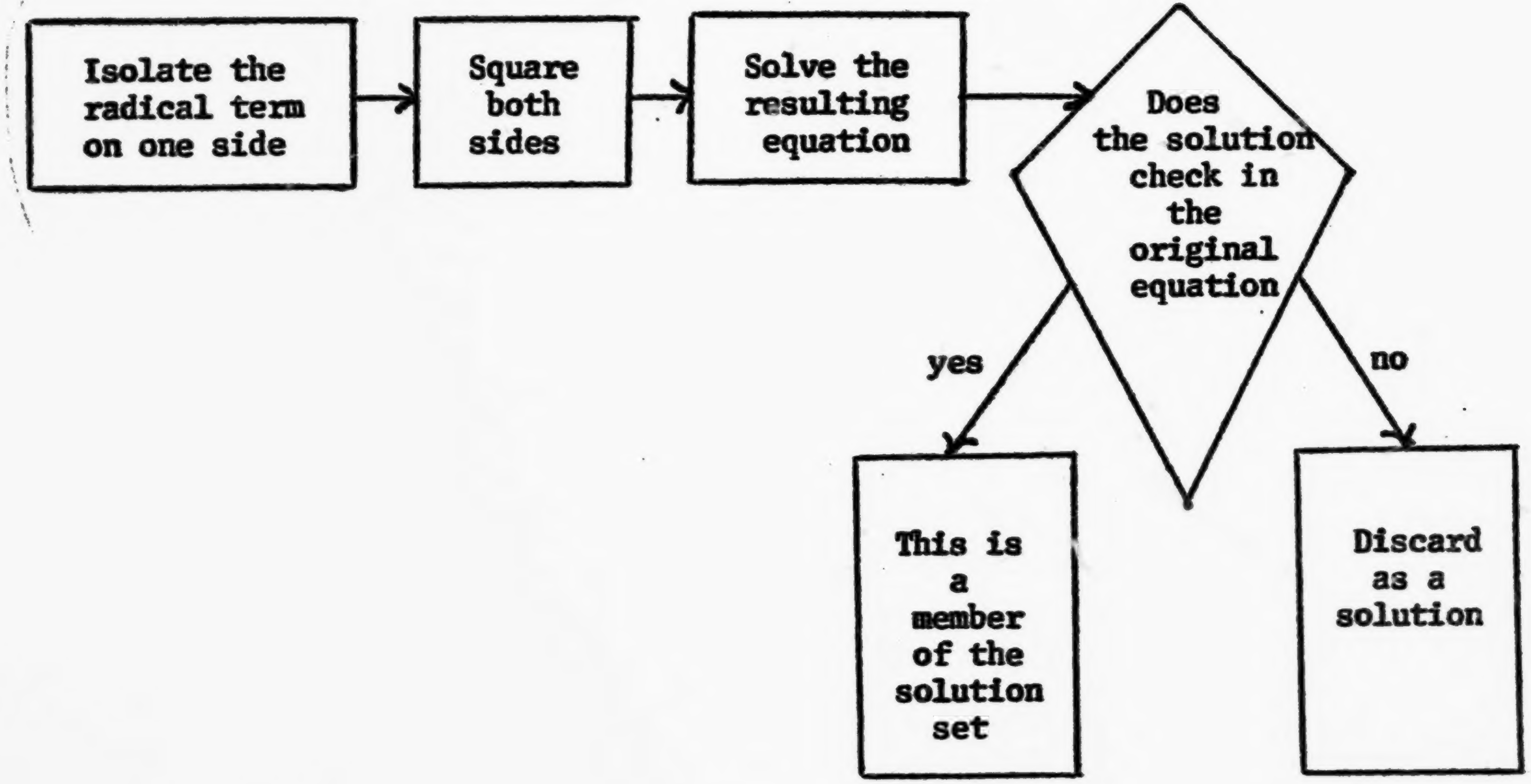
a. Have students look at simple equations like $\sqrt{x} = 2$ and guess the solution. Then look at $\sqrt{2y} - 3 = 3$ to see if guessing is still possible. In this manner justification for a systematic approach may be made.

b. Have students compare the ease of getting the solution of $\sqrt{2y} - 3 = 3$ and $\sqrt{2y} - 3 = 9$. Also look at $2y - 3 = -3$. With many examples, try to have them see that:

If $x = y$ then $x^2 = y^2$ but the converse is no necessarily true:

$$3^2 = (-3)^2 \text{ but } 3 \neq -3.$$

c. A program can be set up to solve these radical equations.


Isolate the radical term on one side
Square both sides
Solve the resulting equation
Does the solution check in the original equation
yes
no
This is a member of the solution set
Discard as a solution

(1) Give many equations where there is no solution.

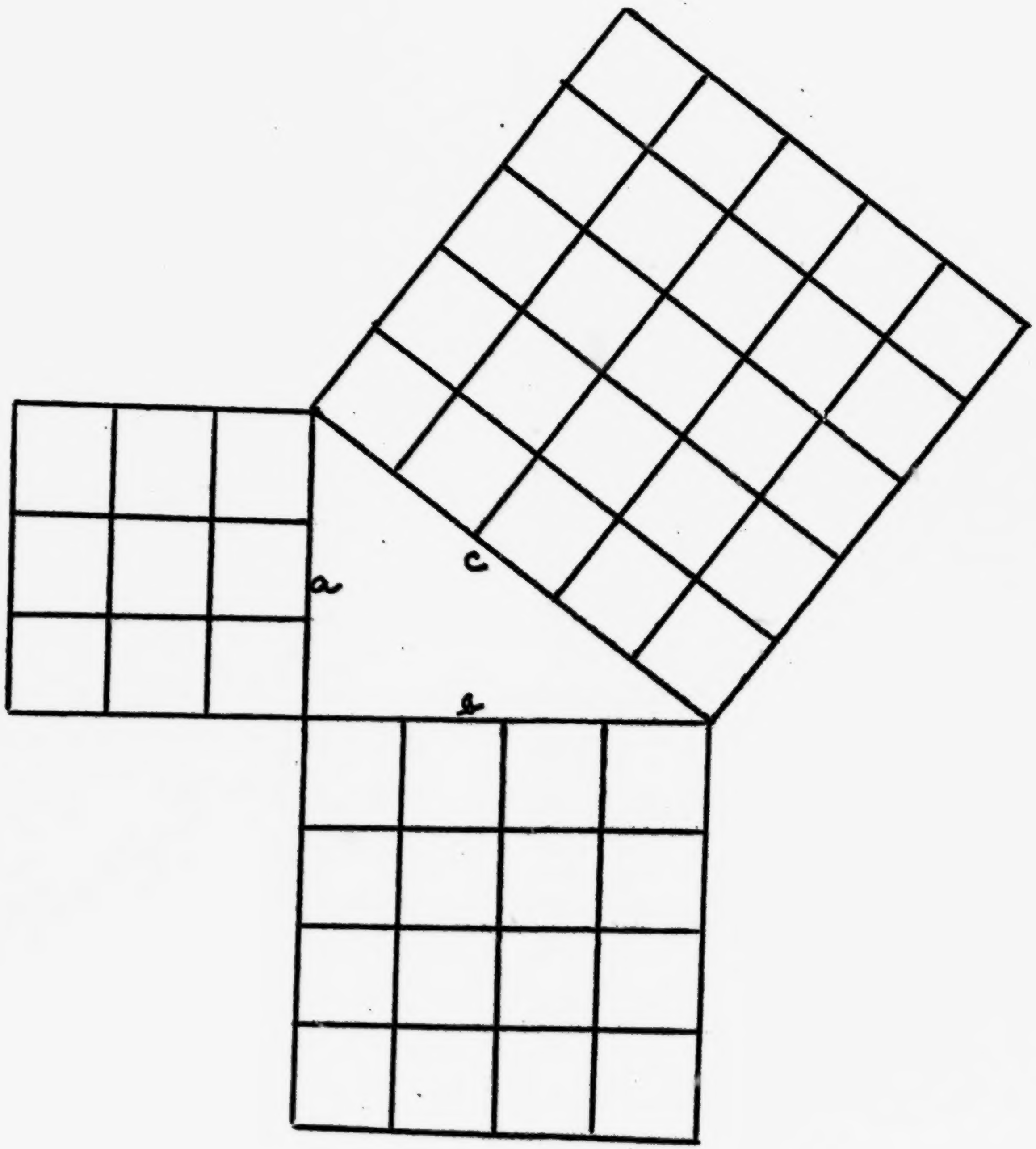
Example: $\sqrt{2y - 3} = -3$

(2) Note: Equations with more than one radical term in them are usually not handled until Algebra 2.

D. Problem solving with irrational numbers

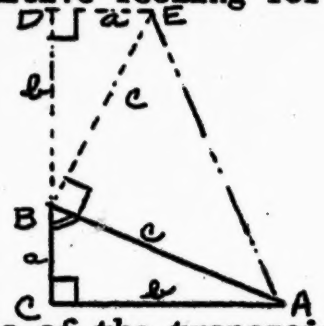
1. Pythagorean theorem

a. Most students are familiar with the formula $a^2 + b^2 = c^2$ and many of them have seen the numerical proof of the theorem by construction of squares on all sides of the triangle.



b. A different approach more algebraic in nature is the one President Garfield submitted. This fact may show that not only "egg heads" do math. It requires the students to know:

(1) Sum of the angles of a $\triangle$ is 180 .
(2) Formula for area of a $\triangle$ and for a trapezoid.
(3) An intuitive feeling for congruent angles and triangles.



Angle EBA is a right angle because angle CBA congruent to angle DEB and angle DBE + angle DEB = 90 . This subtracted from the straight line DBC is then 90 .

The area of the trapezoid ACDE then = the sum of the area of the three right triangles, ABC, BDE, BEA.

$$\tfrac{1}{2}(a + b)(a + b) = \left(\tfrac{1}{2}ab\right) + \left(\tfrac{1}{2}ab\right) + \left(\tfrac{1}{2}c^2\right)$$

$$\tfrac{1}{2}(a + b)^2 = ab + \tfrac{1}{2}c^2$$

$$(a + b)^2 = 2ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

Have students solve for c, a, b in the Pythagorean Theorem

c. Solution of problems using the Pythagorean relationship in right triangles offers a wealth of challenges for the students. A picture to aid in solution should be a must for such problems.

2. Formulas and problem solving.

There are many formulas using irrational numbers. These can be brought into problem situations.

Examples: a. Area of a $\triangle = \sqrt{s(s - a)(s - b)(s - c)}$, where s is $\tfrac{1}{2}$ the perimeter, a, b, c are the sides.

b. d (diameter of each cylinder)$=\sqrt{\frac{H}{1.6n}}$, n is number of cylinders of an automobile with H, horsepower.

For other suggestions, see (1), p. 423.

E. A Look Backward

After this unit, students may appreciate seeing where they have been.

1. Density of a set

   a. Many students enjoy the idea of always being able to find numbers between a pair of numbers (or a point between two points on the number line) no matter how small the difference. Have the students try to find another rational number between any two given rational numbers. This can be done by averaging or by terminating a decimal.

   Example (1): rational number $\frac{3}{29}$ and $\frac{2}{19}$ is $\dfrac{\frac{3}{29}+\frac{2}{19}}{2}$

   (2): A rational number between $2.22\overline{2}$ and $2.22\overline{33}...$ is 2.2229. Many others are also possible.

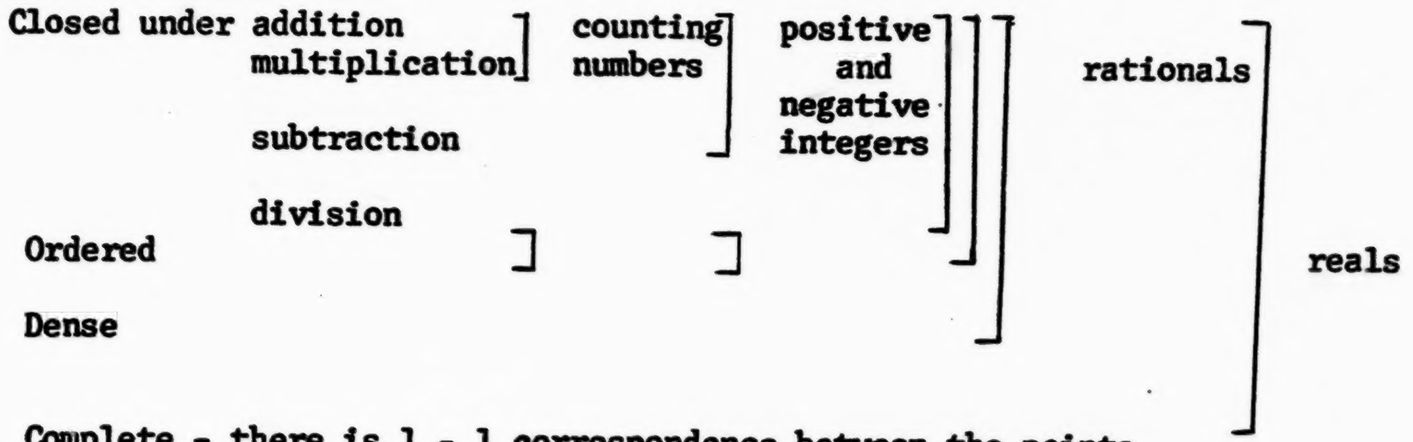
   b. Also have students make up a non-repeating, non-terminating irrational number to fall between two rationals or two irrationals.

   Example: between $2.22\overline{2}...$ and $2.22\overline{33}...$ is 2.22272772777...

   c. Try to show that the property of density is not enjoyed by the integers, counting numbers, etc.

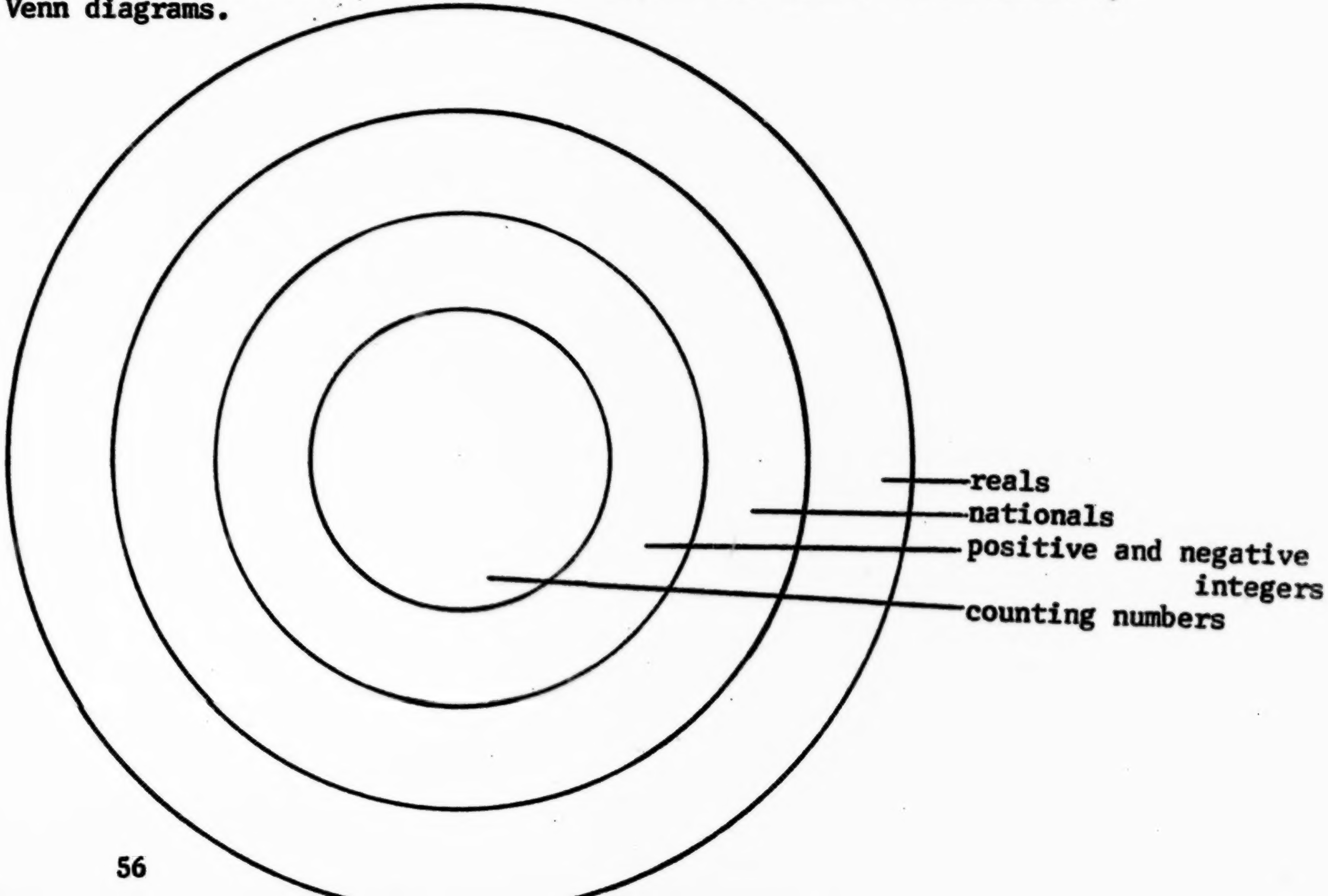
2. Properties of Real Numbers.

a. A summary such as the following can be made:



Complete - there is 1 - 1 correspondence between the points on the number line and real numbers.

b. The proper subset relationships of these sets of numbers can be illustrated by means of Venn diagrams.



c. Students may ask if there is some future extension of the number system. It can be pointed out that the next extension - so equations such as $x^2 + 1 = 0$ can be solved - necessitates giving up a previous property. That is, complex numbers are not ordered.

VI. Graphs - Relations, Functions, Linear Function, Variation

A. Introduction to two dimensional representations - relations.

1. Geometrically, students have been representing a number as a point on the number line - a line is sometimes thought of as having one dimension. Now we want students to extend their ideas to represent a point in a plane - a two dimension idea. Therefore, we will need two numbers - an ordered pair.

   Example: (5, -2). The order is important because of the 1 - 1 correspondence between the points of the plane and the ordered pairs.

2. Bring in some round - or cylindrical objects. Have students make a table of the measurements of the diameter and the circumference (tape measure will be needed.)

C	0	9½"	19"		
d	0	3"	6"	9"	36"

   Since students may already be familiar with this, have them make predictions for missing values.

   Some other formula can be worked with.

   Example: Keep the width of a rectangle the same. Make a table of the length, l, and the area, A.

B. Definition of a relation.

1. This collection - or set- of ordered pairs is then defined to be a relation. A mathematician, or scientist, tries to assign a rule or formula to fit the ordered pairs. In the illustrated case, it is, of course, $C = \pi d$. The only allowable replacement values are non-negative. This is an open sentence in two variables. You can then introduce an open sentence such as $x = y + 5$ in two variables.

2. Review the non-metric geometry idea that when two lines intersect but do not coincide, they determine a plane. Now point out that we are going to perpendicularly intersect two

lines - one horizontal, the other vertical. The point at which they intersect is called the origin. A coordinate system is marked off on each line. The horizontal number line will be marked off as usual. The vertical number line will be positive above the origin and negative below it. To determine any point in the plane determined by the two lines, two directions will have to be given. By common agreement, the first number determines the horizontal direction, the second the vertical.

3. So far, students have been solving equations and inequalities in one variable. Extension is now being made to open sentences in two variables.

C. Definitions of words to be used.

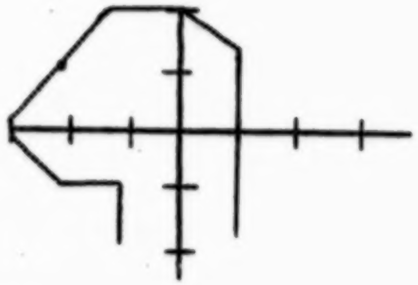
Abscissa, ordinate, axis, graph of a relation, coordinates, ordered pair, quadrants of the real plane, domain and range of the relation, graph of a function are all words or expressions the students should understand because use of them is made so frequently.

D. Graphing - or plotting - in the plane.

1. Points
   a. The place where the point is located in the plane is determined by the values of the numbers in the ordered pair. By agreement the first number determines the horizontal direction from the origin, the second the vertical. Be sure the students know how to do this before continuing. Have them look for the relationships such as when both numbers in the ordered pair are negative, the point corresponding to that ordered pair is located in the third quadrant.

   b. Practice can be given on coordinate pictures. This can fit the time of the year that this unit is being started - e.g. a profile of Lincoln or Washington or a heart in February, an egg, rabbit or lily near Easter. Have an artistic student give a hand with this.

(-1, -3) (-1, -1) (-2, -1) (-3, 0) (-2, 1)
(-1, 2) (0, 2) (1, 1) (1, -3)



c. If the irrational numbers have already been introduced, bring out the idea of the 1 - 1 correspondence between any point in the plane and any ordered pair of real numbers.

Example: $(\sqrt{2}, -\pi)$

d. A tie-in can be made with previous work students have done on graphing - for line and bar graphs they have two axes just as is being used here. Now the divisions of spaces stand for the same thing on both the horizontal and vertical axes.

2. Relations in two variables - the variables raised only to the first power: first degree or linear equations.

a. The concept that a linear equation in two variables has an infinite number of ordered pairs that will satisfy the relation should be brought out. (When students find that graphs of these equations are lines, they have the tendency to think there are only two points on the line - because two points determine a line.) This can be done by graphing several relations.

Example: 1. Plot miles traveled against hours when traveling 30 miles per hour.
$d = 30t$

2. Plot diameter of a circle against the radius of the circle. $d = 2r$

3. $y = \frac{1}{2}x$

b. Point out that there are many ways of showing a relationship between two numbers: table, graph, equation (or formula), statement in words.

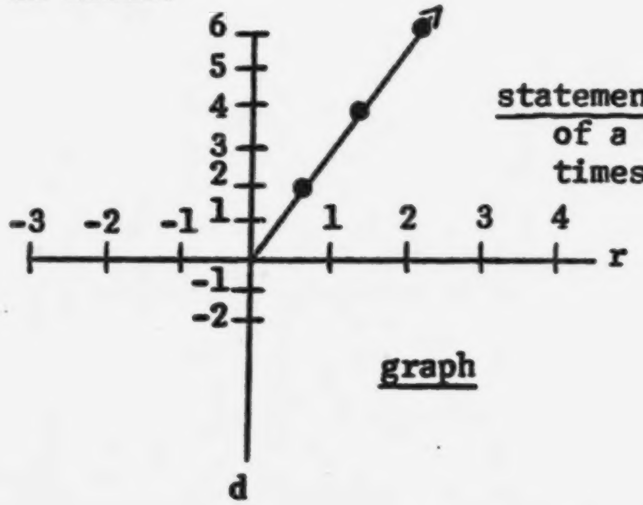
Example:

r	0	1	2	3
d	0	2	4	6

table

$d = 2r$

formula



graph

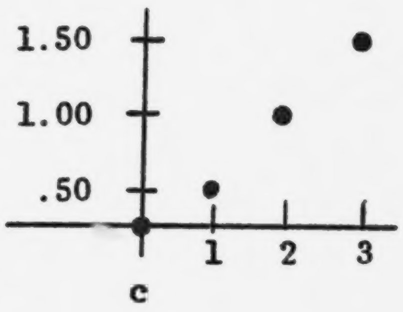
statement: The diameter of a circle is two times the radius

(1) An example such as this also shows that this particular relation can only be located in the 1st quadrant because neither a radius nore diameter are negative. (Domain and range are both non-negative.)

(2) This is not the only statement to have the same graph table and formula.
Example: Total cost in dollars, d, for a racing model, r, when each model costs $2.00.

(3) It can also point up the fact that all points (ordered pairs) between the listed points (ordered pairs) are legitimate choices for the relation. This is not always the case.

Example: Total Cost, c, of pens plotted against number of pens, n, when 1 pen cost $.50.



$c = .50n$

c. In this unit, students are trying to gain facility in interchanging these different ways of representing a relation. Formula to table to graph is the first interchange that they usually work on, with the domain and range the real numbers.

d. Point out that from a formula they are trying to show all the ordered pairs - geometrically - that will make the formula a true sentence.

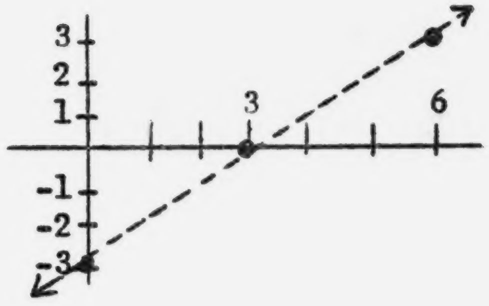
Example: Formula $y = x - 3$

Show all points on a coordinate plane that will make the sentence true.

X	0	3	6
Y	-3	0	3

table

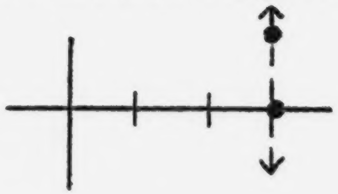
graph



This does not show all the ordered pairs making it true -.just the ordered pairs from the table.

e. Students have difficulty showing that $x = 3$ is also a linear equation when plotting all the points on a plane that make it true. Have them think of it as $x + 0 \quad y = 3$ It will not just be one point in the solution set, but a line.

x	3	3	3
y	0	1	-2



It is all the ordered pairs with the first number of the pair being 3.

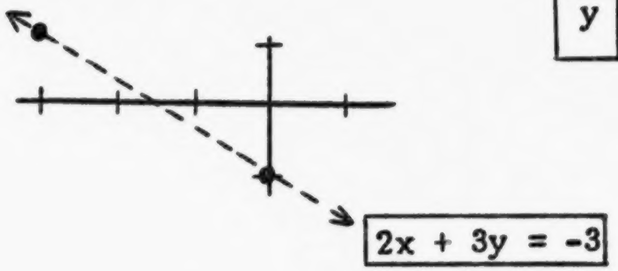
f. Even though 2 points do determine a line, it is a good practice for students to make tables with at least 3 ordered pairs and then plot them - to see if all three do fall on the same line. This is a check.

3. Linear inequalities.

a. To show students the graphing of linear inequalities, use of an overhead projector with some colored overlays is very helpful. If these materials are not available, use of colored chalk can be made.

b. Have students graph the equality $2x + 3y > -3$ first.

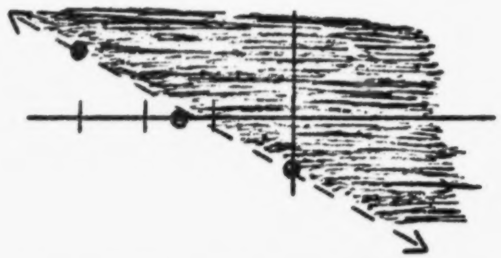
Example: $2x + 3y > -3$



x	0	$\frac{3}{-2}$	-3
y	-1	0	1

A dotted line is used as a boundary for the strict inequality, a solid line if the sentence reads $2x + 3y \geq -3$. Now have students test one point - (0, 0) is usually a convenient one if it is not on the line - to see if that point lies in the half plane making the sentence true or false. In this case: $2 - 0 + 3 \cdot 0 > -3$
$0 + 0 > -3$ is true.

Therefore, the half plane above the line is the truth set. It is generally shown by shading of some sort.



If the point (-3, 0) had been checked, $2(-3) + 3 \cdot 0 > -3$
$-6 + 0 > -3$ is false.

c. Point out that the graph of an inequality on a number line is a half line. Graph of an inequality on the plane is a half-plane.

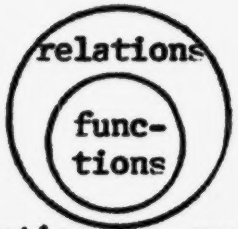
E. The linear function
   1. Definition of a function

a. A function is a relation that has one and only y-coordinate for each x-coordinate.

Example: The set of ordered pairs $(3,2)$, $(3,-1)$ is not a function because 2 and -1 are both paired with the 3.

Have students recognize the difference between $x = 2$ not being the equation of a function and $y = 2$ is an equation of a function, from the definition of function.

b. If students are familiar with Venn diagrams, this picture may be helpful. Functions are a subset of relations.



c. It can be pointed out that a function is sometimes referred to as a rule for which there is only one answer.

Example: (1) The relation of a person and that person's height is a function (person, height). There is only one height for each person. You wouldn't want a person to be two different heights at the same time.

(2) Addition is a function, (two numbers, sum).

$3 + 2 = 5$

$3 + 2 \neq$ any other real number other than 5. It is well-defined.

d. Geometrically, the students should eventually be able to see that any line in a plane other than a vertical one is the graph of a function.

2. $Ax + By = C$.
$B = 0$ - the linear equation

a. This determines a function by:

(1) Replacing any numerical value for x.
(2) Multiplying that number by A - multiplication is a function.
(3) Subtract the product from both sides of the equation. Subtraction is a function. $By = C - Ax$.
(4) Divide both sides by B (this is why $B \neq 0$) Division is a function. $y = \frac{C - Ax}{B}$

The linear function is then the set of all ordered pairs such that the second number (y) is obtained from the first (x) by the relationship $y = \frac{C - Ax}{B}$.

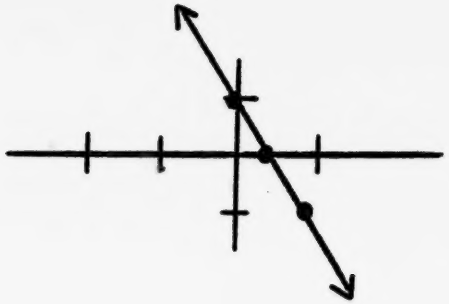
This last phrase is shortened to $(x, y): y = \frac{C - Ax}{B}$, x is real

(5) This gives the numerical value for y which is unique for any distinct values of x.

b. Show that equivalent sentences have the same truth set and therefore the same graph

Example: $2x + y = 1$
$y = -2x + 1$

x	0	$\frac{1}{2}$	1
y	1	0	-1

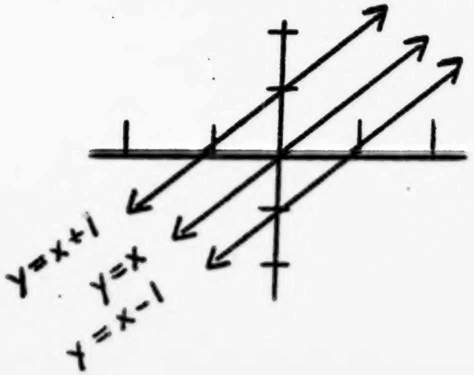


3. $y = mx + b$; a special form of the linear equation.

a. The meaning of m.

Have students at the board and at their desks graph a family of lines where m is held fixed.

Example: $y = x$
$y = x + 1$
$y = x - 1$

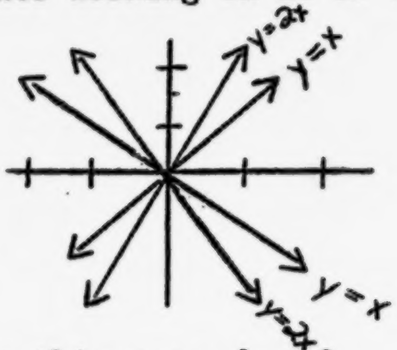


From this sort of exercise: , students should be able to determine that parallel lines are obtained when m, the slope, is fixed.

b. The meaning of b.

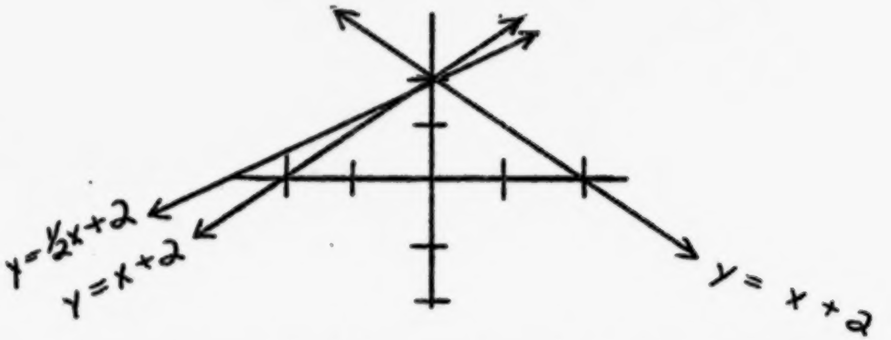
(1) Have students graph families of curves where b is held fixed and m varies. This can be done at the board with several students working on it as well as the rest working at their desks.

Example: (a) $y = x$
$y = -x$
$y - 2x$
$y = -2x$



In this case, b = 0

(b) $y = x + 2$
$y = -x + 2$
$y = \frac{1}{2}x + 2$



Then have students look for the significance of the number b. Not only is it where all the lines intersect, but students should also see that it is where they intersect the y axis (when x = 0)

(2) From these two exercises, holding m and then b fixed, students may see why this form of the linear function is called the "slope - intercept" form.

c. Definition of slope of a line.

(1) Help the students see that when one slope is steeper than another slope, it means it is rising at a faster rate than it is changing horizontally. This

can lead to the formal definition:

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

In relation to graphs of linear functions, this means

$$m = \text{slope} = \frac{\text{y-coordinate change}}{\text{x-coordinate change}}$$

Stress that if a line is horizontal, then its slope is 0, if a line is vertical then the slope is undefined. Have the students graph examples such as $x = -3$ and $y = 2$.

(2) Slope is a ratio or a rate at which a line is changing. If a slope is -2, it can be thought of as $\frac{-2}{1}$.

(3) If two points on a line are given, students can calculate the rate the line is changing by comparing the vertical change to horizontal change.

If one point (x_1, y_1) and another (x_2, y_2) then $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Have students try to see when the slope is a negative number, and when it is positive, or zero. Relate this to the slope - intercept form of the linear function.

(4) Better students may also see that perpendicular lines have slopes that are the negative reciprocals of one another.

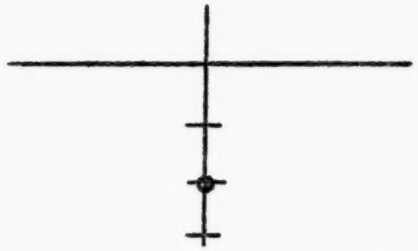
Example: $y = 2x$ and $y = \frac{1}{2}x$ are perpendicular lines.

d. Graph, Equation (Rule), Points on the Line.

(1) Drawing a graph of a line from slope-intercept form.

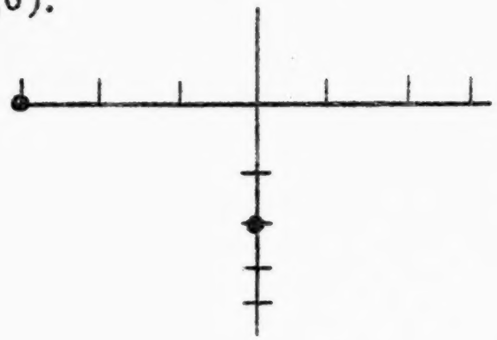
Point out to students that this is an especially handy form of a linear equation because it can be graphed easily.

Example: $y = -\frac{2}{3}x - 2$ Already (0, -2) is known to be an ordered pair in the truth set.



(2) Since the slope is $-\frac{2}{3}$, we can also write this as $\frac{-2}{3}$ or $\frac{2}{-3}$. It is known that another ordered pair is 2 units down (because it is negative) and 3 units to the right $\frac{-2}{3}$(3, -4) or 2 units up and 3 units to the left $\frac{2}{3}$(-3,0).

These points should be checked in the original equation.



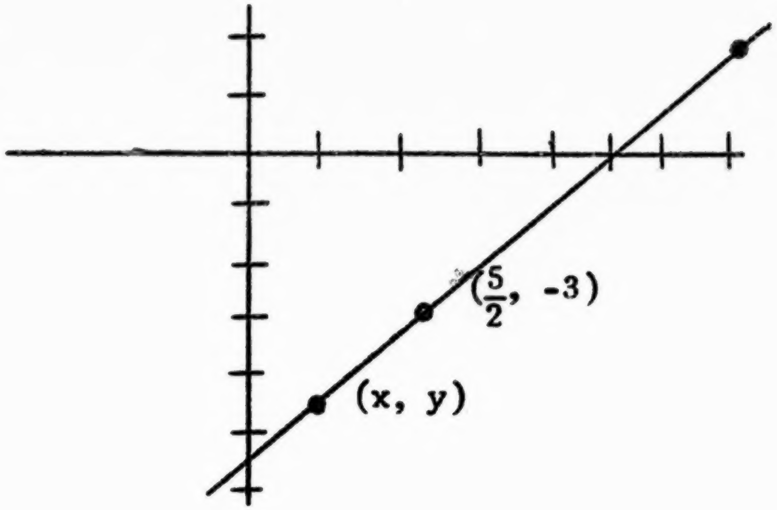
(3) Finding the equation from knowing two points on the line.

x	$\frac{5}{2}$	7
y	-3	2

Sometimes forming the rule (of the linear equation) cannot be done by just inspecting the table. But from two points, ask the students what can be found from that.

$$\frac{-3\ -2}{\frac{5}{2}\ -7} = \frac{-5}{-\frac{9}{2}} = \frac{10}{9} = \text{slope}$$

Now is can say $y = \frac{10}{9}x + b$ and the problem is finding the y-intercept. It can't always be done by graphing.



(4) However, any point on the line can be referred to as (x, y). Then if that point and one other point that is known can be used to express the slope $\frac{y - (-3)}{x - \frac{5}{2}}$, this is then the equation. $\frac{y + 3}{x - \frac{5}{2}} = \frac{10}{9}$

(5) If the point (7,2) were used instead of the $(\frac{5}{2}, -3)$, it can be shown that these are equivalent sentences. They can both be transformed to the same slope intercept equation.

$$\frac{y - (-3)}{x - \frac{5}{2}} = \frac{10}{9} \qquad \frac{y - 2}{x - 7} = \frac{10}{9}$$

It should be pointed out that these are both proportions.

$$y + 3 = \frac{10}{9}(x - \frac{5}{2}) \qquad y - 2 = \frac{10}{9}(x-7)$$

$$y + 3 = \frac{10}{9}x - \frac{50}{18} \qquad y - 2 = \frac{10}{9}x - \frac{70}{9}$$

$$y = \frac{10}{9}x - \frac{52}{9} \qquad y = \frac{10}{9}x - \frac{52}{9}$$

F. Variation

1. Direct variation

a. Many examples of direct variation of two variables have been given in the previous pages. In words, it means "as one variable is increasing (or decreasing), so is the other variable increasing (or decreasing) proportionately."

Examples: (1) As the diameter of a circle increases, so does its radius proportionately. $d = 2r$

(2) As the number of hours, t, increases while traveling at a fixed rate of 40 m.p.h., so does the total distance, d. $d = 40t$.

Have students try to see these direct variations - or make some up - as linear equation.

b. The ratio of one variable to another is a constant. $\frac{d}{r} = 2$; $\frac{d}{t} = 40$; $\frac{x}{y} = 4$. The constant is called the constant variation or the constant of proportionality since the rule is proportion.

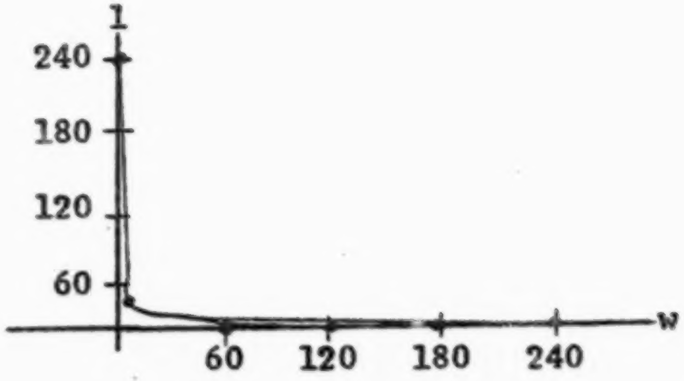
c. Have students graph some direct variations and see if they can relate the constant to the slope of the line.

2. Inverse variation

a. Have students graph the length vs. the width of a rectangle that has an area of 240 sq. ft.

Example: (width, length)

(1, 240)
(2, 120)
(3, 80)
(10, 24)
(24, 10)



b. As one of the variables increases, have students realize that the other variable has to decrease. This is no longer a linear relationship - does not graph to a line.

c. Have students think up examples of inverse variation. Write a rule for it.

Examples:
- (1) $lw = 240$ sq. ft.
- (2) $rt = 300$ miles - the faster you travel the fewer the hours of travel.
- (3) $cn = \$5.00$ - the higher the cost per item, the fewer the number items you can buy.

d. The sciences are rich with examples of inverse variation.

Examples:
- (1) The closer the distance, the more illumination from the same light. This is actually a case of one variable being inversely proportional to the square of the other variable.
- (2) The larger the diameter of a pulley, the less force needed to raise a fixed weight.

e. The reason $xy = k$ is not a linear equation is because with the two variables being in the same term, the term is no longer considered 1st degree - even though the variables by themselves are each only raised to the first power. The degree with respect to x is 1, with respect to y is 1, but with respect to x and y is 2.

VII. Systems of Linear Equations (Simultaneous) and Inequalities

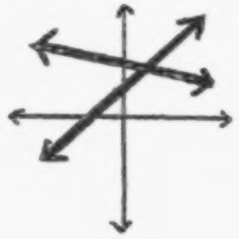
A. Introduction

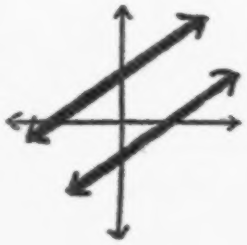
1. Review the students' understanding of the concepts of the intersection of two lines. Develop or review the possibilities:

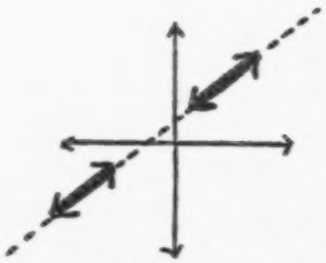
   a. two lines intersect in a point.
   b. two lines are parallel.
   c. "two" lines are actually the same line.

   If this is the first meeting with these ideas, take some time to have students give examples because continual reference can be made to it in this unit. Tying in of the algebraic and geometric methods can be helpful for student understanding.

2. Give the three possibilities of the placement of two lines on sets of coordinate axes.







   Mention that the work of this unit will consist of finding points common to both lines.

3. This is an application of finding the intersection of two sets - points (or ordered pairs) that are elements of both lines (or sentences). Have students check the ordered pair in both equations to see that it makes both of them true.

4. Work in this unit is done with lines - or linear sentences. Point out to the students that use of at least two sentences of the form Ax + By = C constitutes a system of linear equations.

B. Methods of Solution

1. Graphing Equations

a. It is suggested to present this method of solution first because the students should be well-acquainted with how to graph one line already. When an estimate of a solution is not good enough, students can then see the need for an algebraic method of solution.

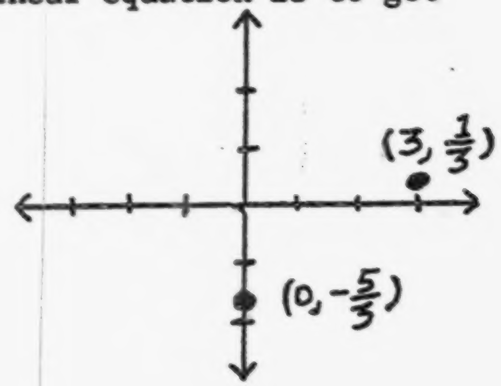
b. Remind the students that a rapid method of graphing a linear equation is to get it in slope-intercept form (from unit on graphing).

Example: $2x - 3y = 5$. Solve for y.

$$2x - 5 = 3y$$

$$\frac{2}{3}x - \frac{5}{3} = y$$

The slope is $\frac{2}{3}$, the y-intercept is $\frac{-5}{3}$.



Refer to Chapter VI for the slope-intercept method, page 67.

c. If students find it difficult to work with the slope-intercept form, have them work from a table. Remind them that a value for one variable can be chosen arbitrarily, then they calculate the other number in the ordered pair. The choice of 0, for x or for y is often a short cut.

Example: $2x - 3y = 5$

x	0	$\frac{5}{2}$
y	$\frac{5}{3}$	0

d. Have students solve systems of all three possibilities.

Example: (1) $\begin{array}{l} x + y = -1 \\ y = 3x - 5 \end{array}$ (2) $\begin{array}{l} 2x - 2y = 4 \\ x - y = 1 \end{array}$ (3) $\begin{array}{l} 6x + 3y = 2 \\ 2x + y = \frac{2}{3} \end{array}$

These three types of systems of equations corresponding to the three possibilities of the two lines are sometimes referred to as follows:

1. consistent
2. inconsistent
3. dependent.

e. If students work with the slope-intercept form to graph, they probably will pick up (or remember from the section on graphing linear equations) that whe the slopes are equal but the intercepts different, parallel lines will resul Also, for dependent systems, the slope-intercept form is the same for both equations.

			slope- intercept form
Examples:	(1)	$2x - 2y = 4$	$y = x - 2$
		$x - y = 1$	$y = x - 1$
		inconsistent	
	(2)	$6x + 3y = 2$	$y = -2x + \frac{2}{3}$
		$2x + y = \frac{2}{3}$	$y = -2x + \frac{2}{3}$
		dependent	

2. Graphing Inequalities

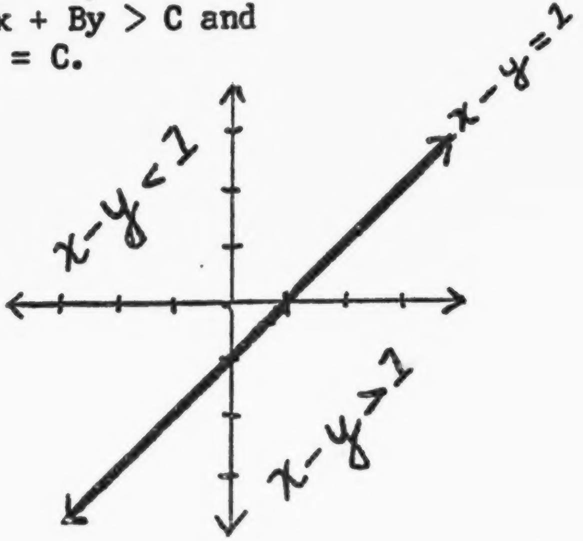
a. To graph systems of linear inequalities, students should be well acquainted with how to graph one linear inequality. For the system, it should be understood by them that they are trying to show all of the points in the plane that will satisfy both (or more than two) inequalities - the intersection of the sets.

b. Use of overhead projector with colored overlays makes an effective way of presenting - and correcting the assignments in this topic.

c. Students can be made aware that a line Ax + By = C divides the plane into three portions: those points of the plane where:

1) The half plane where Ax + By < C
2) The half plane where Ax + By > C and
3) The line where Ax + By = C.

Example: x - y = 1

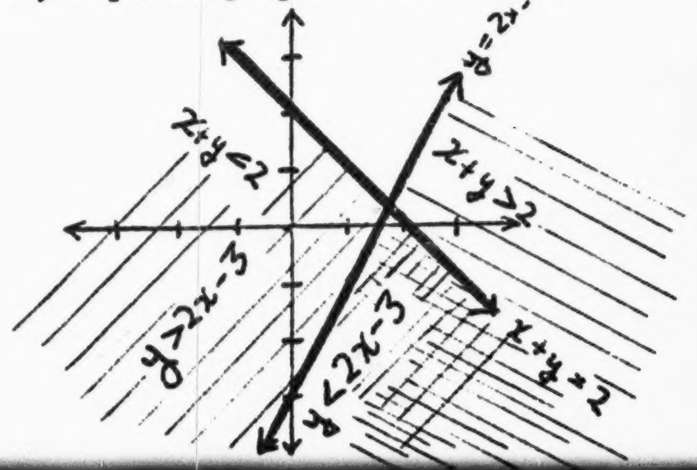


To convince the students that all points on one side of the line are representations of ordered pairs of the truth set for the same inequality - write the equality in slope-intercept form (y = x +-1). Then any real number b >-1 will give an equation for a line whose points will all be on the same side of the original line since the slopes are the same. The union of all possible lines is the half plane.

A system of inequalities then is the intersection of two or more half planes - sometimes including the boundary line and sometimes not, depending upon the use of <, >, ≤, ≥ .

Example: x + y ≤ 2
y < 2x - 3

The double shaded area is the solution set to the system including the points of x + y = 2 and not including those from y = 2x - 3.

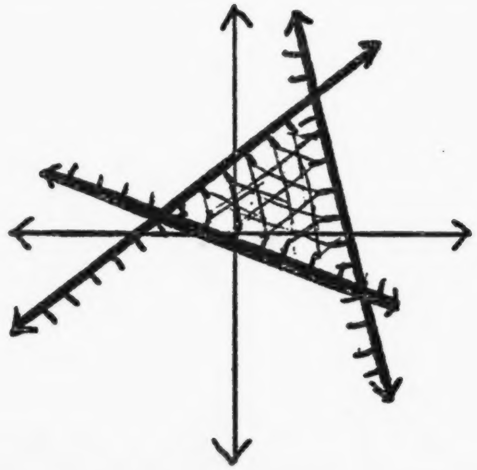


d. The system of inequalities can be extended to more than two sentences. Try to have students see whether there will always be solutions.

Example: Add the inequality $y > 3$ to the system used in 3 above. Also what type of solutions are possible with 3, 4, . . . inequalities?

e. When more than two inequalities are graphed on the same axes rather than shading a large area which can become confusing, only shade a small amount on the proper side of the boundary.

Example:



f. An extension of systems of inequalities is to go into some work on linear programming. This is a term that many students may have heard about. They can now understand the essentials of it. A couple of references for the student level of understanding are (13) pages 252-256, (2) pages 53-60.

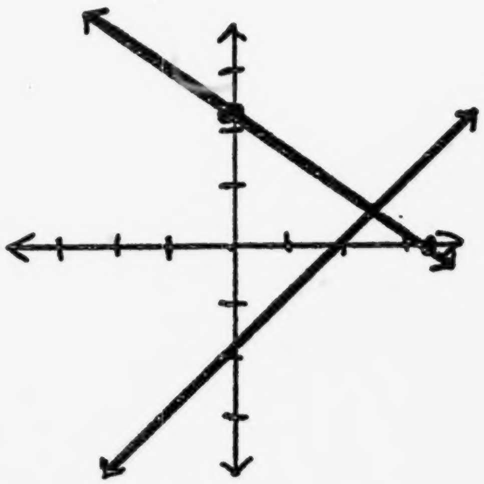
3. Addition and Subtraction

a. A reason for the need for another method for solving systems of equations other than graphing is that graphing is not accurate.

Emphasis can be made concerning the idea that graphing is a form of measurement (which is always approximate) whereas it would be nice to have an approach that will give the exact point of intersection.

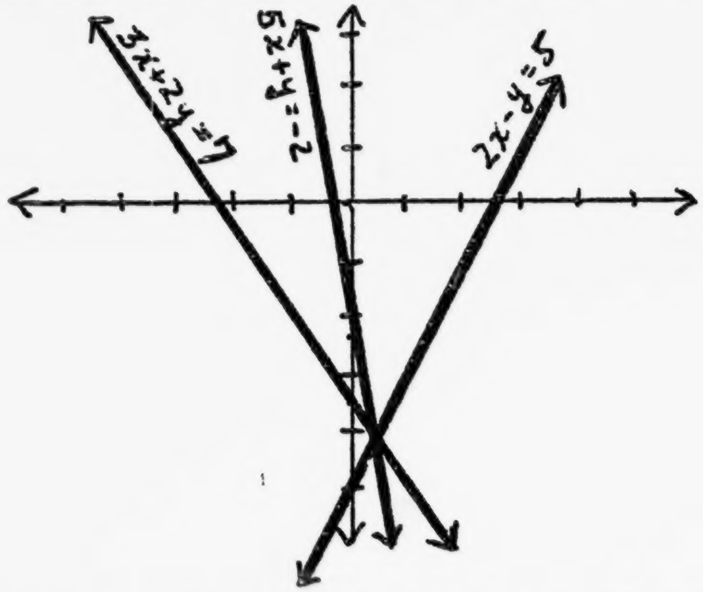
Example: $-2x = 3y - 7$
$-x + y = -2$

The point of intersection
can only be approximated.



b.

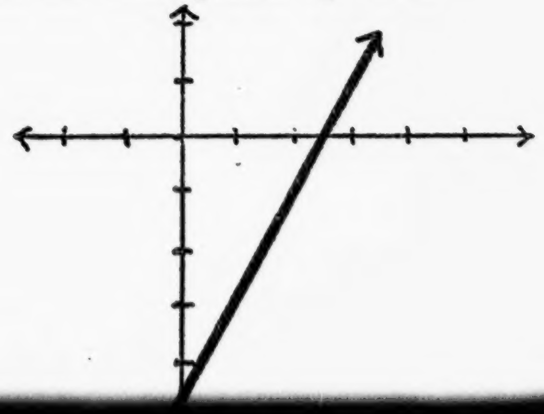
(1) Have students graph two lines that will intersect. Say $2x - y = 5$ and $3x + 2y = -7$. Then have the students add the two equations together. $5x + y = -2$. Have them graph this equation on the same axes. This sum should be seen by the students to pass through the same point of intersection. You could then also have the students subtract the equations.



(2) Try also to have the students see that when an equation is multiplied by some number, the graph is still the same.

Example: $2x - y = 5$

Multiply it by 2 or $\frac{1}{2}$ and the graph will still be



Any equivalent equation (using the axioms of equality for equation solution) was shown in the section on graphs to have the same graph. Review this idea with the students.

(3) With the two notions, equivalent equations have the same solution set, and the sum or the difference of the two equations still passes through the point of intersection of the original pair of equations, sets up the reasons why the addition, subtraction method of solution works.

c. If the two equations in a system are put into some sort of standard form - either $Ax + By = C$ or $Ax + By + D = 0$, the students can develop some facility for recognizing if the two lines will be consistent, inconsistent or dependent by using the ratios of the coefficients of the x, y and constant term.

Example: $2x - 3y = 7$
$4x - 6y = 2$ $\quad \frac{2}{4} \quad \frac{-3}{-6} \quad \frac{7}{2}$

If the ratios of the coefficients of the x and y terms are equal, then the system is either inconsistent or dependent. To determine which, the ratio of the constant terms is examined. You might see if students can figure out why the equality of this ratio to the other two ratios causes dependent equations and inequality causes inconsistent equations.

d. Remind the students of the geometric implications of finding a solution to satisfy two equations at once. They are not done until an ordered pair has been found or proved not to exist.

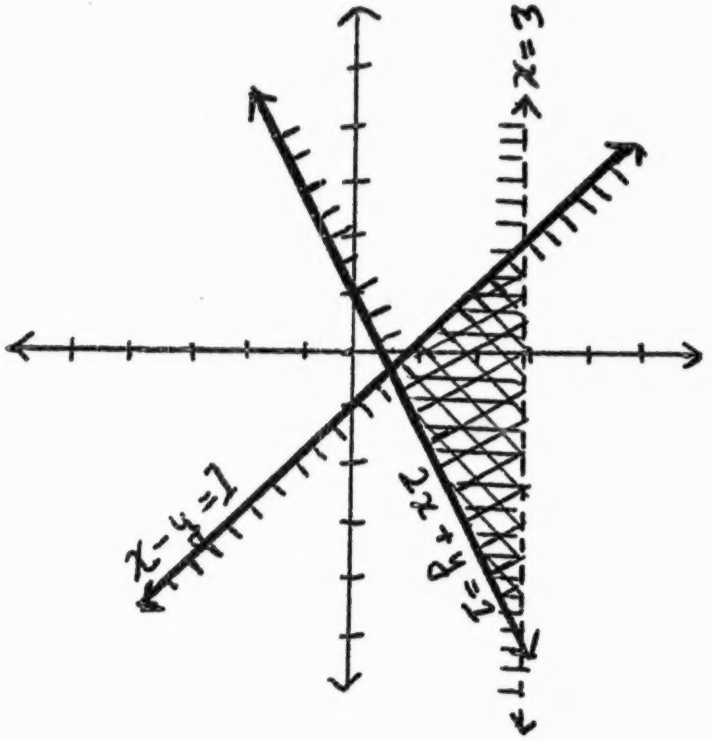
e. The solution of inequalities by means other than graphing is not practical. However, using the addition-subtraction methods to find points of intersections of several lines can be used.

Example: $x \quad 3$
$x - y \quad 1$
$2x + y \quad 1$

To find point B, this is the intersection of $x - y = 1$

$$2x + y = 1,$$

so these two equations can be solved by the addition, subtraction method.



4. Substitution

a. It is recommended that you point out to students that the addition-subtraction method will work for any system. Another method is presented only because in certain problems this method is easier to handle. This usually occurs when one of the variables appears in one of the equations with a coefficient of 1. Solution of that equation in terms of that variable can then be easily accomplished.

Example: $3x - 2y = 1$
$2x + y = 5$

Solve for y in the second equation. $y = -2x + 5$. Then y and $-2x + 5$ are expressions meaning the same thing. Hence, substitution can now be used.

b. Point out that in both of the algebraic methods the object is to go from two equations with two variables to one equation with one variable.

c. Leave the method of algebraic solution to the student.

5. Iterative procedures (optional)

For a very able class where you might be far ahead of schedule, students may appreciate looking at an approach to solution of a system of linear equations used on a computer. Refer to a write-up in (6) pages 12-15.

C. Problem Solving.

1. Point out that many problems that had been solved using only one variable can now be solved using two variables. However, two equations (a system) is generally needed for solution.

Example: The sum of two numbers is 10 and their difference is 2. What are the numbers?

one variable	two variables
x - one number	x - one number
$10 - x$ - other number	y - other number
$x - (10 - x) = 2$	$x + y = 10$
	$x - y = 2$

2. Give problems that correspond to a system with no solution and one that has infinite solutions (inconsistent and dependent).

Example: (1) The sum of two numbers is 5. If two times the first is added to two times the second and then 12 (or 10) is subtracted from it, the difference is 0. What are the numbers?

$x + y = 5$ $2x + 2y - 12 = 0$ inconsistent	$x + y = 5$ $2x + 2y - 10 = 0$ dependent

3. Have students write down what each of the two variables is representing. In problems, they should be looking not only for the ordered pair satisfying both sentences, but

also the answer to the problem.

D. Indeterminate Equations (optional) (sometimes called Diophantine equations)

A natural extension of this topic is to have students solve a system of equations that has more variables than numbers of equations. Students enjoy problems that have more than one solution. It also gives a clue on how to work many puzzle problems.

Example: A famous puzzle problem is of Chinese origin: If a hen is worth 3 yen; rooster, 5 yen; and 3 chickens, 1 yen, how many of each could you buy if the total number you spent a 100 yen? Assume at least 5 roosters are purchased.

1. Some references where problems of this type are available are (1) pages 393-395, (11) pages 343-345.
2. A systematic approach to the solution of some of these problems is given in (7) pages 223-225.

VIII. Quadratic Functions

A. Introduction

1. Point out to students that so far in Algebra, their work with equality sentences (equations) has been of the following order:

Number of Equations	Number of Variables	Highest Degree of Variables	Example	Number of Solutions
1	1	1	$3x + 2 = 5$	1
1	1	2	$x^2 = x + 2$	2 (at most)
1	2	1 for both	$x + y = 5$	infinite
2	2	1 for both	$\begin{cases} x + y = 2 \\ x - y = 5 \end{cases}$	1 (for consistent case)
1	2	1 for one 2 for one	$y = x^2$	infinite
2	2	linear 1 for one 2 for one	$\begin{cases} \left(\begin{matrix} \text{only} \\ y = 0 \end{matrix}\right) \\ y = x^2 \end{cases}$	2 (at most)

The last two cases are the ones to be worked on for this unit. Students sometimes enjoy seeing a logical sequence of development--even though a lot of this chart is a look backward.

2. As in the case of the work with linear equations, a good approach to this topic would seem to be through graphing some quadratic equations in a plane. Then the "solution" of a quadratic equation being the intersection of the x-axis and the parabola might make more sense with the graphic representation in mind. This is not the order of approach in most texts.

3. **Have students graph:**

$y = x^2$ $y = x^2 + 1$ $y = x^2 + x + 1$

$x = y^2$ $x = y^2 + 1$ $x = y^2 + y + 1$

Remind students about the method of graphing going from an equation (rule, formula) to a table to a graph. Since these graphs are in two variables, the object is to look for all the ordered pairs that satisfy the rule. Point out that this will not be a straight line and, therefore, many more than three ordered pairs are needed to see the configuration. Connect the points with a smooth curve.

B. The quadratic function

1. Definitions

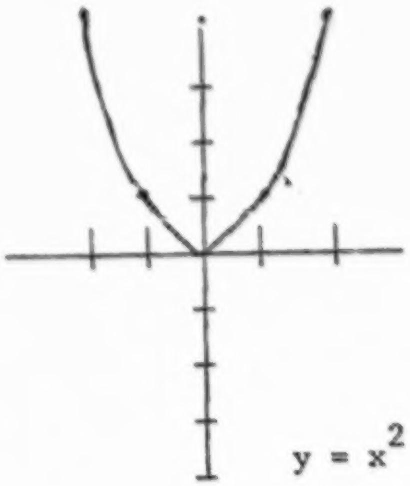
The meaning of the words parabola, vertex of the parabola, quadratic equation, quadratic function, axis of symmetry should be clear to the student.

2. Recognition of the function

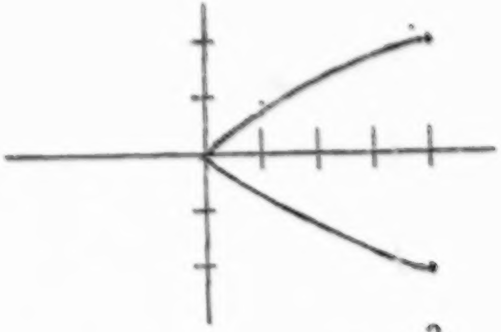
$y = Ax^2 + Bx + C$

$A \neq 0$

a. Review the idea of function and the difference between $y = x^2$ and $x = y^2$ graphically:



$y = x^2$



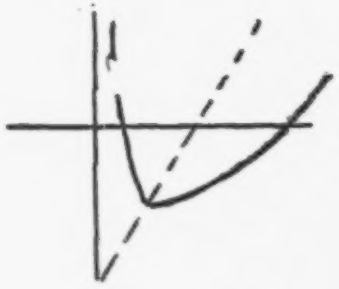
$x = y^2$

The truth set of $x = y^2$ is not a function because for each $x > 0$, there is more than one y value to be paired with it.

Example: (1,1) and (1,-1) are both ordered pairs in the set belonging to the truth set for $x = y^2$. Graphically this is shown to the students to be the fact that vertical lines can intersect the graph in more than one point.

b. From the activity in 3 of the introduction, having students graph the pairs of equations, students may be able to see that those which are equations of functions have x to the 2nd power and y to the 1st. This is the one with which we will be working.

c. Students may be curious about the relations occurring with the shape shown. It can be pointed out that this occurs when an xy term is present in the equation and that this is handled in more advanced courses.

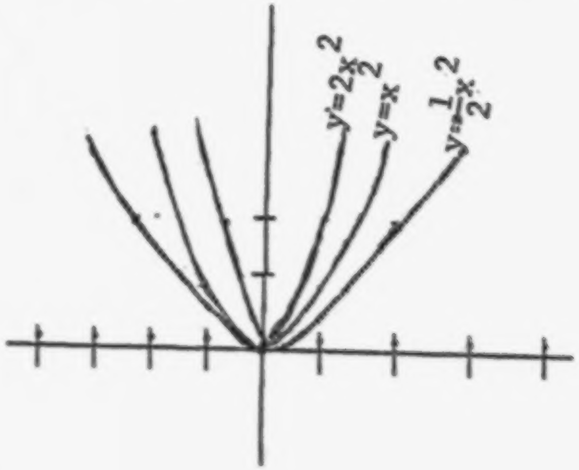


3. Some relationships of the quadratic function

a. Have students graph "families" of parabolas on the same set of coordinate axes.

Example:

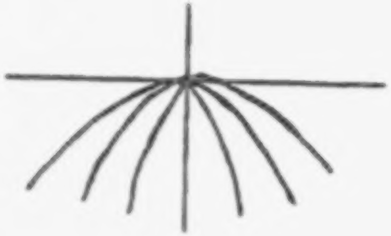
$$\begin{cases} y = x^2 \\ y = \frac{1}{2}x^2 \\ y = 2x^2 \end{cases}$$



Example: $y = -x^2$

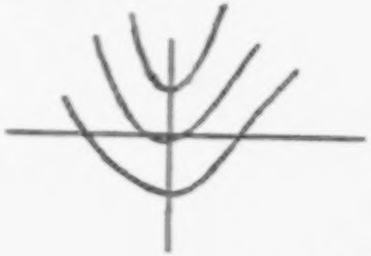
$y = -\frac{1}{2}x^2$

$y = -2x^2$



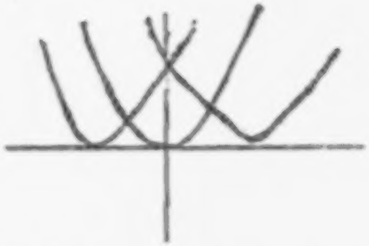
b. Due to the length of time it takes most students to make a graph, this work can be dwelled on too long. If there is time available, more families of curves of the following variety can be done.

$$\begin{cases} y = x^2 + 1 \\ y = x^2 \\ y = x^2 - 1 \end{cases}$$



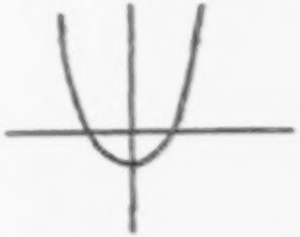
This shows shifting along the y-axes.

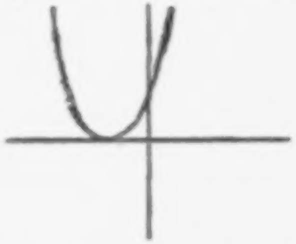
$$y = (x + 1)^2$$
$$y = x^2$$
$$y = (x - 1)^2$$

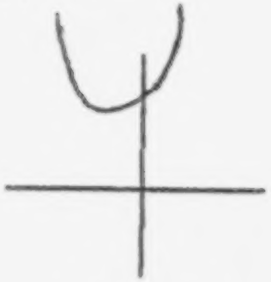


This shows shifting along the x-axes.

c. Have students draw freehand the possible locations of a parabola $y = Ax^2 + Bx + C$ $A \neq 0$ with relation to the x-axes. Try to have them see the locations as:







This is a leadup to the possible solutions of the system of equations

$$\begin{cases} y = 0 \text{ and} \\ y = Ax^2 + Bx + C \qquad A \neq 0 \end{cases}$$

d. If time allows, some work with quadratic inequalities could be done. The idea, as with lines, that the equation $y = Ax^2 + Bx + C$ divides the plane into three portions:

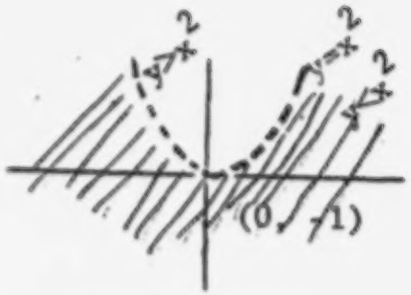
a. The points where $y < Ax^2 + Bx + C$

b. The points where $y > Ax^2 + Bx + C$

c. The points where $y = Ax^2 + Bx + C$

can tie in graphing with what has already been learned.

Example: $y < x^2$



Test point (0,-1) shows that it is part of the truth set of this sentence.

C. Methods of solution of the $y = 0$ and $y = Ax^2 + Bx + C$

Suggest activity c under section B part 3 of this unit be done--or something similar to it--to lead up to what we mean by solution of this system.

1. Graphing

a. Point out that this system of equations can be written in another way $y = 0$ and $y = Ax^2 + Bx + C$ is equivalent to (has the same solution set as) $0 = Ax^2 + Bx + C$. Ask students "where does $y = 0$?" Graphically, this means the points where the parabola is intersecting the x-axis.

b. Give students graphic work with the three variables of intersection; 2, 1 and 0 points of intersection.

Example:	$0 = (x + b)(x + c) \quad b \neq c$	2 solutions
	$0 = (x + b)^2$	1 solution
	$0 = x^2 + a \quad a > 0$	0 solutions

c. It can be brought out that "real" solutions to this system of equations are being sought. The last example, the parabola $y = x^2 + a \quad a > 0$, does not cross the real number line and, therefore, has no real solutions.

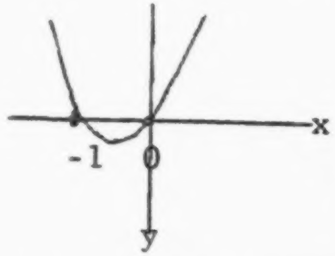
2. Factoring

   a. As with graphing solutions of systems of linear equations, this graphing method is not exact since it is based on measurement. The rest of the methods can be pointed out as algebraic in nature.

   b. Review (from Chapter III on Powers, Polynomials, Products, Factors) or introduce the principle that if $ab = 0$, then either $a = 0$ or $b = 0$ (or both).

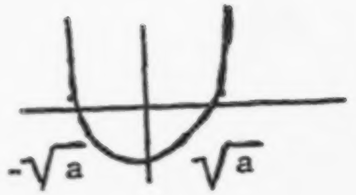
   c. This is a good place to review factoring--especially different forms of quadratic expressions. Whenever students seem to be getting lost in the mechanics rather than understanding that they are looking for the points of intersection of a parabola and the x-axis, take them back to graphic illustrations.

Example: Systems such as $y = 0$ and $y = x^2$ or $0 = x^2 + x$ sometimes gives students difficulty. After factoring $x^2 + x$ into $x(x + 1)$, remind students that $x = 0$ and $x = -1$ are the two intersecting points of the x-axis and the parabola.

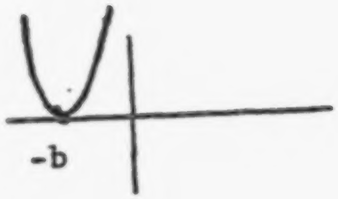


You can have students observe that the axis of symmetry is halfway between the solutions.

Example: Systems such as $y = 0$ and $y = X^2 - a$ or $0 = x^2 - a \quad a > 0$ means the factors of $x^2 - a$ are $(x - \sqrt{a})\ (x + \sqrt{a})$. Therefore, the parabola is intersecting the x-axis in two points equi-distant from the y-axis.

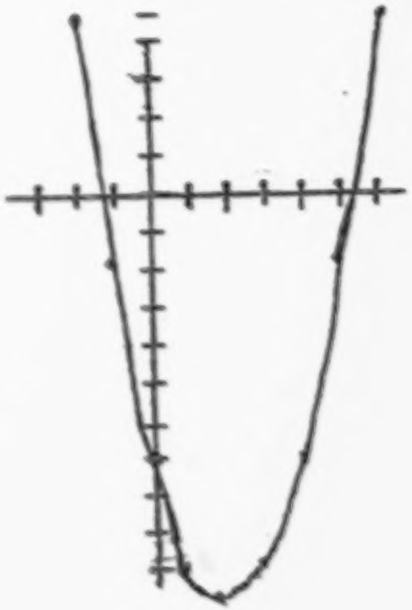


Example: $y = 0$ and $y = (x + b)^2$ or $0 = (x + b)^2$ means the parabola is intersecting the x-axis in one point--is tangent to it.



3. Completing the square

a. Give students a graphing problem of the system $y = 0$ and $y = x^2 - 4x - 7$. Their solutions could look something like this:



x	0	-1	-2	1	2	3	4	5	6
y	-7	-2	5	-10	-11	-10	-7	-2	5

Then have them try to factor $x^2 - 4x - 7$. From the graph, it can be seen the parabola does intersect the x-axis but the solutions cannot be calculated exactly by the methods of factoring that have been developed. This can give reason for developing a technique that will give solutions when the quadratic expression can not be factored over the rational numbers.

b. Review (from Chapter III section D part 2 on Powers, Polynomials, Products, Factors) what a perfect square trinomial is:

$$\square^2 + 2\square\triangle + \triangle^2 = (\square + \triangle)^2$$

c. Give students a preview that when a quadratic expression cannot be factored over the rational numbers, and the graph of the parabola does intersect the x-axis, then the expression can be factored over the real numbers and that is what is going to be done in this section.

d. Take the terms that contain the variable--the x^2 and x terms--and make a perfect square trinomial out of it by adding a constant of appropriate size.

Example: $x^2 - 4x - 7 = 0$

Take $x^2 - 4x$ and add 4 to it to complete the square. If work with the pattern of the perfect square trinomial has been done, students are able to see that ½ of the coefficient of the x term is squared to complete the square.

$x^2 + 2\triangle x$ $\qquad$ $(\frac{1}{2}\, 2\triangle)^2 = \triangle^2$

Since 4 (or $\triangle^2$) has been added to the equation, 4 (or $\triangle^2$) also has to be subtracted--so another form of 0 has been added. $x^2 + 2\triangle + \triangle^2 - \triangle^2 + c = 0$

Example: $x^2 - 4x + 4 - 7 - 4 = 0$

Then the perfect square trinomial and the other constant term are written as squares.

Example: $(x - 2)^2 - (\sqrt{11})^2 = 0$

Since we now have a "difference of two squares," factoring can be accomplished and solutions for x can be calculated.

$$\left((x - 2) - \sqrt{11}\right)\left((x - 2) + \sqrt{11}\right) = 0$$

$$\begin{array}{l|l} x - 2 - \sqrt{11} = 0 & x - 2 + \sqrt{11} = 0 \\ x = 2 + \sqrt{11} & x = 2 - \sqrt{11} \end{array}$$

e. Point out that to use the pattern of a perfect square trinomial in x

$$x^2 + 2\Delta x + \Delta^2 = (x + \Delta)^2$$

it is necessary to have the coefficient of the x^2 term be 1. If it is not, divide the equation by that number to make it 1.

Example: $2x^2 + 4x - 5 = 0$

$$x^2 + 2x - \frac{5}{2} = 0$$

Then continue as outlined in previous section.

f. Point out to students that the problem is no longer the same when a "sum of two squares" instead of a "difference of two squares" is obtained just before the factoring step.

Example: $x^2 + 2x + 3 = 0$

$$x^2 + 2x + 1 + 3 - 1 = 0$$

$$(x + 1)^2 + (\sqrt{2})^2 = 0$$

You can point out that this corresponds to the graph where there is no intersection of the parabola $x^2 + 2x + 3 = y$ and the x-axis. In other words, the sum of two squares cannot be factored over the real numbers. This graphing can be performed to convince the student of this fact. It can be pointed out that since squares of real numbers are always positive or zero, the sum of two squares which are not equal to zero cannot be equal to zero.

Example: $(x + 1)^2 + (\sqrt{2})^2 = 0$

If any discussion of complex numbers has been done in a particular class, you may mention that a sum of two squares can be factored over the complex numbers.

4. The Quadratic Formula

   a. Although completing the square can give a solution to every quadratic equation, point out that a formula has been devised to make the computations easier, Show how the quadratic formula is developed by means of completing the square. The proof of this is in most Algegra I books. Point out that all solutions for x are being sought that will make the general equation $Ax^2 + Bx + c = 0$ true.

   b. It is a good idea to have students go through the development of the formula so they can see that either the formula or completing the square will give solutions to all quadratic equations.

   c. Give students the preference of methods now that both completing the square and the formula method have been developed.

   d. Point out that in the formula $x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$ or $x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$

   when the value of $B^2 - 4AC$ is negative, this corresponds to no real solutions; when $B^2 - 4AC = 0$ this corresponds to one real solution; when $B^2 - 4AC > 0$ this corresponds to two real solutions. Graphing can be done to show this if time permits.

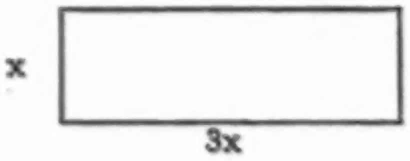
   e. From the formula an able class may be shown that the sum of the roots is $\frac{-B}{A}$ and the product is $\frac{C}{A}$.

D. Problem solving needing quadratic equations

1. Some suggestions for problem solving are given in Chapter III on Powers, Polynomials, Products, Factors.

2. Use of quadratic equations is often made when working with areas. This will come up often in geometry. If time allows, many problems of this variety can be given. This also provides a place for students to practice rejecting of inappropriate roots.

Example: A rectangle of 81 square feet is 3 times as long as it is wide. What are the dimensions?



x = width

$3x$ = length

$$3x^2 = 81$$

$$x^2 = 27$$

$$x = 3\sqrt{3} \text{ or } x = -3\sqrt{3}$$

The negative value is rejected because it is inappropriate as a dimension. Therefore, its width is $3\sqrt{3}$ ft. and its length is $9\sqrt{3}$ ft.

E. Solution of quadratic inequalities in one variable (optional)

1. This is quite an advanced topic but would be a natural extension of this unit. This can be thought of as solutions of sentences of the variety $y > Ax^2 + Bx + C$ (or $<$ or $\leq$, or $\geq$).

2. If any work had been done on suggestion 4 of section 2.3 in this unit, the applications of the graphing can be used.

3. A discussion of this topic is given in (1) pages 479–483.

IX. Nature of Proof (See Preface, Algebra I)

A. Introduction

1. For some classes of able students, formal proofs may have been used throughout the course. For an average class of Algebra I, it is recommended that this be reserved for late in the year. The reasons for this are:

   a. The students are more mature.

   b. The look back at the real number system may make more sense to the student than getting involved with the mechanics of proof as they are going along with the algebraic manipulation.

   c. As one of the late topics to be handled in the year, some carry over to geometry may be made. The danger of this timing is that in many classes, you, as the teacher, will not be able to do much, if anything, with this topic.

2. There is fine discussion of "proof" in (3) pages 111-181. It shows proof to be of two varieties--the probable inference (induction) and the necessary inference (deduction). The Algebra I student may enjoy picking examples of how he has used some of the forms of induction--authority, seeing is believing, measurement, analogy, patterns, and hunches. Some of the tools of deduction are Venn diagrams, formulas of validity, truth tables. Several strategies of proof are discussed. Two of these are presented under the next suggestion part C.

3. Deductive reasoning--which is the basis of most of mathematical proof has some important features which can be highlighted for the students. See (12) pages 112-113 for further discussion.

   a. Definitions are agreements to use words, phrases, or symbols as substitutes for others that are generally longer.

      Example: The opposite or additive inverse of a number is defined to be that number which when added to the original number gives a sum of 0. Since a definition is an agreement it cannot be proved or disproved.

b. Impossibility of defining all technical terms is evident when students think of an example of having looked up certain words in the dictionary. Several definitions later may have ended them at their starting point. Some words have to be understood more basically--a picture or the actual object. These are undefined terms. Set ordering (less than), and number are examples in Algebra. Point, line, plane are examples of undefined terms in geometry.

c. Proof is a chain of statements--the first in the chain is the hypothesis, the last is the conclusion.

Example: If a = b is the hypothesis then a + c = b + c is the conclusion. Statements that are proved are called Theorems. Two types of proof used often in algebra proofs are the modus ponens--if p then q, and p; therefore q.--and rule of contraposition--if p, then q; therefore if (not q) then (not p).

Example: modus ponens

If $\underbrace{\boxed{a = b}}_{p}$ then $\underbrace{\boxed{a + c = b + c}}_{q}$

and $\boxed{\underbrace{\boxed{x + 3}}_{a} = \underbrace{\boxed{5}}_{b}}$

therefore $\boxed{\underbrace{\boxed{x + 3}}_{a} + \underbrace{\boxed{-3}}_{c} = \underbrace{\boxed{5}}_{b} + \underbrace{\boxed{(-3)}}_{c}}$

Example: contraposition

If $\underbrace{\boxed{\sqrt{2} \text{ is a rational number}}}_{p}$

Then $\boxed{\sqrt{2} = \frac{a}{b} \quad a, b \text{ integers with no common factors } b \neq 0}$

q

Therefore if $\boxed{\sqrt{2} \neq \frac{a}{b}} \ \boxed{a, b \text{ integers with no common factors } b \neq 0}$

not q

then $\boxed{\sqrt{2} \text{ is not rational}}$

not p

d. It is impossible to prove every statement of a system because proof is based on some assumed hypothesis. Therefore, some statements are accepted as unproved. These are called axioms or postulates or sometimes properties and are used as the basis of the structure of that system. The word postulate is generally reserved for geometry. Among the first postulates in geometry are:

Example (1) Two lines can intersect in only one point.

Example (2) Through two points, there is one and only one line.

4. Because all the properties about real numbers stated in a geometry text as well as those that can be proved from those stated are taken as postulates, this provides some motivation for looking at some of the proofs in algebra.

B. Properties (postulates) and definitions of algebra

1. Point out that different texts may start out with different properties and different definitions. In a geometric setting, for example, Euclid's postulates can be compared to Birkhoff's postulates or Hilbert's axioms.

2. A suggestion for a class that has done a considerable amount of work with proofs during the year and is ready to do more is to use (9) pages 126-154 as a unit outline. The definitions and the properties are a little different than in most Algebra I texts. This

could give the able students an appreciation for what the change of properties mean in a system of proof. The reference also gives complete proofs.

3. An approach that is advocated for teaching proof is to collect the student's textbooks and just provide them with the needed properties and definitions as they work along in the unit.

4. A possible list of properties and definitions for the real numbers is provided here in very brief form. Some theorems for the students to prove follow in the next section of this unit.

   a. Properties of operations:

      (1) Addition operation

         (a) closure

         (b) commutative

         (c) associative

         (d) There exists an identity element (zero).

         (e) For every real number a there exists an inverse such that $a + (-a) = 0$

      (2) Multiplication operation

         (a) closure

         (b) commutative

         (c) associative

         (d) There exists an identity element (one).

         (e) For every real number a, $a \neq 0$ there exists an inverse such that $a \cdot \frac{1}{a} = 1$.

      (3) Multiplication is distributive over addition. The previous eleven properties are often called the field axioms.

b. Properties of equality
    - (1) reflexive
    - (2) symmetric
    - (3) transitive
    - (4) addition
    - (5) multiplication
    - (6) subtraction
    - (7) division

c. Properties in inequality
    - (1) transitive
    - (2) addition
    - (3) subtraction
    - (4) $a < b$, $a = b$, or $a > b$
    - (5) multiplication by positive real numbers
    - (6) division by positive real numbers

d. Definitions
    - (1) subtraction: $a - b = a + -b$
    - (2) division: $\frac{a}{b} = a \cdot \frac{1}{b}$
    - (3) factor: a is a factor of b if there exists an integer c such that $b = ac$

(4) absolute value: $|x| = x$, if $x \geq 0$

$= -x$, if $x < 0$

C. Theorems

1. The list of theorems that follow is not complete. In some cases students might be able to help make up a list of theorems to be proved. Point out that once a statement has been proven true for all real numbers, it can also join the list of properties and be used in subsequent proofs. Many proofs appear in the Algebra I texts in the references (1), (10), (11).

2. Assume that students know that a number has a unique inverse.

Examples:

a. Prove $-(a + b) = (-a) + (-b)$ for all real numbers a, b

We already know that

$-(a + b) + (a + b) = 0$	addition of opposites
$(a + b) + \big((-a) + (-b)\big) = a + b + (-a) + (-b)$	associative property of addition
$= a + (-a) + b + (-b)$	commutative property of addition
$= \big(a + (-a)\big) + \big(b + (-b)\big)$	associative property of addition
$= 0 + 0$	addition of opposites
$= 0$	identity of 0

But this means $\big((-a) + (-b)\big)$ is also the opposite of $(a + b)$. But an opposite is unique. Therefore $(-a) + (-b)$ must be another name for $-(a + b)$ so $-(a + b) = (-a) + (-b)$. In some cases a proof can be given, but let the students supply the reasons. This process can be reversed, you providing reasons and the students providing the steps.

A partial list of some other theorems follows: (assume a, b, c, d stand for any real number except where specified; denominators are not 0)

b. $a \cdot 0 = 0$

$$\left.\begin{array}{ll} \text{c.} & \text{If } ab = 0 \text{ then } a = 0 \text{ or } b = 0 \\ \text{d.} & \text{If } a \neq 0 \text{ and } b \neq 0 \text{ then } ab \neq 0 \end{array}\right\} \text{contrapositives}$$

e. $(-a)\ (-b) = ab$

f. $-(a - b) = b - a$

g. If $a > b$ then $a + c > b + c$

h. If $a > b$ and $b > c$ then $a > c$

i. If $a > b$ and $c < 0$ then $ac < bc$

j. $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

k. $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

l. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = cb$

m. If a is a factor of b and b is a factor of c, then a is a factor of c (a, b, c integers).

n. If a is an odd integer, then a^2 is an odd integer.

o. If $S = \begin{cases} ax + by + c = 0 \\ dx + cy + f = 0 \end{cases}$ and

$$T = \begin{cases} ax + by + c = 0 \\ k_1\ (ax + by + c) + k_2\ (dx + cy + f) = 0 \end{cases}$$

where $k_2 \neq 0$, then S is equivalent to T (has the same solution set).

p. If a is any real number $\sqrt{a^2} = |a|$

Point out that some theorems may have been presented to the students as definitions or as properties to be accepted as unproved, but true statements.

3. It may be interesting to the student that these theorems that follow from the 21 initial properties used to be stated as rules to be memorized by many Algebra I students. The object now is for the student to see how the system of algebra can be interrelated.

4. The interrelation of algebra and geometry can also be done to some degree.

Example: The theorem: if a = b and c = d then a + c = b + d can be put in geometric terms.

$$\text{if } \angle a \cong \angle b \text{ and } \angle c \cong \angle d$$

$$\text{then } \angle a + c \cong \angle b + d$$

D. Extension ideas

More work with Venn diagrams, truth tables, or methods of proof can be done. References already mentioned for this unit can be consulted, especially: (3) pages 111-181, (10) chapter 5.

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O.C.S.E.I.P. SYLLABUS

Algebra II

SE 006 070

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OFFICE OF EDUCATION

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PREFACE

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of the recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

The first draft of this syllabus was written during an 8 week session at University of California, Irvine during the summer of 1966 by:

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I. Foundations of Algebra II - Time: 15 days

A. Glossary

Number
Natural number
Integer
Real number
Rational number

Numeral
Digit, constant, positive
negative numbers
Prime and composite numbers
Zero

Students and teachers who approach a course in algebra 2 may well be overwhelmed by the amount of material to be covered in one year. Of late years more and more topics have been added, and the need for simplification is evident.

B. Axioms-properties

1. Properties of real numbers

2. Axioms of equality

3. Properties of order

Compounding the dilemma is the fact that students come to the second course in algebra with widely varying backgrounds, depending upon the texts and methods used in previous mathematics courses. For example, some students have been exposed to the terminology and concepts of the "new" mathematics in greater depth than others.

C. Sets

Finite & infinite sets
Subset, proper subset
Element, domain, intersection of sets, union of sets,
Complement of a set

Null set, Empty set $\emptyset$
Universal set
Disjoint sets, closure
Venn diagrams

It is suggested that the teacher review the vocabulary dealing with the foundations of algebra by using a check list of words such as this. Introduce as examples are met.

The student should probably know the meaning of all these words, and be able to use them. The teacher will of course use his own judgment as to which ones need more attention than others, according to the previous training and ability of his students.

D. Algebraic expressions

Term, like terms
Unlike terms
Formula, evaluation of formulas
Variable, literal number
Monomial, binomial
Trinomial, polynomial
Degree of a monomial
Degree of polynomial
Coefficient of a monomial

The list presented here includes the vocabularies given in the most widely used textbooks, in the introductory chapters.

E. Fundamental

Addends, minuend
Subtrahend, dividend
Multiplicand
Multiplier, difference divisor, quotient, product
Absolute value, factor, prime factorization
Simplest form (of fraction)
Indeterminate form, $\frac{0}{0}$
Reciprocal, (Multiplicative inverse)
Equality, inequality, equivalent expression
Identity element for addition
Identity element for multiplication
Order of operations
Operation
Binary operation, unary

F. Powers and Roots

Squares, square root
Cubes, cube root
Index, base, power
Principal root
Radical
Radicand
Exponent

It is helpful for students to keep a vocabulary list of words used in algebra 2, probably in alphabetical order, adding to the list as new material is introduced.

G. Symbols in Inclusion or Grouping Symbols

Parentheses
Bars (vinculum)
Braces
Brackets

The idea of system as related to mathematics. Requirements of a System:

1. A set of symbols
2. A set of operations
3. A set of undefined terms
4. A set of axioms
5. Theorems developed from the above

H. Other Symbols

$\in$ is an element of

$\emptyset$ empty set

$\{\}$ set

$\subset$ is a proper subset of (In some texts "any subset")

$\cap$ intersection

$\cup$ union

PROPERTIES OF REAL NUMBERS

If a, b, c, are real numbers

ADDITION		MULTIPLICATION
$a + b =$ real number	closure	$a \cdot b =$ a real number
$a + b = b + a$	commutative property	$ab = ba$
$a + (b + c) = (a + b) + c$	associative property $a\,(b + c) = ab + ac$ (a distributive property)	$a \cdot (bc) = (ab) \cdot c$
$a + 0 = a$ or 0 is the identity element for addition	identity elements	$a \cdot 1 = a$, or 1 is the identity element for multiplication
$a + (-a) = 0$, or $-a$ is the additive inverse or negative of a	inverses	$a \cdot \frac{1}{a} = 1$, or $\frac{1}{a}$ is the multiplicative inverse or reciprocal of a.

Using these 11 properties as a foundation, some examples of theorems which can be proved deductively are the following.

$0 \cdot a = 0$ and $a \cdot 0 = 0$, the multiplication property of zero

$ab = 0$ if and only if $a = 0$ or $b = 0$

$-(a + b) = (-a) + (-b)$, negative of a sum

$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$, if $a \neq 0$, $b \neq 0$

BASIC PROPERTIES OF EQUALITY

a, b, c, are real numbers

$a = b$, means a and b stand for the same number

$a = a$	- reflexive property
If $a = b$, then $b = a$	- symmetric property
If $a = b$, and $b = c$, then $a = c$	- transitive property
If $a = b$, and $a + c = d$, then $b + c = d$ If $a = b$, and $ac = d$, then $bc = d$	- substitution property
If $a = b$, then $a + c = b + c$	- addition property
If $a + c = b + c$, then $a = b$ or if $a = b$, then $a - c = b - c$	- subtraction property
If $a = b$, then $ac = bc$	- multiplication property
If $ac = bc$, $c \neq 0$, then $a = b$	- division property

PROPERTIES OF ORDER

a, b, c are real numbers

Either $a < b$, $a = b$, or $a > b$	- comparison property
If $a < b$ and $b < c$, then $a < c$ If $a > b$ and $b > c$, then $a > c$	- transitive property
If $a < b$, then $a + c < b + c$ If $a > b$, then $a + c > b + c$	- addition property
If $a < b$, then $ac < bc$, if $c > 0$ If $a < b$, then $ac > bc$, if $c < 0$	- multiplication property
If $a < b$, then $a - c < b - c$ If $a > b$, then $a - c > b - c$	- subtraction property
If $a < b$, then $\frac{a}{c} < \frac{b}{c}$, if $c > 0$ If $a < b$, then $\frac{a}{c} > \frac{b}{c}$, if $c < 0$	- division property
If $a < b$, then there is a positive number c, such that $b = a + c$	

II. Linear Equations in One Variable, Inequalities - Time: 15 days

A. Solution of equations

1. Perform indicated operations to remove symbols of grouping
2. Collect terms whenever possible
3. Transform equation by addition
4. Transform equation by multiplication

By the time a student is in second-year algebra, the solution of linear equations in one variable should be almost automatic. However, a review of the principles involved, and additional experience in solving a variety of equations, containing parentheses and/or fractions, are essential at this stage.

It is advisable for the student to give the reasons for each step in the solution (distributive principle, addition axiom, etc.) especially during the first presentation and explanation so that he can proceed on firm ground.

Example: Solve and check $x + 4(2x - 3) = 5x - 4$

$x + 8x - 12 = 5x - 4$ Dist. (distributive law)

$9x - 12 = 5x - 4$ C. T. (collect like terms)

$9x - 5x = -4 + 12$ A(-5x) a(12) (add -5x, add 12)

$4x = 8$ C.T.

$x = 2$ M (1/4) (multiply by 1/4)

The student should be urged to collect terms continually, to simplify the work enroute.

B. Checking the solution

r is a root of $f(x) = 0$, Domain: $\{x : a < x < b\}$ if and only if $f(r) = 0$

[Explain $f(r)$]

The student must develop the habit of checking the solutions to all equations. The check *must* be made on the *original* of the equation, not on some transformation of the equation, he may have done incorrectly.

If there are parentheses, after substituting the value of the solution in the equation, he should combine terms within the parentheses first.

Example: $2(x + 3) = 14$

Check: $2(4 + 3) = 14$

$x = 4$ $\quad 2(7) = 14$

$14 = 14$

not $\quad 2(4 + 3) = 14$

$8 + 6 = 14$

$14 = 14$

The reason for this is that the checking should, if possible, be done in a *different* way from that in which the problem was worked out.

C. Literal equations

$ax + bx = c$

$x(a + b) = c$

$x = \frac{c}{a + b}$

Literal equations are often more difficult than those containing only one letter, and provide good practice in the operations of equation-solving. The use of the distributive principle in extricating the variable is a necessity when the variable appears in more than one term. Especially important are problems involving the solution of formulas for the various letters in the right member.

Example: $A = 1/2\ h(b + b')$; Solve for b

$$2A = hb + hb'$$

$$2A - hb' = hb$$

$$\frac{2A - hb'}{h} = b$$

D. Word or verbal problems - changing from English to algebraic expressions

1. Changing from English to algebraic expressions

When beginning the unit on word problems, practice on simple expressions before tackling equations. Such phrases as these should be put into algebraic form; and making word problems from the algebraic form will help to fix these ideas in mind.

a. The sum of the squares of two numbers a and b. $a^2 + b^2$

b. The square of the difference of a and b $(a - b)^2$

c. 4 more than the difference between the cubes of x and y $4 + (x^3 - y^3)$ etc.

2. Writing equations (or open sentences) from verbal statements

At first it is good to have concentrated practice on writing the equations, only.

a. Draw a figure, if possible. Label all parts indicated.

b. Use a symbol to represent one of the unknown quantities asked for.

d. Write an equation using the given relationships

e. Solve the equation

f. Show that the solutions satisfy the conditions required by checking in the original problem

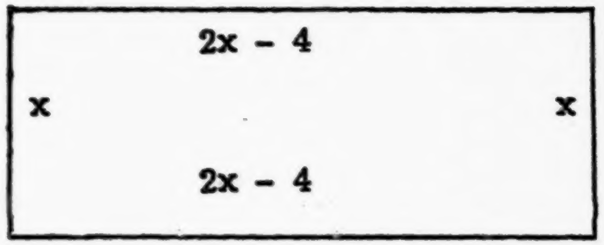
At first it is good to have concentrated practice on writing the equations, only.

When the student gains confidence through success, work on the complete solution of such equations. Steps b and c are often stumbling blocks. The student must have step c written down in black and white before attempting to write the equation. (After he gains facility, he will, of course, be able to condense the solution somewhat.)

Example: The length of a rectangle is 4 ft. shorter than twice the width. Find the dimensions if the perimeter is 118 feet.

(b) Let x = width

(c) $2x - 4$ = length



(d) $2x + 2(2x - 4) = 118$
$2x + 4x - 8 = 118$

(e) $6x = 126$
$x = 21'$ = width
$2x - 4 = 38'$ = length

(f) $2(21) + 2(38) \overset{?}{=} 118$
$46 + 76 \overset{?}{=} 118$
$118 = 118$

3. Types of verbal problems

a. Number relationships

b. Geometry

c. Motion

d. Mixtures

e. Investment

f. Work problems

The accent should perhaps be on problems that "make sense" to the student, such types as a, b, and c, although the others given here, d, e, being the ones presented in most texts, provide excellent experience in equation-writing. Special attention should be given to a, number relationships. (Care must be exercised here lest the pupil try to "type" all problems.)

This topic, verbal problems, should receive as much attention as time permits. Even after the class has moved on to other topics, a few verbal problems should be given frequently, and on weekly tests. The rules of order (see outline in Chapter 1, Foundations of Algebra) should be reviewed here.

E. Solving Inequalities

Example: Solve: $|x - 2| < 3$

Case I: $x \geq 2$, $|x - 2| = x - 2$

$x - 2 < 3$

$\underline{x < 5}$

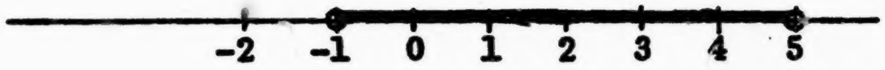
Case II: $x \leq 2$, $|x - 2| = 2 - x$

$2 - x < 3$

$-x < 1$

$\underline{x > -1}$

Graph the inequalities:



$|x - 2| \quad 3 \quad (x \quad -1 \text{ and } x \quad 5)$

Note: The inequality A resolves into two dependent inequalities.

E. Graphing Inequalities

Example: Solve: $|x - 2| > 3$

Case I: $x \geq 2, \ |x - 2| = x - 2$

$x - 2 > 3$

$\underline{x > 5}$

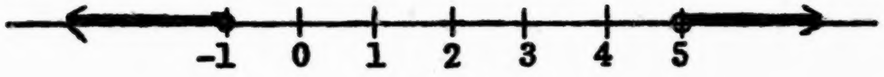
Case II: $x \leq 2, \ |x - 2| = 2 - x$

$2 - x > 3$

$-x > 1$

$\underline{x < -1}$

F. Graph the inequalities:



$|x - 2| < 3 \Leftrightarrow (x > -1 \text{ or } x < 5)$

Note: The inequality E resolves into two distinct inequalities. Current usage regards a a solution of an inequality to produce numbers defined by an inequality or inequalities which represent intervals on the number line.

The concept of absolute value, and its applications, is a difficult one for many students. Especially good are some of the graph8ng exercises.

Problems involving absolute value:
Absolute value:

a. Develop slowly and clearly with easy examples, always starting with a definition of Cases.

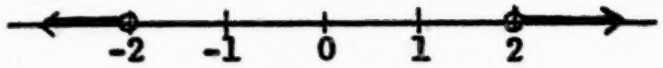
Example I:

a) $|x| > 2$

Case I: $x \geq 0, \ |x| = x$
$\underline{x > 2}$

Case II: $x < 0, \ |x| = -x$
$-x > 2$
$\underline{x < -2}$

Graph:



Note: A graph of a plane region can result when working with a linear inequality as $y \geq x + 2$.

Example III: $|x| + |y| = 3$

Case I: $x < 0, \ |x| = -x$
$y < 0, \ |y| = -y$
$-x - y = 3$
$y = -x - 3$

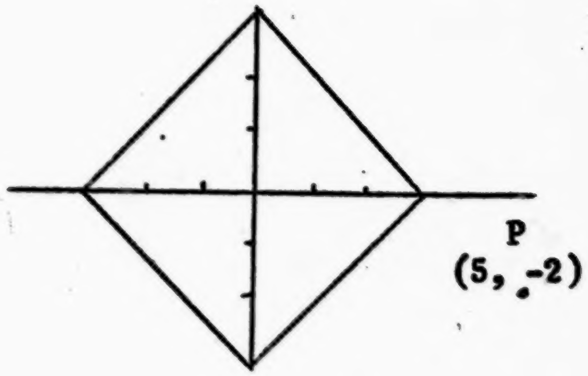
Example II:

b) $y = |x - 3|$

Case I: $x \geq 3, \ |x - 3| = x - 3$
$y = x - 3$

Case II: $x \leq 3, \ |x - 3| = 3 - x$
$y = -x + 3$

Case II: $x > 0, \ |x| = x$
$y < 0, \ |y| = -y$
$x - y = 3$
$y = x - 3$



Case III: $x < 0, \ |x| = -x$
$y > 0, \ |y| = y$
$-x + y = 3$
$y = x + 3$

Case IV: $x < 0, \ |x| = x$
$y > 0, \ |y| = y$
$x + y = 3$
$y = -x + 3$

x and y must be chosen from only those numbers whose absolute values have a sum of 3, as the equation states. The point P(5, -2), for example, lies on the line $x + y = 3$, but $|5| + |-2| \neq 3$.

Example IV: Solve $\frac{x + z}{4 - x} > 0$

a. $4 - x = 0$, not possible

b. $4 - x > 0$
$\frac{x + z}{4 - x} \cdot \frac{(4 - x)}{1} > 0$

$x + z > 0$
$x > -2$
$x < 4$
$-z < x < 4$

c. $4 - x < 0$
$\frac{x + z}{4 - x} \cdot \frac{(4 - x)}{1} < 0$
$x + z < 0$
$x < -2$
$x > 4$
not possible

d. Combining parts a., b., and c., the final solution is $-2 < x < 4$.

III. Type Products, Factoring, Equations - Time: 15 days

A. Type products of the general form: $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$.

1. In the above general form, if $a = c$ the result is a quadratic trinomial, or

$$(a+b)(a+d) =$$
$$a(a+d) + b(a+d) =$$
$$a^2 + ad + ab + bd, \text{ or}$$
$$a^2 + (b+d)a + bd$$

Or:

$$(ax + by)(cx + dy) =$$
$$acx^2 + (ad + bc)xy + bdy^2$$

The general form of the product of any two binomials $(a+b)$ and $(c+d)$ involves the use of the distributive property of multiplication with respect to addition.

Example 1: $(x + 2)(x + 5) = x^2 + 7x = 10$.

Example 2: $(3x + 2y)(4x + 5y) =$
$12x^2 + 8xy + 15xy + 10y^2$, or
$12x^2 + 23xy + 10y^2$

$$(3x + 2y)(4x + 5y)$$

F	O	I	L
Product of first terms	Product of outer terms plus the product of inner terms		Product of last terms

2. $(a \pm b)(a \pm b) =$
$a^2 \pm 2ab + b^2$

also

$(ax \pm by)(ax \pm by) =$
$a^2x^2 \pm 2abxy + b^2y^2$

Students should learn to recognize the perfect-square trinomial, in which the middle term equals $\pm$ twice the product of the square roots of the first and last terms.

3. $(a + b)(a - b) = a^2 - b^2$

also

$(ax + by)(ax - by) =$
$a^2x^2 - b^2y^2$

Most students need a wealth of practice on the various forms of binomial products, until a reasonable degree of facility results.

B. Factoring

1. Factoring monomials into positive integral factors

   a. Finding all possible factors
   b. Finding the prime factors

The factors of 20 are 1,2,4,5,10,20 but the prime factors are 2, 2, 5.

An efficient way to find the prime factors of a number is to divide successively by prime numbers.

Example: Find the prime factors of 420.

$$\begin{array}{l|l} 2 & 420 \\ \hline 2 & 210 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline & 7 \end{array}$$

The prime factors of 420 are 2,2,3,5, and 7.

Review or teach divisibility rules for 2, 3, and 5.

c. Greatest common factor

Find the prime factors, as above, then find the largest group common to both.

Example: Find the greatest common factor of 420 and 500.

The prime factors of 420 are 2, 2, 3, 5, 7 (above).
The prime factors of 500 are 2, 2, 5, 5, 5.
The greatest common factor is $2^2 \cdot 5$, or 20.

d. Lowest common multiple

Proceed as before to find the prime factors, then use every factor the greatest number of times it is used in one place.

Example: Find the L.C.M. of 420 and 500.

$420 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 2^2 \cdot 3 \cdot 5 \cdot 7$
$500 = 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 = 2^2 \cdot 5^3$

$LCM = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7 = 2^2 \cdot 3 \cdot 5^3 \cdot 7$
$LCM = 10{,}500$

2. Factoring Polynomials

a. Removing greatest common monomial factor: $axy + bxy = xy(a + b)$

In factoring polynomials, the type forms listed in 3.1 above are used in reverse. However, the problems often have a different aspect when factorization is asked for, and it is helpful to classify the given polynomials according to appearance, -- as binomials, trinomials, etc.

It is of great importance to remove the greatest common monomial factor first, when factoring. Otherwise, either the polynomial will appear to be unfactorable, or the final answer will almost surely contain factors.

Example 1: $16a^3 - 9a = a(16a^2 - 9) =$
$a(4a + 3)(4a - 3)$

Until the common factor, a, was removed, the other factors would probably not be discovered.

Example 2: Factor $4a^2 + 8ab + 4b^2$

$4(a^2 + 2ab + b^2) = 4(a + b)^2$

But if the factor 4 is not removed first, --$4a^2 + 8ab + 4b^2 = (2a + 2b)^2$, and 9 times out of 10 the student will present this as his answer, failing to notice that

$(2a + 2b)(2a + 2b) = 2(a + b)(2)(a + b) =$
$4(a + b)^2$, the correct answer in prime factors.

3. Factoring binomials

a. The difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

The student should be taught to recognize the difference of two squares: sometimes simple squares, sometimes the squares of binomials or even trinomials, such as:

Example 1: $(a+b+c)^2 - (2x-y)^2 = (a+b+c+2x-y)(a+b+c-2x+y)$

Example 2: $a^2 + 4ab + 4b^2 - 9 = (a + 2b)^2 - 9 = (a+2b+3)(a+2b-3)$

b. The sum of identical powers of two different expressions: $a^n + b^n$

Case I.
If n = 1,2,4,8, etc. (Powers of 2), the binomial $a^n + b^n$ is prime (over the set of real numbers).

Case II.
If n is not a power of 2, and n is prime, $a^n + b^n$ is always divisible by $(a + b)$ to obtain the other factor, which will be prime.

Example: $(a^3 + b^3) \div (a + b) = a^2 - ab + b^2$

Case III.

If n is not a power of 2, and n is composite, write $a^n + b^n$ in the form,

$$a^{cm} + b^{cm} \quad \text{or} \quad (a^c)^m + b^c)^m = (a^c + b^c)(........)$$

Dividing $(a^{10} + b^{10})$ by $(a^2 + b^2)$ yields the other factor,

$$(a^{10} + b^{10}) = (a^2 + b^2)(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8)$$

Notice that we do not write $a^{10} + b^{10} = (a5)^2 + (b5)^2$, since the exponent of 2 is a power of 2 (Case I above).

c. The difference of identical powers of two different expressions: $a^n - b^n$

Case I.

If n is even, write $a^n - b^n = a^{2m} - b^{2m} = (a^m)^2 - (b^m)^2$.

Example: $a^{10} - b^{10} = (a^5)^2 - (b^5)^2 = (a^5 + b^5)(a^5 - b^5)$.

These binomials must be factored again, by dividing by (a+b) and (a-b) respectively, as in Case II above.

$$(a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)(a-b)(a^4 + a^3b - a^2b^2 + ab^3 - b^4)$$

Note again that we do not write $a^{10} - b^{10} = (a^2)^5 - (b^2)^5$.

In this form the expression can be factored into $(a^2 - b^2)(.......)$.

Introduce the factor theorem, and thence into $(a + b)(a - b)(.....)$but the third factor will not be in a form which the student can factor. Only by factoring first as the difference of two squares, as above, will all four factors be discovered.

Case II.

If n is odd, and prime, (a-b), is one factor. Divide by (a-b) to find the other factor.

Example: $(a^3 - b^3) == (a-b)(a^2 + ab + b^2)$

Case III.
If n is odd, and composite, treat $a^n - b^n$ in the same manner as $a^n + b^n$ (Case III).

Example: $a^9 - b^9 = (a^3)^3 - (b^3)^3 =$
$(a^3 - b^3)(a^6 + a^5b^3 + y^6) =$
$(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + y)$

d. A special case $a^{4n} + 4b^{4n}$, where n is a positive integer.

These can be solved by completing the square (add and subtract $(4a^4b^4)$.

Example:

$$\begin{array}{llll} a^8 & & +\ 4b^8 & \\ & +\ 4a^4b^4 & & -4a^4b^4 \\ \hline (a^4 & + & 2b^4)^2 & -4a^4b^4 \end{array}$$

or

$(a^4 + 2b^4 + 2a^2b^2)(a^4 + 2b^4 - 2a^2b^2)$

Note: The method of completing the square has not as yet been reviewed in this syllabus, but was supposedly a part of the student's first course in algebra.

4. Factoring trinomials

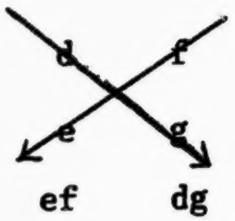
a. Quadratic Trinomials: $ax^2 + bxy + cy^2$

To factor this type of expression (the one the student is probably most often confronted with), work the multiplication problem in reverse, finding coefficients d,e,f, and g such that $d \cdot e = a$, $f \cdot g = c$, but also $dg + ef = b$.

Have the students picture the letters as follows:

$$\begin{array}{cccc} d & x & f & y \\ e & x & g & y \\ \hline \end{array}$$

$$\textcircled{a}x^2 + \textcircled{b}xy + \textcircled{c}y^2$$



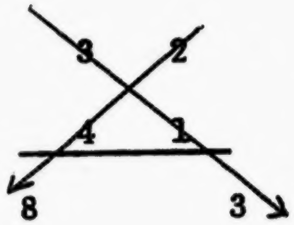
This may take several trials, but with practice, will usually be done easily unless a and c have several factors.

At first, it is good to have the student draw all the possible ways of arranging the factors, putting them down in logical sequence in order to locate them all.

Example: Factor the expression $12x^2 + 5x - 2$.

All the possibilities are:

$$\begin{array}{ccccccccc} 2 & 2 & & 3 & 2 & & 12 & 2 \\ \underline{6} & \underline{1} & & \underline{4} & \underline{1} & & \underline{1} & \underline{1} \\ \\ 2 & 1 & & 3 & 1 & & 12 & 1 \\ \underline{6} & \underline{2} & & \underline{4} & \underline{2} & & \underline{1} & \underline{2} \end{array}$$



Only this arrangement can possible yield +5 as the sum of the cross products dg and ef: 3 and 8 can produce +5, if 3 is negative and 8 is positive. Keeping 3 and 4 positive (since 12 is positive) we have $(3x + 2)(4x - 1)$. The student should be alert to the arrangement of signs. (When the constant term is negative, the constant terms of the factors will have opposite signs, etc.) A careful final check is extremely important.

b. Trinomial Square: $a^2x^2 + 2abxy + b^2y^2 = (ax + by)^2$

These must have perfect squares for their first and last terms, and the middle term must be twice the product of the square roots of the first and last terms, in both halves of the problem.

Sometimes it helps to substitute a single letter for the common group:

Example: $ax + ay + az + bx + by + bz = a(x+y+z) + b(x+y+z)$

Substituting R for $(x+y+z)$, $aR + bR = R(a+b) = (x + y + z)(a+b)$

b. **Difference of two squares:** $(ax \pm by)^2 - c^2$

Some quadrinomials (polynomials having four terms) can be arranged as the difference of two squares:

Example: $a^2 + 2ab + b^2 - 9c^2 = (a+b)^2 - 9c^2 = (a+b+3c)(a + b - 3c)$

Students may have trouble putting the following type of problem in the proper form. (It should be stressed that if necessary, a trinomial square must be assembled and placed in a parentheses preceded by a minus sign so that the difference of 2 squares is formed).

Example: $1 - x^2 + 2xy - y^2 = 1 - (x^2 - 2xy + y^2) = 1 - (x-y)^2 =$

$$\left[1 + (x-y)\right]\left[1 - (x-y)\right] = (1+x-y)(1 - x+y)$$

c. Trinomials factorable by completing the square $ax^{4n} + bx^{2n} + 1$, where $\underline{a}$ is a perfect square and $\underline{b}$ is such that if kx^{2n} is added to bx^{2n}, the trinomial will become a perfect square, provided also that k is a perfect square.

These are usually polynomials of the fourth degree.

Example:
$$\begin{array}{cc} 4x^4 + 3x^2 + 1 & \\ + x^2 & -x^2 \\ \hline (2x^2 + 1)^2 & -x^2 \end{array} \qquad \text{or} \qquad (2x^2 + 1 + x)(2x^2 + 1 - x)$$

Here the term x^2 was added, then subtracted, in order to produce a trinomial square, $4x^4 + 4x^2 + 1$.

This is actually the same type of factoring problem as described under binomials, 3d.

5. Factoring Polynomials of 4 or more terms.

a. Expressions having a common polynomial factor: $ax + ay + bx + by =$
$a(x+y) + b(x+y) = (a+b)(x+y)$

Some expressions having 4 or more terms can be grouped to obtain a common polynomial factor. Sometimes the student must devise ways of rearranging the expression to discover the common factors.

Example: $a^3 + a + a^2 + 1 = a(a^2 + 1) + 1(a^2 + 1) = (a + 1)(a^2 + 1)$

Note that the factor, 1, is removed from the second binomial group, in order to have the same binomial factor $(a^2 + 1)$ in both groups.

6. Factoring Polynomials of any length, by the factor theorem.

Using the Factor Theorem, if $f(a) = 0$, $x - a$ is a factor of $f(x)$, some polynomials which are different in any other way can be factored.

Example: $x^3 + x^2 + 4$.

Substituting $x = -2$, we have $-8 + 4 + 4 = 0$; hence, $f(-2) = 0$. Therefore $(x + 2)$ is a factor. Divide the given polynomial by $(x + 2)$, obtaining $(x^2 - x + 2)$.

Since this is prime, the answer is $(x + 2)(x^2 - x + 2)$. $(x^2 - x + 2)$ is unfactorable, by trying the only possible factors. However, the method using $b^2 - 4ac$ (the discriminant) is a quick test for primeness. Explain the use of the discriminant.

In fact, it is often possible to factor polynomials usually done by some other method, by means of the Factor Theorem. For example, $a^3 + a + a^2 + 1$. It is easy to see that $f(-1) = 0$. Therefore, $(a + 1)$ must be a factor, and by division the other factor can be found. (Note that this polynomial can also be factored by grouping as shown in B5.)

C. Solution of equations by factoring.

The importance of this topic should be made clear to the student, since much of his time will be spent in finding solutions to equations, especially quadratics. The student must master the fundamental principle that $ab = 0$ if and only if $a = 0$ or $b = 0$. This tells his that he must transform the equation to an equivalent one having 0 as one member of the equation, and that after factoring the other member completely he must set every factor equal to zero and solve the resulting equations.

Example: $x^3 = 4x$

$x^3 - 4x = 0$ $\qquad \{0, 2, -2\}$

$x(x^2 - 4) = 0$

$x(x + 2)(x - 2) = 0$

If he had divided the original equation by x, he would have lost the solution $(x = 0)$. This is a common error, eliminated if the student has the factor theorem firmly fixed in his mind.

Emphasize, too, the perpetual need to check all solutions, but also the fact that even if all solutions check, there still may be others not discovered because of such error as the above. Be sure student knows that an equation of third degree must have 3 solutions of some kind, he should not be content with only two.

D. Solution of inequalities by factoring.

1. Algebraic solution

The solution of inequalities by factoring demands a thorough grasp of the theorems on inequality. The student can lose one set of solutions if he does not remember that the product of two positive numbers is positive, but also the product of two negative numbers is positive.

Example: $x^2 - 2x > 8$ $\qquad$ The left member will be positive if

$x^2 - 2x - 8 > 0$ $\qquad$ (1) both factors are positive

$(x-4)(x+2) > 0$ $\qquad$ (2) both factors are negative.

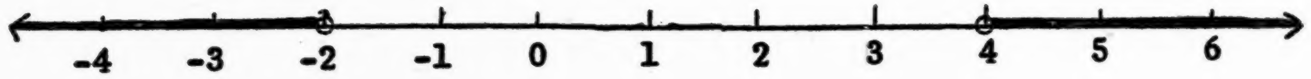
If both factors are positive, $x > 4$ and $x > -2$, the intersection of these solution sets is $\{x : x > 4\}$

If both factors are negative, $x < 4$ and $x < -2$, the intersection of these solution sets is x: $\{x < -2\}$

Therefore the solution set of the inequality $x^2 - 2x > 8$ is $\{x: \ x > 4\} \cup \{x: \ x < -2\}$

In graphing solution sets of inequalities, emphasize that all parts are graphed.

Example: graph of the above inequality.



IV. Fractions and Fractional Equations - Time: 10 days

A. Fractions

1. Definition

An indicated quotient of 2 algebraic expressions, defined, only when the denominator $\neq 0$.

Example: $\frac{3}{x-2}$ is not defined when $x = 2$

2. The multiplication property of fractions: $\frac{ac}{bc} = \frac{a}{b}$, $c \neq 0$

Dividing or multiplying the numerator and denominator of a fraction by the same nonzero number produces a fraction equal to the given one.

Example: $\frac{20a^2 + 5ab}{16a^3 - ab^2} = \frac{5a(4a + b)}{a(4a-b)(4a+b)} = \frac{5}{4a-b}$

3. Reduction of Fractions to lowest terms.

One of the greatest sources of error here is the failure to factor before dividing numerator and denominator by some number or expression. For example, some students may want to "cancel" or reduce as at the beginning:

$$\frac{20a^2 + 5\cancel{a}b}{16a^3 - \cancel{a}b^2} \qquad \text{etc.!!!}$$

They must be made to see clearly that "cancelling" (reducing) can take place only when something is divided by itself. Whenever this operation is questioned, the teacher can write on the board, $\frac{a + b}{b} = \frac{1}{b}(a + b)$. Then ask, "Is b̲ a factor of (a + b)?"

"NO." "Then can you cross out the b's?" "NO!" "Why?" "Because crossing out indicates the presence of the identity element." $\frac{a}{b} + \frac{b}{b} = \frac{a}{b} + 1$.

At this point, write on the board, $\frac{ab}{b}$.

"Can you cross out the b's?" "Yes, because b i̲s̲ a factor!" $\frac{a}{1} \cdot \frac{b}{b}$

This point will have to be hammered home for some students in every class, and a dialogue similar to the above, repeated when needed, will in time do the trick.

4. Addition of fractions having equal denominators: $\frac{a}{c} + \frac{b}{c} = ?$ $\quad a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = \frac{1}{c}(a+b) = \frac{a+b}{c}$

The distributive principle is the basis here. Also, the "denominator" nominates, or names, the kind of thing, just as $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$. (We have 2 of these things called sevenths, added to 3 of them, getting 5 of them). So here we have a̲ of the c'ths added to b̲ of them, getting (a + b) of them.

Example: $\frac{2x}{3x+y} - \frac{5x}{3x+y} - \frac{(x-1)}{3x+y} \rightarrow \frac{2x - 5x - x + 1}{3x + y} \rightarrow \frac{-4x + 1}{3x + y}$

Some students will want a method of checking such a problem as this. The final expression will be true of any values of x and y (Denominator $\neq 0$). By setting the first variable equal to 2, the second variable equal to 3, etc. in both members of the equation, and carrying out the designated operations, we have a fairly **good verification** of the final result.

In the above problem letting $x = 2$ and $y = 3$:

$$\frac{2x}{3x+y} - \frac{5x}{3x+y} - \frac{(x-1)}{3x+y} \qquad \frac{-4x+1}{3x+y}$$

$$\frac{4}{9} - \frac{10}{9} - \frac{1}{9} \qquad \frac{-7}{9}$$

$$\frac{-7}{9} \qquad \frac{-7}{9}$$

This method of checking can be used to advantage in problems involving the addition, multiplication, and division of algebraic expressions, especially those involving fractions.

5. Addition of fractions having unequal denominators: $\frac{a}{c} + \frac{b}{d} = \frac{(a)}{c}\frac{(d)}{d} + \frac{(b)}{d}\frac{(c)}{c} = \frac{ad + bc}{cd}$

This operation involves the multiplication property of fractions, and the distributive principle; we must have denominators that are alike before addition takes place.

Example: Add $\frac{2}{3a} + \frac{5}{6ab} - \frac{4}{15a^2}$

The student should write the fractions with space between to make room for the multipliers (building factors):

(1) $\frac{2}{3a}\left(\frac{10ab}{10ab}\right) + \frac{5}{6ab}\left(\frac{5a}{5a}\right) - \frac{4}{15a^2}\left(\frac{2b}{2b}\right)$

(2) $\frac{20ab}{30a^2b} + \frac{25a}{30a^2b} - \frac{8b}{30a^2b}$

(3) $\frac{20ab + 25a - 8b}{30a^2b}$

To find LCD:

$$3a = 3\cdot a$$
$$6ab = 2\cdot 3\cdot a\cdot b$$
$$15a^2 = 3\cdot 5\cdot a^2$$

$$LCD = 2\cdot 3\cdot 5\cdot a^2\cdot b$$
$$LCD = 30a^2b$$

Each of the original fractions has been multiplied by a fraction (such as $\frac{10ab}{10ab}$, etc) whose value is 1, (the identity element for multiplication), thereby changing only the appearance of the fraction, not its value.

It is important, at the beginning of this unit, to have the students write in the multipliers $\frac{10ab}{10ab}$, etc. both as a check (will the denominator actually become $30a^2b$?) and as an insurance against "denominator losing" $(\frac{2}{7} + \frac{3}{7} = 5)$. After students have studied the solution of fractional equations, they may become "denominator losers" as follows:

$$\left(\frac{2}{3a} + \frac{5}{6ab} - \frac{4}{15a^2}\right) 30a^2b = 20ab + 25a - 8b!!!$$

Here the student is, of course, thinking of the problem as an equation to be solved, not as an addition-of-fractions problem.

As students become more fluent with the addition of fractions, the form showing the multipliers 10ab etc. may, of course, be condensed, - but only if the student is thinking the multipliers. This is a good example of the fact that each person should develop the style of operation that is as concise as he is capable of using without danger of error.

The addition of fractions having unlike denominators is a difficult one for many students, even in second-year algebra, and demands careful teaching procedures and a generous allowance of time.

6. Multiplication of fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Example: $\frac{2a-4}{a^2-1} \cdot \frac{a^3+a^2}{4-a^2} = \frac{2(2-a)}{(a+1)(a-1)} \cdot \frac{a^2(a+1)}{(2-a)(2+a)} = \frac{-2a^2}{(a-1)(a+2)}$

In order to match the factor (a-2) in the numerator with (2-a) in the denominator, the factor (-2) instead of (2) was removed from (2a-4). $\frac{a-b}{b-a}$ is one, except for the sign.

The resulting fraction must be expressed in a form in which the numerator and denominator are relatively prime (contain no common factors) - that is, in simplest form.

7. Multiplication of mixed expressions: $(a + \frac{b}{c})(d + \frac{e}{f}) = \left(\frac{ac}{c} + \frac{b}{c}\right)\left(\frac{df}{f} + \frac{e}{f}\right) =$

$$\frac{ac+b}{c} \cdot \frac{df+e}{f} = \frac{(ac+b)(df+e)}{cf}$$

Just as in the multiplication of mixed numbers: $(1\frac{2}{3})(2\frac{1}{4}) = \frac{5}{3} \cdot \frac{9}{4} = \frac{15}{4}$ each mixed expression must be expressed as a fraction, so that the multiplication principle for fractions can be applied: numerators multiplied by numerators, etc. We must have numerators!

8. Division of Fractions: $\frac{r}{s} \div \frac{x}{y} = \frac{r}{s} \cdot \frac{y}{x} = \frac{ry}{sx}$

Since $\frac{a}{b}$ means $a \cdot \frac{1}{b}$, division of a by b means that a is multiplied by the reciprocal of b.

However, in the division of fractions, it is sometimes better to write the first fraction over the second, instead of multiplying the first fraction by the reciprocal of the second.

The numerator and denominator can then be cleared of fractions by multiplying by the LCM of all denominators.

$$\frac{\frac{r}{s} \cdot sy}{\frac{x}{y} \cdot sy} = \frac{ry}{sx}$$

9. Division of mixed expressions: $\left(a + \frac{b}{c}\right) \div \left(d + \frac{e}{f}\right)$ (Complex Fractions)

Method 1: Write the first expression over the second, multiply all terms by the L.C.M. of all denominators:

$$\frac{a + \frac{b}{c}}{d + \frac{e}{f}}\left(\frac{cf}{cf}\right) = \frac{acf + bf}{cdf \quad ce}$$

This automatically clears all fractions and is usually much simpler than method 2, in which each mised expression is changed to a pure fraction, the first one being multiplied by the reciprocal of the second.

Method 2:

$$(a + \frac{b}{c}) \div (d + \frac{e}{f}) = \frac{ac + b}{c} \cdot \frac{f}{df + e} = \frac{acf + bf}{cdf + ce}$$

B. Fractional equations

This is a less difficult topic than the addition, multiplication or division of fractions. It is necessary to factor all polynomial denominators at the start, in such a way as to produce like polynomial factors (such as x - 3 below):

Example:

$$\frac{x + 2}{2x - 6} + \frac{3}{3-x} = \frac{x}{2}$$

$$\left[\frac{x + 2}{2(x-3)} + \frac{-3}{x-3} = \frac{x}{2}\right] \qquad \begin{matrix}2(x-3)\\ \text{LCD}\end{matrix}$$

Multiply by the LCD:

$$x + 2 - 6 = x^2 - 3x, \text{ or } x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0, \; x = 2,2.$$

As usual, each solution must be checked. When each member of an equation is multiplied by an expression involving the variable, as here, a redundant equation may result. Define a redundant equation (Has roots which are not roots of the original equation).

$$\left[\frac{x}{x-3} + \frac{3}{x+3} = \frac{x^2+9}{x^2-9}\right] x^2 - 9 \text{ or}$$

$$x(x+3) + 3(x-3) = x^2 + 9 \text{ or}$$

$$x^2 + 3x + 3x - 9 = x^2 + 9$$

$$6x = 18,\ x = 3$$

Check: $\frac{3}{3-3} + \frac{3}{3+3} = \frac{9+9}{9-9}$

Since division by zero is impossible, there is no solution to this equation. The transformed equation was not equivalent to the original equation.

IV. Ratio, Proportion.

1. Ratio: an indicated division. $\frac{a}{b}$, where a and b are like quantities. Also written a:b, or a:b:c, etc.

   Example: Find the angles of a triangle if they are in the ratio 2:3:5

$$\begin{array}{rl} 2x + 3x + 5x = 180 & \quad 2x = 36 \\ 10x = 180 & \quad 3x = 54 \\ x = 18 & \quad 5x = 90 \end{array}$$

2. Proportion, two equal ratios

   (a) Direct proportion: $\frac{x_1}{x_2} = \frac{y_1}{y_2}$

   Example: The perimeters of two similar polygons are (directly) proportional to any two corresponding sides. $\frac{p_1}{p_2} = \frac{s_1}{s_2}$

   (b) Inverse proportion: $\frac{x_1}{x_2} = \frac{y_2}{y_1}$

   Example: The number of machines needed for a given job is inversely proportional to the time required. $\frac{m_1}{m_2} = \frac{t_2}{t_1}$

In direct proportion, as one variable increases, the related variable also increases, at the same rate. That is, if one variable doubles, the other does, etc. In inverse proportion as one variable increases, the other decreases at the same rate. That is, if one variable doubles, the other is cut in half, etc.

Problems of this type should be given in which the student has to decide whether he has direct or inverse proportion, and then write the proportion with correct symbols.

(c) Solution of proportions by extremes-means rule: $\frac{x_1}{x_2} = \frac{y_1}{y_2}$, $x_1y_2 = x_2y_1$

This topic should have been mastered in previous mathematical study, but a little review is usually needed.

(d) Mean proportional: if $\frac{x}{a} = \frac{a}{y}$, a is the mean proportional between x and y; $a^2 = xy$, and $a = \sqrt{xy}$.

Example: Find the mean proportional between 3 and 27.

$$\frac{3}{x} = \frac{x}{27} \qquad x^2 = 81 \qquad \underline{\underline{x = \pm 9}}$$

Example: Write in symbols: the tangent to a circle from an external point is the mean proportional between the entire secant and the external segment. $\frac{s}{t} = \frac{t}{e}$

Relations, Graphs of Linear Equations in Two Variables: Time - 15 days

A. Relations

Here we are studying the relationship between quantities; as one quantity varies, the value of one or more quantities varies.

1. Ways of expressing relations:

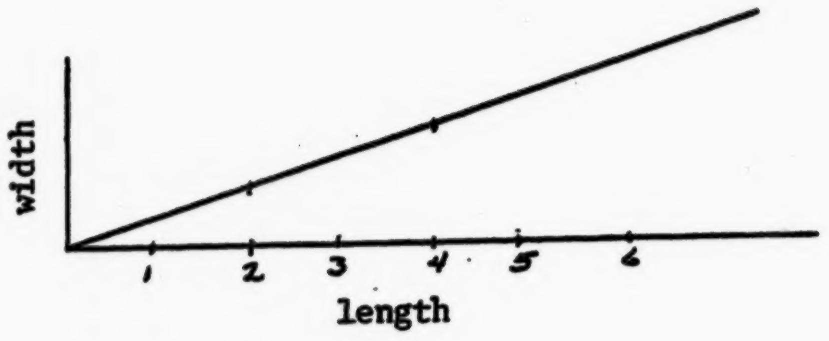
(a) Statement
(b) Algebraic expression for the statement, called a formula
(c) Table (Collection of ordered pairs)
(d) Graph

If there are two variables, the value of one is determined by the value assigned to the other. A relation is a set of ordered pairs.

Example: The length of a rectangle is to be twice the width.

$\ell = 2w$

2	1
4	2
6	3
8	4
10	5
12	6
14	7



2. Translate statements to open sentences (formulas).

Example: To find the number of degrees in the sum of the measures of the angles of a polygon, subtract 2 from the number of sides and multiply the difference by 180.

$$D = (n-2)\,180$$

3. Forming equations from tables

Example:

x	-1	2	5	8 ......
y	0	1	2	3 ......

Form an equation (open sentence) from this table.

Method 1

As y increases by 1, x increases by 3, or x increases 3 times as fast as y increases. This would be $y = 1/3\,x$ or $x = 3y$, but $x = \underline{-1}$ when y is 0. $\therefore\ x = 3y - 1$.

This equation must be tested on each ordered pair given. (See Ratio and Proportion)

Method 2

Use the general form for the linear equation in two variables: $y = mx + b$ where y and x are variables.

(here it should be explained that if any linear equation in y and x is solved for y, it will appear in this form, where m and b are constants.)

Substitute any two ordered pairs such as: $\left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\}$ $\left.\begin{matrix} x = 5 \\ y = 2 \end{matrix}\right\}$

$$y = mx + b$$

(1) $mx + b$ (2) $2 = 5m + b$

Subtract (1) from (2), to eliminate b.

$$\begin{array}{rl} (2) & 2 = 5m + b \\ (1) & \underline{-1 = 2m + b} \\ & 1 = 3m \\ & m = 1/3 \end{array}$$

To find b: Substitute $m = 1/3$ in either of the above equation. (1) was used

$$\begin{array}{rl} (1) & 1 = 2m + b \\ & 1 = 2/3 + b \\ & 3 = 2 + 3b \\ & 1 = 3b \\ & b = 1/3 \end{array}$$

Substituting $m = 1/3$, $b = 1/3$ in the equation $y = mx + b$:

$$\begin{array}{l} y = 1/3x + 1/3 \\ 3y = x + 1 \\ x = 3y - 1 \end{array}$$

This method may appear lengthy but is actually easy to use if the student has the guiding plan firmly in mind: in the formula $y = mx + b$, consider m and b as constants, x and y as variables; form two equations, solve for a and b.

B. Functional notations: $y = f(x)$, $y = F(x)$, $Y = G(x)$, $y = \phi(x)$.

f: Function: a relation such that no two ordered pairs have the same first element.
(a rule relating variables x and Y)

Domain set - the set of x's
Range set - the set of y's

Domain
Range

$y = f(x)$

The set of x's, abscissas: the domain set
The set of y's, ordinates: the range set

$f(x)$: "f of x;" or "the value of the function at x."

Example: $y = 2x - 3$; also written $f(x) = 2x - 3$.

Find: $f(4)$: read f at 4 (The value of the function at 4)

$$\left.\begin{aligned} f(x) &= 2x - 3 \\ f(4) &= 2(4) - 3 \\ f(4) &= 5 \end{aligned}\right\} \quad f(4) = 5$$

In the preceding example, if the domain is the set of all real numbers, the range is also the set of real numbers. But, the domain of the function $y = \frac{2}{x-1}$ is all real numbers except 1, since $\frac{2}{1-1} = \frac{2}{0}$.

Set form of functional notation $\{(x,y): y = kx\}$

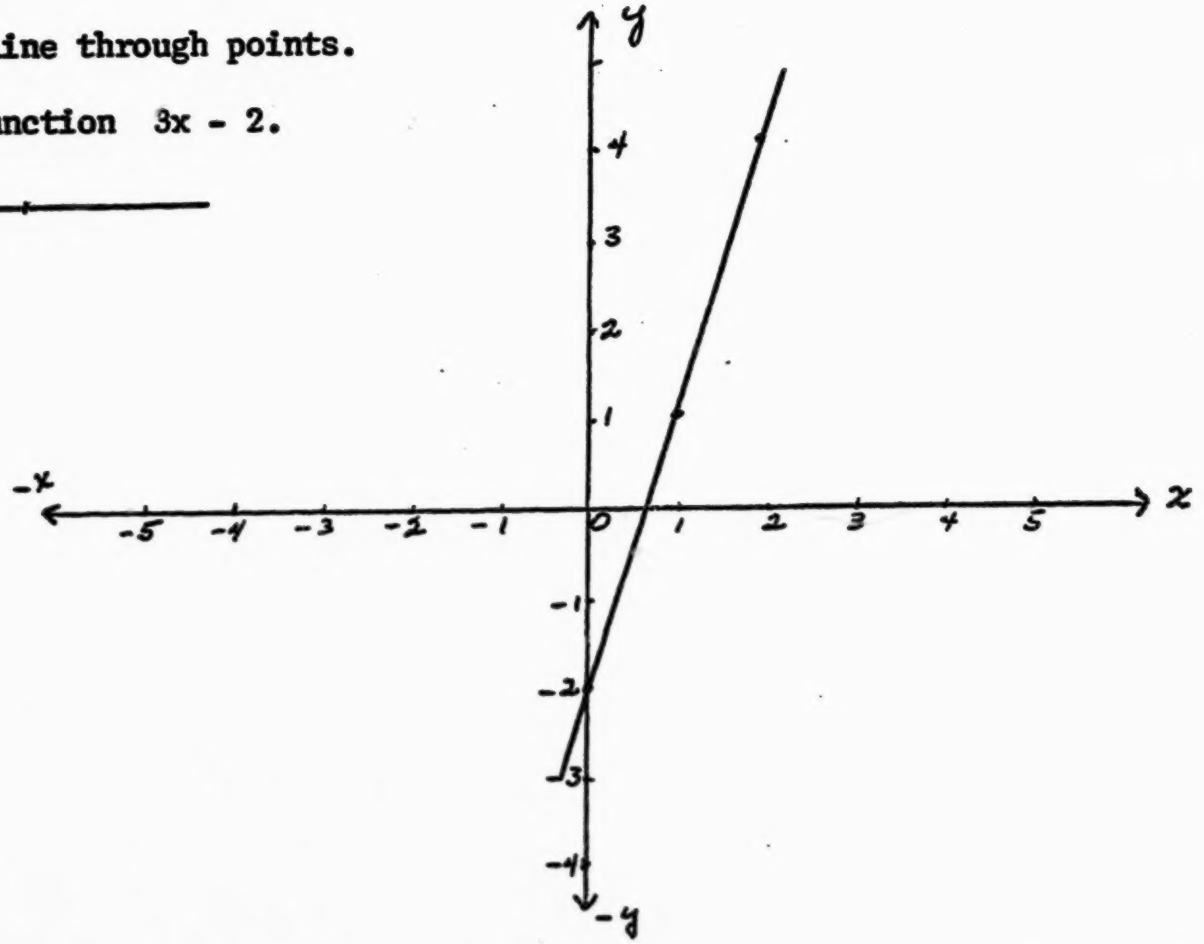
Example: $\{(x,y): y = 3x\}$ Find $f(4)$
$f(4) = 3(4) = 12$

C. Graphing linear functions

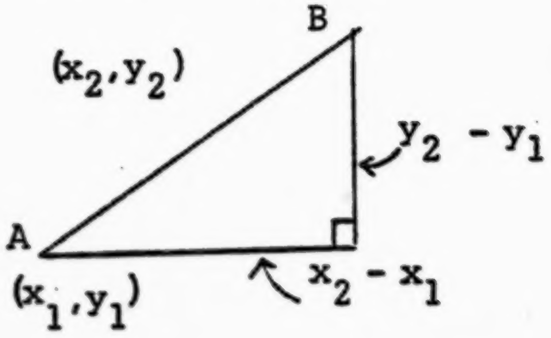
1. Set y = to the range of the function
2. Make a table of 3 or more ordered pairs
3. Draw x and y-axes
4. Plot points
5. Draw a straight line through points.

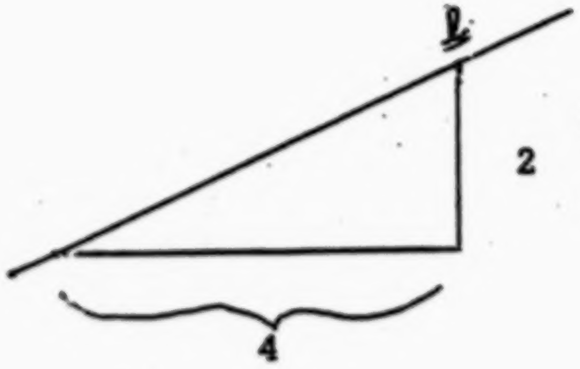
Example: Graph the function 3x - 2.

$x =$	0	1	2
$y =$	-2	1	4



D. Slopes of lines: 1. Slope = $\frac{\text{rise}}{\text{run}}$ 2. $\frac{y_1 - y_2}{x_1 - x_2} = m$ or $y_1 - y_2 = m(x_1 - x_2)$

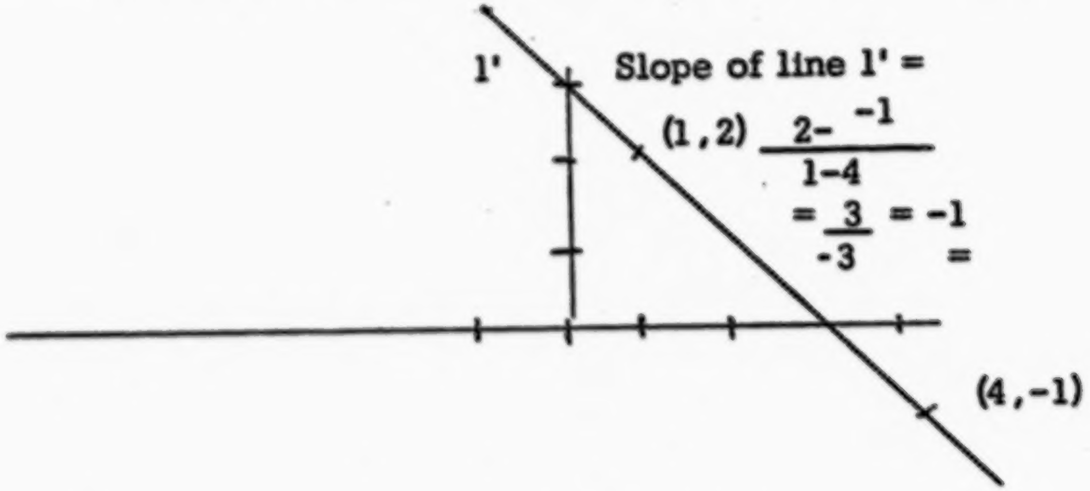




Slope of line ℓ

equals $\frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2}$

Example: Given the line 1' through points (1,2) and (4,-1), find the slope of 1'.



Slope of line 1' =

$$\frac{2-{}^{-}1}{1-4}$$

$$= \frac{3}{-3} = -1$$

=

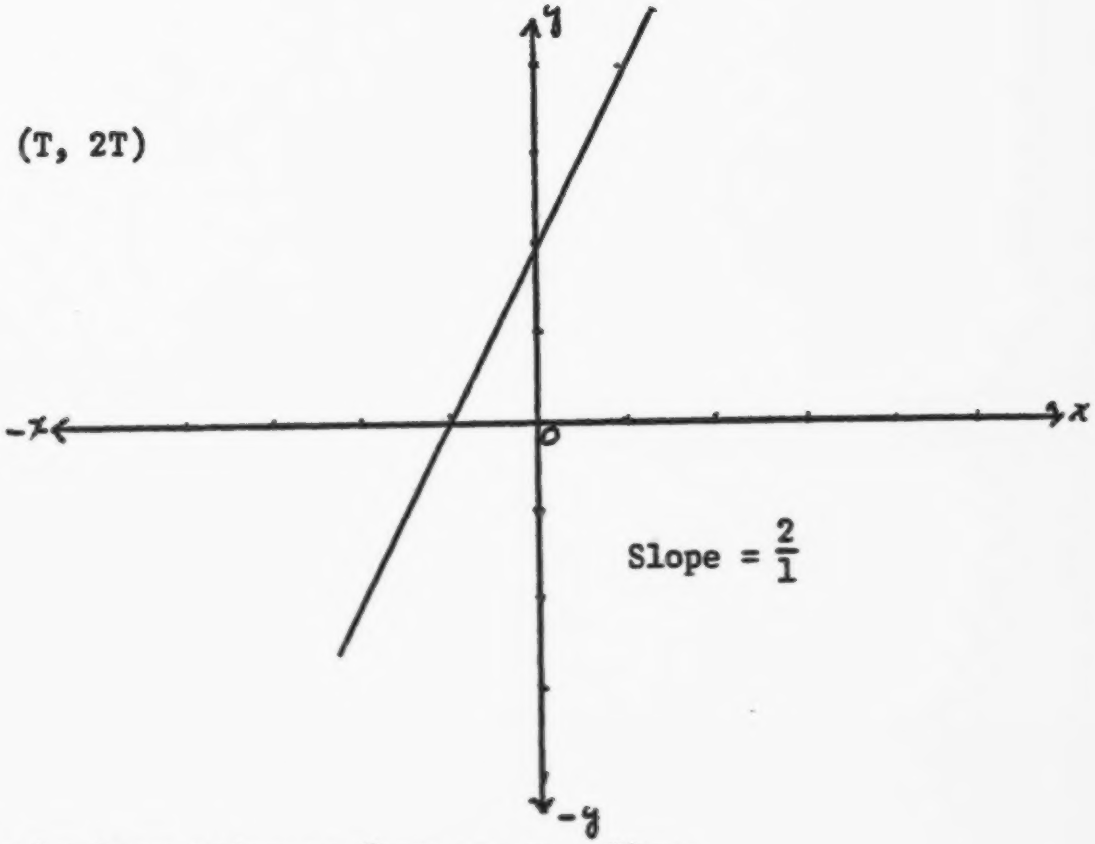
In writing the formula for the slope, $\frac{y_1 - y_2}{x_1 - x_2}$, special comment should be made on the fact that this could just as well be written $\frac{y_2 - y_1}{x_2 - x_1}$, since the slope is "the difference of the y's divided by the difference in the x's. It is important that the order be the same in the numerator and the denominator: if y of first point minus y of second point, then x of first point minus x of second point, or vice versa.

3. Slope equals average rate of change of y with respect to x.

Example:

$$x = 0 \overset{1}{\frown} 1 \overset{1}{\frown} 2 \overset{1}{\frown} 3 \quad ..6.. \quad (T, 2T)$$

$$y = 2 \underset{2}{\smile} 4 \underset{2}{\smile} 6 \underset{2}{\smile} 8$$

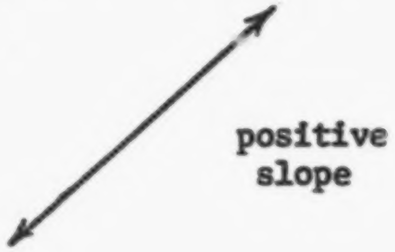


Slope = $\frac{2}{1}$

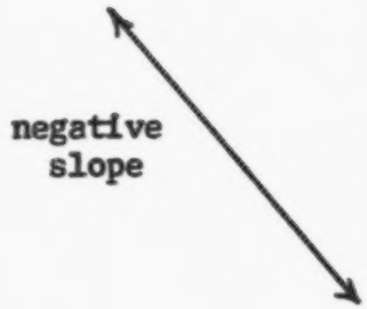
As x increases by one, y is increasing by 2 or twice as fast as x. Since the slope = $\frac{\text{the increase in y}}{\text{increase in x}}$, it equals the rate of change of y with respect to the change in x.

4. Positive and negative slopes

(1)



(2)

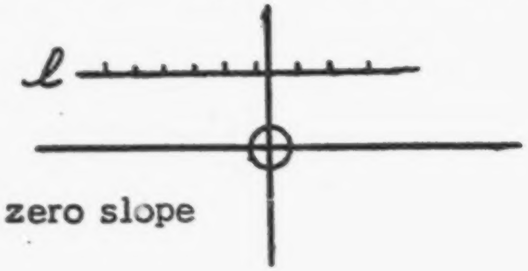


Since the slope = $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{y increase}}{\text{x increase}}$ if both x and y are positive, (resulting in a positive slope) the line will rise to the right as x increases. If y decreases while x increases (resulting in a negative slope) the line will fall toward the right as x increases.

5. Zero slope and slope undefined

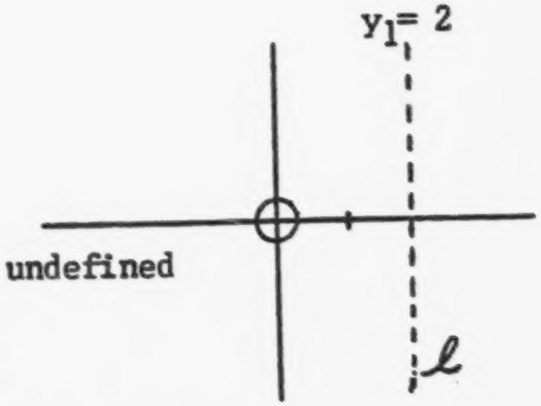
If the slope of a line is zero, $\frac{y_1 - y_2}{x_1 - x_2} = 0$, and $y_1 - y_2$ must = 0 or $y_1 = y_2$.

Drawing a line through y_1 and y_2 we have a horizontal line || to the x-axis.



zero slope

$$\frac{y_1 - y_2}{x_2 - x_2} = m$$



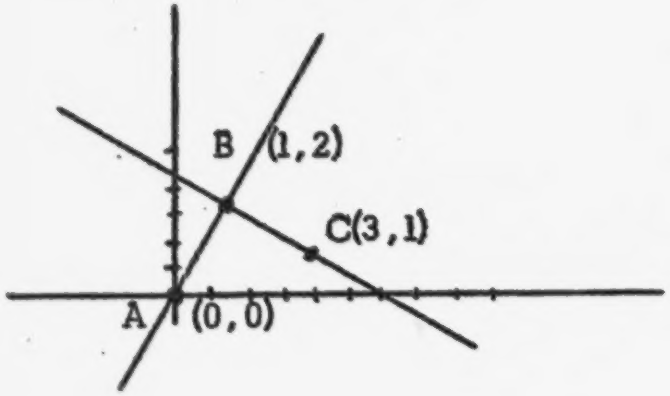
slope undefined

If the difference in x's of line ℓ is zero (or x is constant, as y varies), line ℓ is vertical, $\parallel$ to the y-axis. Since the denominator of the above fraction is zero, it is undefined.

6. Slopes of parallel and perpendicular lines

Parallel lines obviously have the same slope. $\perp$ lines have slopes that are the negative reciprocals of each other.

Demonstration, on graph board.



Slope of AB = $\frac{2}{1}$

Slope of BC = $\frac{-1}{2}$

AB $\perp$ BC

This is no proof, but it illustrates one case of perpendicularity. The proof can easily be done by the use of similar triangles. (The $\perp$ on the hypotenuse of a right $\triangle$ divides the $\triangle$ into two $\triangle$s, each one similar to the given $\triangle$, and similar to each other.

E. Writing the equation of a line and graphing the line.

1. Given the slope, m, and the y-intercept b. $y = mx + b$ (slope-intercept formula)

$y = mx + b$, the equation of a line in terms of the slope, m and the y-intercept, b, defined as the ordinate or the point at which the line crosses the y-axis. When $b = 0$, $y = mx$, or $m = \frac{y}{x}$, the formula for the slope of the line.

When $x = 0$, $y = b$, $\therefore$ b is the ordinate value of the y-intercept, since its abscissa is 0, ordinate = b.

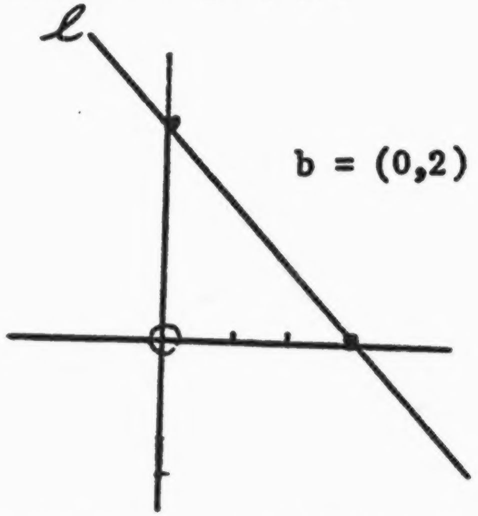
Example: Write the equation of a line, given the slope = $-\frac{2}{3}$ the y-intercept = 2.

$$y = mx + b$$
$$y = -\frac{2}{3}x + 2, \text{ or } 3y = -2x + 6 \text{ or } 2x + 3y = 6$$

The formula $y = mx + b$ is fundamental to the work with slopes of lines; the student should acquire a firm grasp of its meaning and application. Graphing the above problem, where $m = \frac{2}{3}$ and $b = 2$:

(a) Locate the y-intercept, 2.

Since the slope is $-\frac{2}{3}$ or $-\frac{2}{3} = \frac{\text{rise}}{\text{run}}$, move down 2 units, over to the right 3 to the point (3,0). Draw line through (3,0)



2. Given the slope, m, and the coordinates (x_1, y_1) of any point on the line,
$\frac{y - y_1}{x - x_1} = m$ or $y - y, = m(x - x_1)$

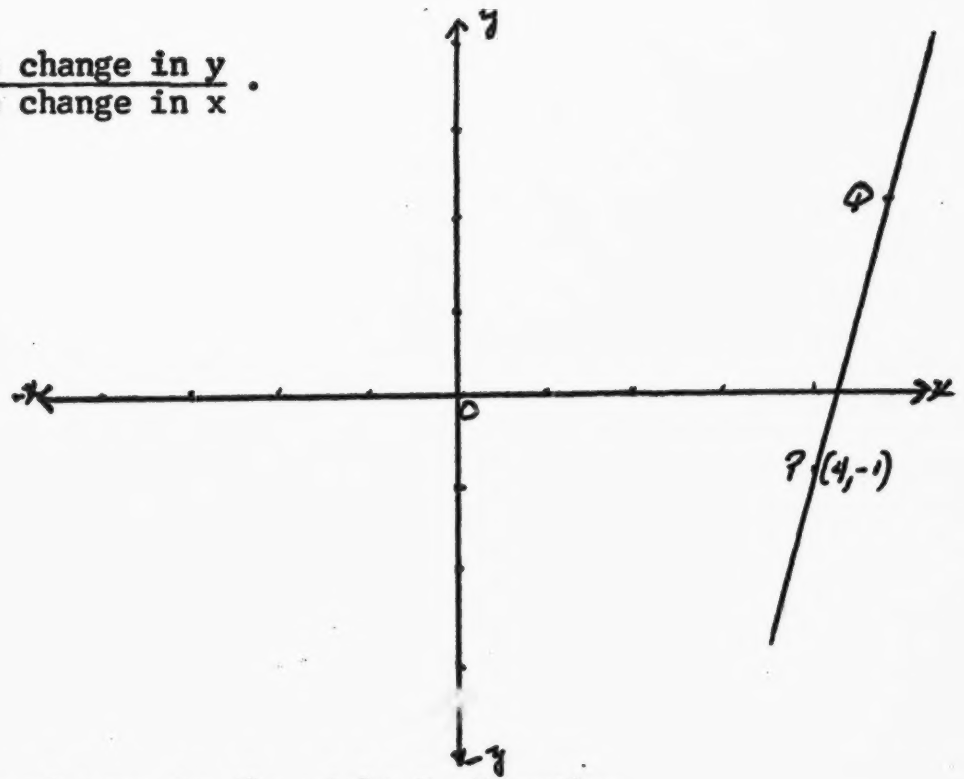
Example: Write the equation and draw the graph of a line through the point P (4, -1) and having the slope, 3.

$$\frac{y - y_1}{x - x_1} = m \qquad \frac{y - (-1)}{x - 4} = 3$$

$$y + 1 = 3x - 12, \text{ or } 3x - y = 13$$

Graph: Plot the point P (4,-1)

Since the slope = 3 or $\frac{3}{1} = \frac{\text{the change in } y}{\text{the change in } x}$.



From point P, move up 3 units and over to the right 1, locating point Q. Draw PQ. Note: When a line is required to pass through a certain point, a final check should be made to see that the coordinates of the point satisfy the equation of the line:

Check (4,-1) in $3x - y = 13$

$$\begin{aligned} 3(4) - (-1) &= 13 \\ 12 + 1 &= 13 \\ 13 &= 13 \end{aligned}$$

3. Given the coordinates of any two points (x_1, y_1) and (x_2, y_2), write the equation of the line passing through the two points;

Example: Write the equation of the line through the points A(2,4) and B(-3, 1)

Students will usually have more success with this type of problem if they use the point slope formula (a) rather than the expanded two-point formula (b).

(a) $\dfrac{y - y_1}{x - x_1} = m$

First, find the slope: $\dfrac{y - y_1}{x - x_1} = m$

$$\frac{1 - 4}{-3 - 2} = m = \frac{-3}{-5} \text{ or } 3/5$$

(b) $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$ (two-point formula)

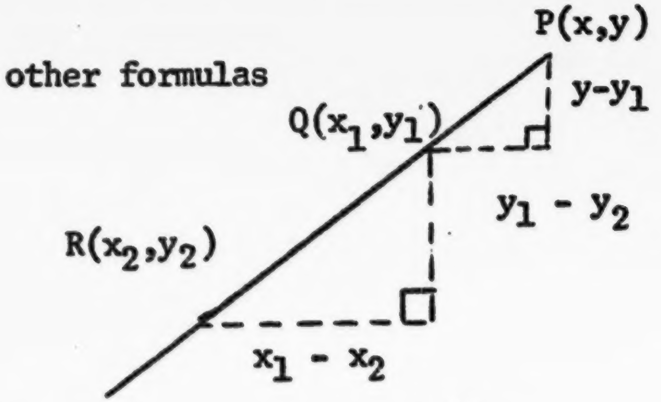
Let C(x,y) be another point on the required line; then, using the coordinates of either point, say:

A (2,4), $\dfrac{y - 4}{x - 2} = \dfrac{3}{5}$ or $5y - 20 = 3x - 6$, or $3x - 5y = 14$

Check: A (2,4) $\longrightarrow 6 - 20 = -14$
B (-3,1) $\longrightarrow 9 - 5 = -14$

The derivation of the other formulas should be emphasized.

F. Derivation of other formulas

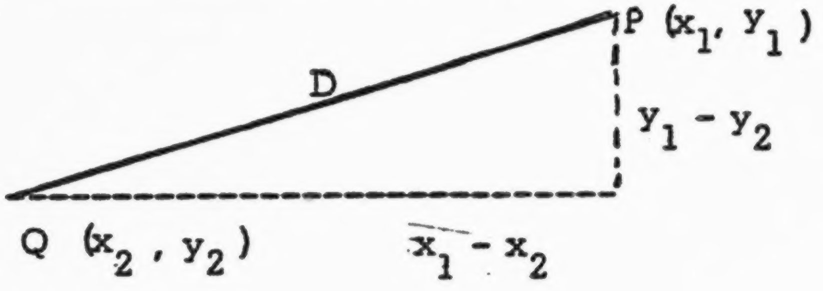


The coordinates of P and Q are given, these are all constants. Point P is any point on the line, placed at random. Since the coordinates of P are not yet known, they are represented by the variables x and y. Since all parts of this straight line have the same slope, the slope of RQ = the slope of PQ.

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y - y_1}{x - x_1}$$, the two point formula.

At best, however, the two-point formula is more difficult to remember and to use; students at this level usually have more success with the point-slope formula, used twice, as in the above example.

G. Finding the distance between two given points: $D = (x_1 - x_2)^2 + (y_1 - y_2)^2$

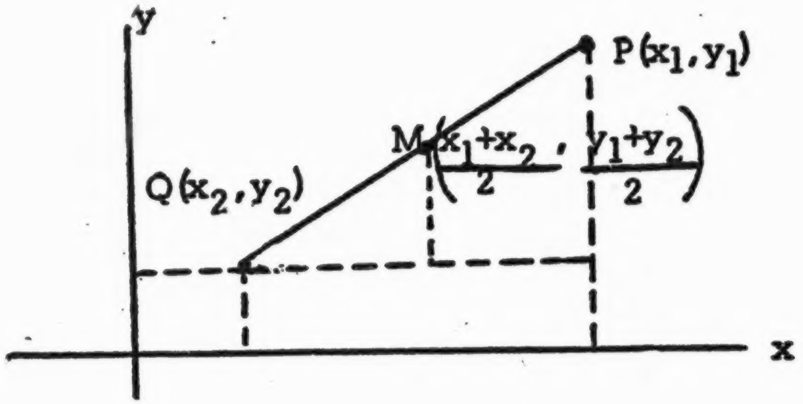


By the Pythagorean Theorem,

$$D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \text{ , or}$$

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1. Finding the coordinates of the midpoint of a given line segment.



The coordinates of the midpoint = $\frac{x_1 + x_2}{2}$, $\frac{y_1 + y_2}{2}$.

Since point M is the midpoint of line PQ, the abscissa of M is $x_2 + \frac{1}{2}(x_1 - x_2)$ and the ordinate of M is $y_2 + \frac{1}{2}(y_1 - y_z)$.

VI. Systems of Linear Open Sentences

A. Solution of two equations in two variables

Example: Solve the system of equations

(1) $2x + 3y = 5$ Multiply by 5
(2) $5x - 2y = 2$ Multiply by -2

1. Addition-subtraction method

To eliminate x, multiply equation (1) by 5, equation (2) by -2; and add:

(1) $10x + 15y = 25$
(2) $\underline{-10x + 4y = 4}$
(3) $19y = 21$ or $y = \frac{21}{19}$

To find the solution for x, one can substitute the value of y, $\frac{21}{19}$, in one of the equations, or eliminate y in the original equations in the same way in which x was eliminated above. By either method, x = 16/19. The habit of checking each equation is of extreme importance.

$$2\left(\frac{16}{19}\right) + 3\left(\frac{21}{19}\right) = 5$$
$$5\left(\frac{16}{19}\right) - 2\left(\frac{21}{19}\right) = 2$$

$$\frac{32}{19} + \frac{63}{19} = \frac{95}{19}$$

$$\frac{80}{19} + \frac{42}{19} = \frac{38}{19} = 2$$

2. Substitution method

It is often difficult at first to get students to use the substitution method, but when the student gains a degree of facility through practice, he should turn to the substitution method naturally, when it is appropriate (i.e., if a simple non-fractional substitution can be made).

Example: (1) Solve the system

$$\begin{aligned} x + 3y &= 10 \\ y &= 8 \end{aligned}$$

$$\begin{aligned} x + 3(8) &= 10 \\ x &= -14 \end{aligned}$$

(2) Solve the system

(a) $x - 2y = 9$
(b) $3x + y = 13$

In equation (a) $x = 9 + 2y$

In equation (b)

$$\begin{aligned} 3(9 + 2y) + y &= 13 \\ 27 + 6y + y &= 13 \\ 7y &= -14 \\ y &= -2, \quad x = 5 \end{aligned}$$

3. Determinant method (later)

The determinant method of solving linear equations may be approached now or later in the year. It is well that students become proficient with the elementary methods(Unit VI, A-1 and A-2) before using a condensed, short-cut operation.

4. Graphic method

The graphic method is not usually a practical one because of the time involved. However, graphs should frequently be used as illustrations.

5. Solution sets

(a) Dependent and Independent Equations; null set, finite set, infinite set.

Example: Dependent (or equivalent) equations

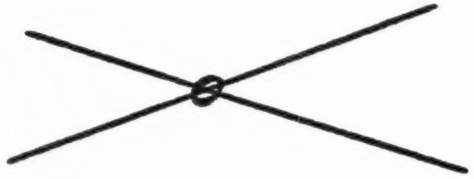
$$\begin{aligned} x + 3y &= 8 \\ 2x + 6y &= 16 \end{aligned}$$

The graphs of these are the same, the solution is the infinite set of numbers. Independent equations are not equivalent; their graphs are either parallel or intersecting.

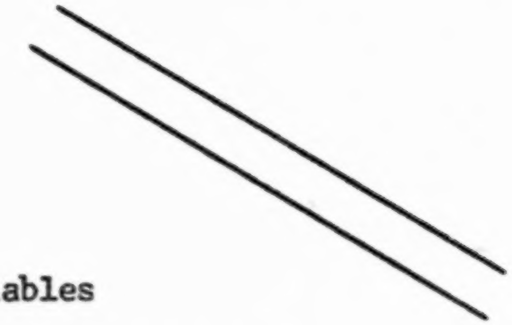
(b) Consistent and Inconsistent Equations

Consistent equations must have at least one point in common.

(1) Equivalent equations are consistent.
(2) Equations with one solution pair are consistent, independent, and their graphs intersect in one point.



Inconsistent equations have no common solutions, and their graphs are parallel lines.



3. Solution of 3 equations in 3 variables

   1. Addition, subtraction, substitution method.

The solution of 3 equations in 3 variables can do much to develop algebraic ingenuity in the student. There are almost always several choices of approach open to him, but his guiding principle remains: reduce the problem to two equations in the same two variables. After solving for these two he can then, by substitution, obtain the solution value of the third variable. If one or more of the original equations has only two variables, the pattern is set for producing a second equation in the same two variables, from the other two equations.

If all three equations have all three variables, the same variable should be eliminated from 2 different pairs of equations (1 & 2, 1 & 3, 2 & 3).

This will yield the needed 2 equations in the same 2 variables; after solving these, the student can proceed by appropriate substitutions to find the solution value of the third variable.

Example: $2x + 3y + z = 5$ (1)
$3x - y + 4z = -11$ (2)
$4x + 2y + 3z = 1$ (3)

Plan: since the equation (1) has $1z$ in it, eliminate z from (1) and (2) and from (1) and (3):

$$\begin{array}{rl} 8x + 12y + 4z = 20 & (1) \text{ multiplied by } 4 \\ \underline{3x - y + 4z = -11} & (2) \\ 5x + 13y = 31 & (4) \end{array}$$

$$\begin{array}{rl} 6x + 9y + 3z = 15 & (1) \text{ multiplied by } 3 \\ \underline{4x + 2y + 3z = -1} & (3) \\ 2x + 7y = 16 & (5) \end{array}$$

$$\begin{array}{rl} 5x + 13y = 31 & (4) \\ 2x + 7y = 16 & (5) \end{array}$$

$$\begin{array}{rl} 10x + 26y = 62 & (4) \text{ multiplied by } 2 \\ 10x + 35y = 80 & (5) \text{ multiplied by } 5 \\ \hline -9y = -18 & \\ y = 2 & \end{array}$$

Substituting $y = 2$ in equation (4), $5x + 26 = 31$

$5x = 5$

$x = 1$

Substituting $x = 1, y = 2$ in equation (1), $2x + 3y + z = 5$

$2 + 6 + z = 5$, or $z = -3$

Answer: $\{(x,y,z): 1,2,-3\}$

Check:

$$\begin{aligned} 2x + 3y + z &= 5 \\ 2 + 6 - 3 &= 5 \\ 4x + 2y + 3z &= -1 \\ 4 + 4 - 9 &= -1 \\ 3x - y + 4z &= -11 \\ 3 - 2 - 12 &= -11 \end{aligned}$$

2. Determinant method

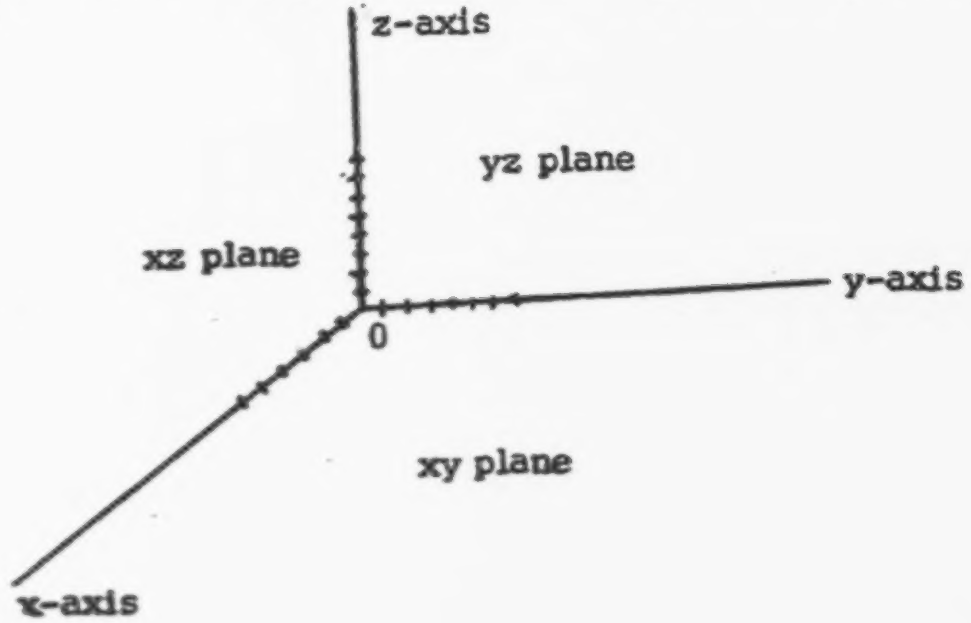
This method may be studied now or later in the year (See Unit 12).

3. Graphic method

As with 2 variables, so with three, graphs are useful as illustrations.

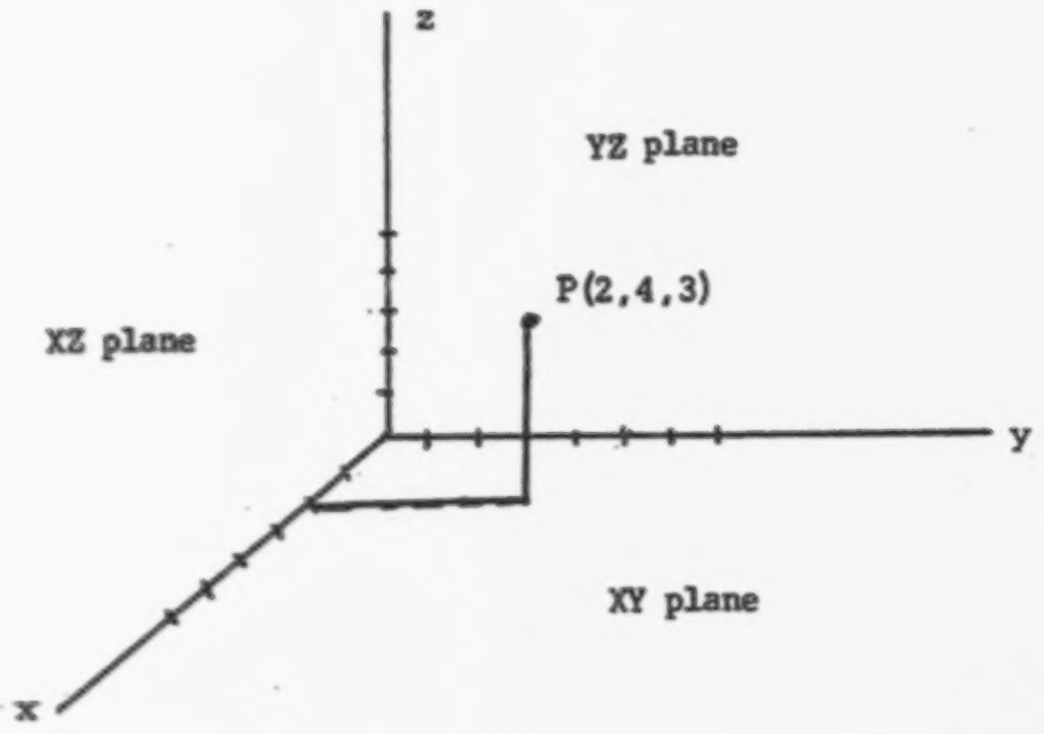
(a) The graphing system in 3-space

The teacher can make drawings or a model to explain the plotting of points in 3-space:



Mention the negative values on each axis (not shown) extending past the origin.

(b) Plotting points in 3-space



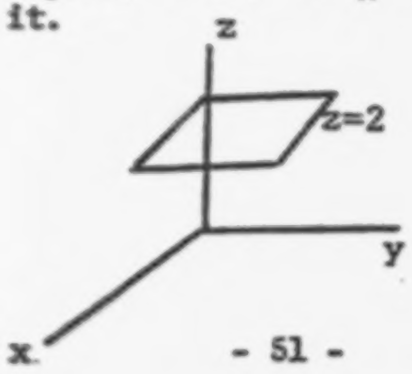
To plot a point in space, such as P ($x = 2$, $y = 4$, $z = 3$) or P (2, 4, 3) move 2 units in a positive direction on the x-axis, then 4 units to the right, parallel to the z-axis.

(c) Plotting planes in 3-space

(1) Plane to one of the coordinate planes

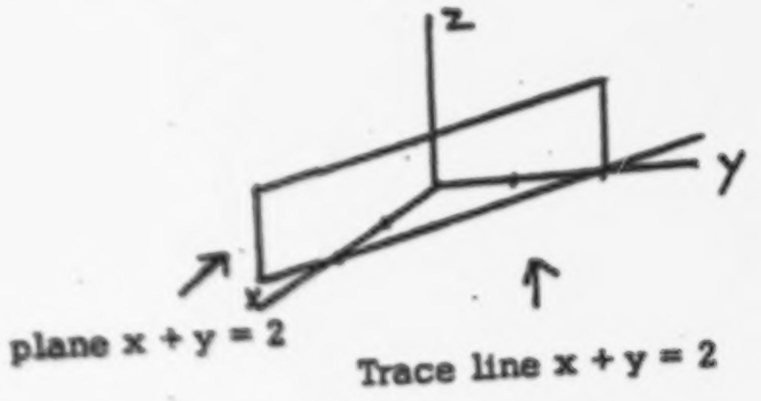
The plane determined by the x-axis and the y-axis is called the XY-plane; the YZ and the XZ planes are similarly formed.

The plane, $Z = 2$, is a plane such that x and y are undetermined; z is always 2. The plane $Z = 2$ is $\|$ to the X-Y plane, at a distance of 2 units from it.



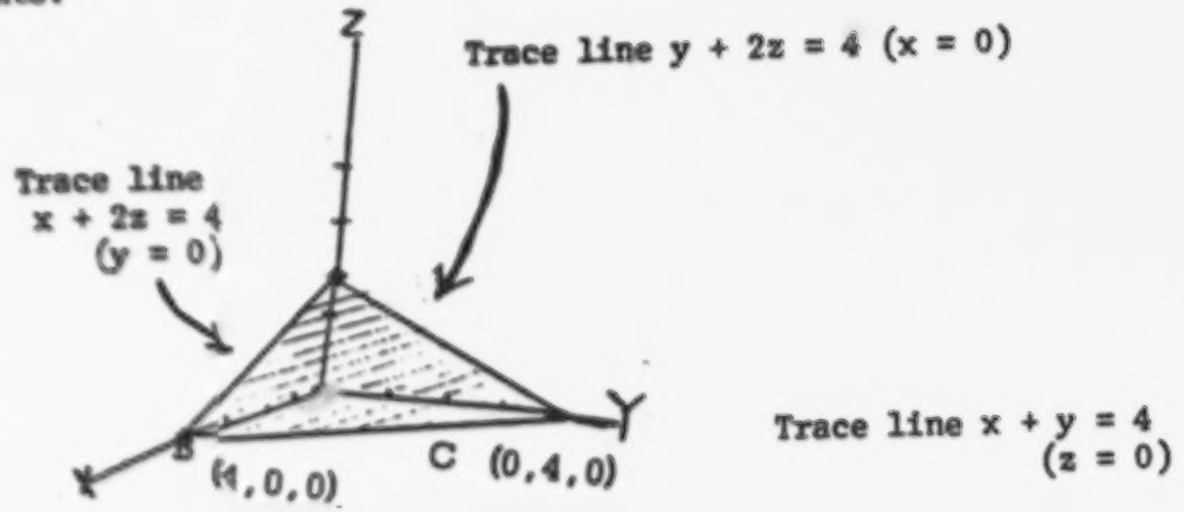
(2) Plane parallel to one of the axes

Example: The equation $x + y = 2$ in 3-space becomes a plane || to the z-axis, intersecting the XY plane in the "trace" line $x + y = 2$. (z is undetermined and can have any value.)



(3) Plane oblique to all 3 axes

Example: the equation $x + y + 2z = 4$ is located simply by finding the intercepts on the axes and plotting these points.



x	0	4	0
y	0	0	4
z	2	0	0
	↓	↓	↓
	A	B	C

The equation $x + y + 2z = 4$ becomes a plane (shaded) determined by the 3 points A, B, and C. Any 3 noncollinear points satisfying the given equation. would do, but these intercepts are usually the most easily located ones. (Note that for a set of 3 equations in 3 variables to have a common solution they would all have to intersect in one point.)

C. Verbal problems involving linear systems

Here is another opportunity for experience in changing from verbal to mathematical symbols

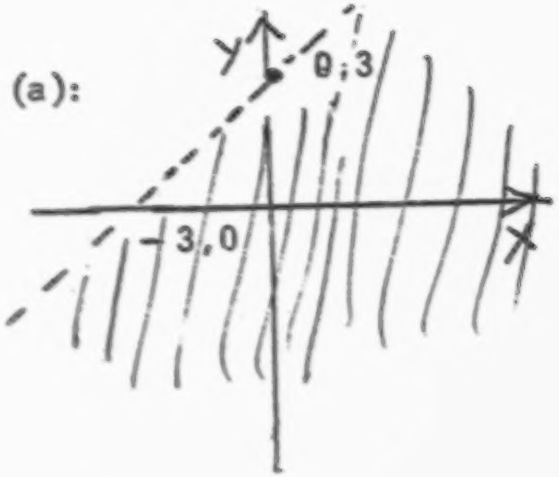
1. Number relations
2. Motion
3. Mixtures
4. Coins
5. Geometry
6. etc.

Students should be required to , at first, solve these problems with two or more variables, even though it is usually possible to solve them with only one. Later on, when they are familiar with either method, they should choose the most suitable one.

D. Solution of systems of linear inequalities by graphs.

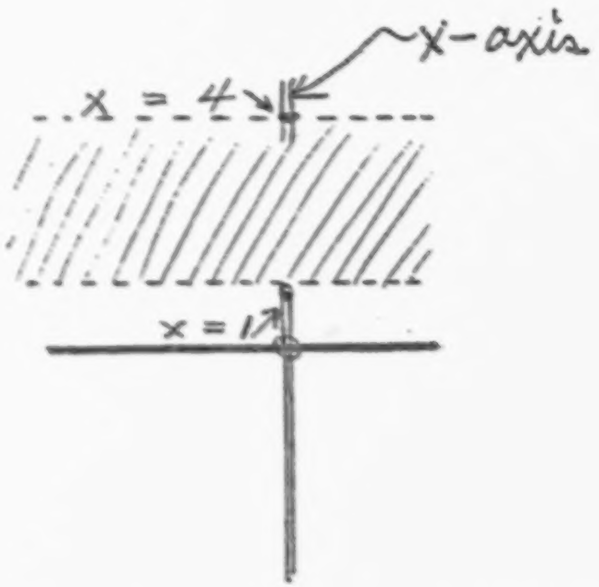
Example: $y < x + 3$ (a)
$1 < x < 4$ (b)

The graph of (a):



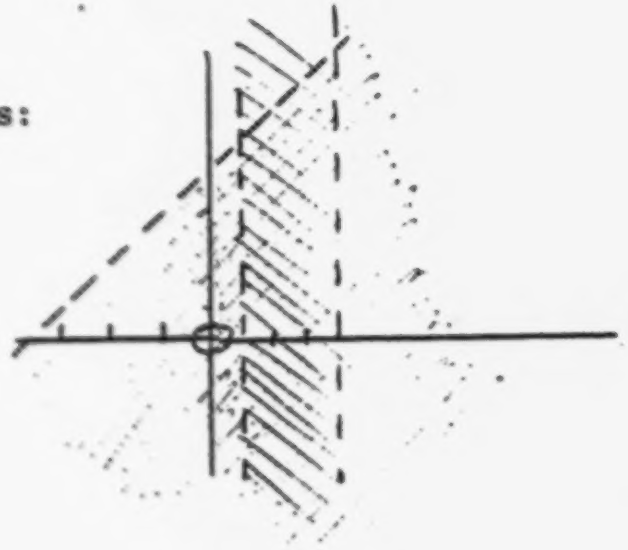
Graph the equation $y = x + 3$. Make this a dotted line since the line itself is not part of the solution. All of the region below the line is shaded, since y is less than $(x + 3)$.

The graph of (b):



Graph the equations $x = 1$ and $x = 4$, in dotted lines. Shade the area between, since $1 < x < 4$.

Combine the two graphs:



The overlapping shows the solution, which is the intersection of the sets:
$\{(x, y): y < x + 3\} \cap \{(x,y): 1 < x < 4\}$

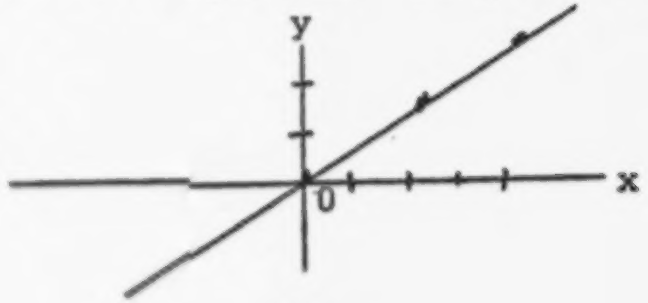
After a little practice, the two graphs need not be done separately first, although a preliminary rough draft of each one is helpful. Note: Colored pencils are useful here, using different colors for each graph. The overlapping areas will show plainly.

E. Direct variation: $x = ky$

1. Direct variation is involved with direct proportion; the fact that $x = ky$ follows from the proportion $\frac{x_1}{x_2} = \frac{y_1}{y_2}$.

2. Graph of direct variation

   The graph of $x = ky$ is a straight line. Example: $x = 2y$

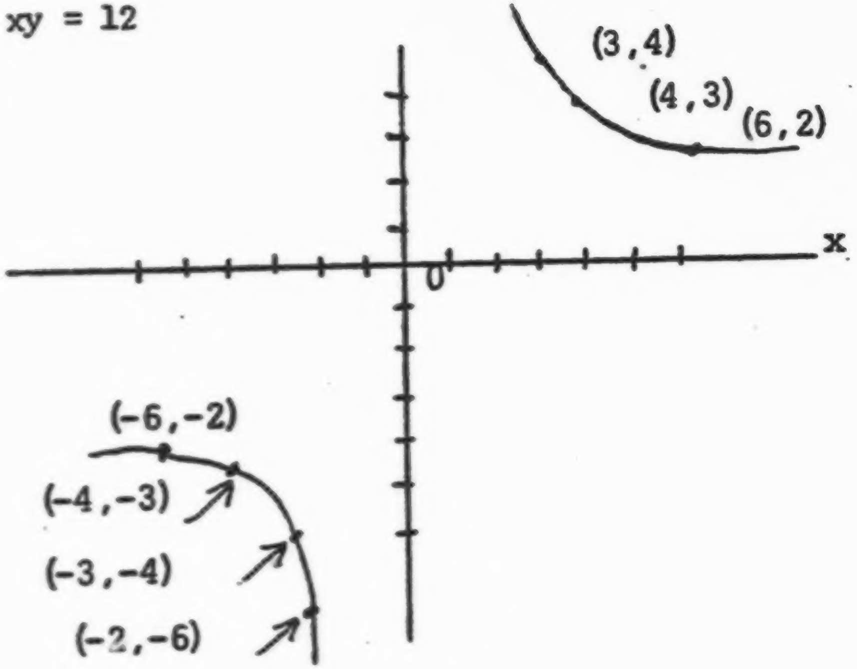


3. Inverse variation: $xy = k$

The graph of $xy = k$ is a hyperbola.

4. Graph of inverse variation

Example: $xy = 12$



5. Joint variation: $A = kxy$

Here A is directly proportional to x (varies directly as x); A is also directly proportional to y (varies directly as y). Therefore A varies jointly as x and y. Note that if A becomes constant, x and y vary inversely with each other.

6. Combined variation: $y = \frac{kx^2}{z}$, etc.

Variations such as this are combinations of direct and inverse variation "y varies directly as the square of x, and inversely as z."

7. Variations written in set notation: $\{(x,y): y \text{ varies inversely as } x\}$ etc.

Example: $\{(4, 3), (-2, a)\} \subset \{(x,y): y \text{ varies inversely as } x^2\}$

Find a

$$\begin{array}{rlrl} x^2y &= k & 4a &= k \\ 16 \cdot 3 &= k & 4a &= 48 \\ 48 &= k & a &= 12 \end{array} \qquad \text{Ans., } \underline{12}$$

8. Verbal Problems

There are many fine examples of variation in physics and geometry, suitable for practice in writing and solving variation problems. In some cases, writing the statement as a variation, $xy = k$, is best, sometimes as a proportion, $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. The student should be able to write in either form and carry on to a solution.

9. Choice of form used: Write the problem as a proportion, $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

Problems in simple direct variation are usually more workable in proportion form but inverse variation should most often be put into the variation form, $xy = k$. Find k by multiplying one set of x and y, then use k to find the unknown x or y.

Example: If x is 3 when y is 5, find y when x is 7.

$$\begin{array}{rlrl} xy &= k & 7y &= 15 \\ 3(5) &= k & y &= 15/7 \\ 15 &= k & & \end{array}$$

VII. Introduction to the Development of Exponents and Radicals

A. Definitions and "conventions" for exponents.

1. $a^0 = 1$
2. $a^1 = a$
3. $a^{-m} = \frac{1}{a^m}$

$\left.\right\}$ $a \in R, a \neq 0, m \in I$

B. Rules of Calculation

1. $(abc\ .\ .\ .)^m = a^m b^m c^m .\ .\ .$
2. $(\frac{a}{b})^m = \frac{a^m}{b^m}$
3. $(a^m)(a^n) = a^{m + n}$
4. $(a^m)^n = a^{mn}$
5. $\frac{a^m}{a^n} = a^{m-n}$

SHOULD BE MEMORIZED

C. Relations

1. $a = b \Leftrightarrow a^{2m+1} = b^{2m} + 1$
2. $|a| = |b| \Leftrightarrow a^{2m} = b^{2m}$
3. $a > 1 \Rightarrow a^m > a$
4. $a < b \Leftrightarrow a^{2m+1} > b^{2m+1}$
5. $|a| < |b| \Leftrightarrow a^{2m} \quad b^{2m}$
6. $0 < a < 1 \Rightarrow a^m < a$

D. The arithmetic for radicals.

1. $\sqrt[m]{a} = x \Leftrightarrow a = x^m$
2. $\sqrt[m]{a^m} = a$

$\left.\right\}$ a,b,c. . . are positive arithmetic numbers and m is a positive integer.

E. Rules of Calculation with Radicals

1. $\sqrt[m]{abc \ldots} = \sqrt[m]{a}\ \sqrt[m]{b}\ \sqrt[m]{c \ldots}$
2. $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$
3. $\left(\sqrt[m]{a}\right)^R = \sqrt[m]{a^R}$
4. $\sqrt[m]{\sqrt[R]{a}} = \sqrt[mR]{a}$
5. $\sqrt[mP]{a^{pq}} = \sqrt[m]{a^q}$

THESE SHOULD BE MEMORIZED

6. $a = b \Longleftrightarrow \sqrt[m]{a} = \sqrt[m]{b}$
7. $a < b \Longleftrightarrow \sqrt[m]{a} < \sqrt[m]{b}$
8. $a > 1 \Longrightarrow 1 < \sqrt[m]{a} < a$
9. $a < 1 \Longrightarrow a < \sqrt[m]{a} < 1$

RELATIONS
Easy Problems Involving
These Should Follow.

F. $\sqrt{A \pm \sqrt{B}} = \sqrt{x} \pm \sqrt{y}$ with $x = \frac{A + B}{2}$, $y = \frac{A - C}{2}$

$A, B \in$ Rat., B not a perfect square
$A^2 - B$ = a perfect square $(A^2 - B = C^2)$

Lots of Problems Are
Needed Here.

G. Fractional Exponents

1. $\sqrt[m]{a^P} = a^{\frac{p}{m}}$
2. $\sqrt[m]{a} = a^{\frac{1}{m}}$

$A \in$ Arithmetic Numbers
$m \in I^+$
$p \in I$

These Should be Memorized.

H. Rules of calculation for fractional exponents.

$$\left.\begin{array}{l} 1.\ a^{\frac{p}{m}} \cdot a^{\frac{p^1}{m^1}} = a^{\frac{p}{m} + \frac{p^1}{m^1}} = a^{\frac{Pm^1 + mp^1}{mm^1}} \\ 2.\ a^{\frac{p}{m}} : a^{\frac{p^1}{m^1}} = a^{\frac{p}{m} - \frac{p^1}{m^1}} = a^{\frac{Pm^1 - mp^1}{mm^1}} \\ 3.\ \left(a^{\frac{p}{m}}\right)^{\frac{p^1}{m^1}} = a^{\frac{pp^1}{mm^1}} \end{array}\right\} a > 0$$

These should be Memorized

Note: Observe the two cases, $a > 0$ and $a < 0$.
The latter case requires algebraic notation and rules.

I. The Algebra of Radicals

$$\left.\begin{array}{l} 1.\ x^{2m} = a \Longrightarrow x = \pm \sqrt[2m]{a} \\ 2.\ x^{2m+1} = a \Longrightarrow x = \sqrt[2m+1]{a} \end{array}\right\} a > 0,\ m \in I^+$$

$$\left.\begin{array}{l} 3.\ x^{2m} = a \ \left[\text{is impossible}\right] \\ 4.\ x^{2m+1} = a \Longrightarrow x = -\sqrt[2m+1]{-a} \end{array}\right\} a < 0,\ m \in I^+$$

Note: Where $a < 0$, $-\sqrt[2m+1]{-a}$, not $\sqrt[2m+1]{-a}$

1. If m is odd, $\sqrt[m]{a^m} = a$
2. If m is even, $\sqrt[m]{a^m} = |a|$
3. Observe: $\sqrt{4^2} = |x| = \begin{cases} x \text{ if } x > 0 \\ -x \text{ if } x < 0 \end{cases}$

Lots of Problems are needed

5. When A and B are algebraic expressions and m is an even integer,

$$\sqrt[m]{A^mB} = |A|\sqrt[m]{B} = \begin{cases} A\sqrt[m]{B} & \text{if } A > 0 \\ -A\sqrt[m]{B} & \text{if } A < 0 \end{cases}$$

6. Note: $$A\sqrt[m]{B} = \begin{cases} \sqrt[m]{A^mB} & \text{if } A>0 \\ -\sqrt[m]{A^mB} & \text{if } A<0 \end{cases}$$

Lots of Problems are needed.

VIII. Exponents, Radicals, Complex Numbers: Time - 15 days

A. Exponents; rules of operation

A review of the rules of operation with exponents is in order, with extension to negative and fractional exponents.

$x^a \cdot x^b = x^{a+b}$

$\frac{x^a}{x^b} = x^{a-b}$

$(x^a)^b = x^{ab}$

$(xy)^n = x^ny^n$

$(\frac{x}{y})^n = \frac{x^n}{y^n}$

1. Fractional exponents: $\sqrt[n]{x^m} = x^{m/n}$

   Example 1: $\sqrt{x^5} = (x^5)^{\frac{1}{2}} = x^{5/2}$

   Example 2: $\sqrt[3]{a^4} = (a^4)^{1/3} = a^{4/3}$

2. Negative Exponents: $\frac{1}{x^n} = x^{-n}$ $(x \neq 0)$

Example 3: $\frac{x^3}{x^5} = \frac{1}{x^2}$; but $\frac{x^3}{x^5}$ also $= x^{-2} \therefore \frac{1}{x^2} = x^{-2}$

Facility in the use of fraction al and negative exponents is much to be desired. Enough time should be spent on this unit to achieve mastery of these operations through use of the good problem material available in most textbooks.

B. Principal Roots

1. Rules for the sign of roots: $\sqrt[n]{x} = y$

   (a) If n is odd and x is $\oplus$ y is $\oplus$

   (b) If n is odd and x is $\ominus$ y is $\ominus$

   (c) If n is even and x is $\oplus$ y is $\oplus$

   (d) If n is even and x is $\ominus$ y is not a real number.

   Mathematicians have agreen that the symbol $\sqrt[n]{a}$ means the principal (or positive, real) nth root of a when n is even. When n is odd you are interested in the real root. When n is even, there are always two real nth roots, one positive and the other negative. There are two square roots of x^2, x and -x. $\sqrt[3]{-x^3} = -x$

   In solving an equation such as $x^2 = 9$, both roots are reported: $x = 3$, $x = -3$.

   $$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x = 3,\ x &= -3 \end{aligned} \qquad \begin{aligned} \sqrt{x^2} &= x \\ \sqrt[4]{x^4} &= x \end{aligned}$$

2. Square root

   (a) By table

Finding square roots is a necessary, frequently-used operation. If a table is available, it should be used, and the student should have some practice in getting square roots in this way. The greatest limitation of any table is of course the limitation of extent: Interpolation beyond the limits of the table is not reliable.

(b) By arithmetic

Example: Find $\sqrt{115}$.

```
    20         10.724
     0        115.00
     0        1
   207        015
     7          0
   214        1500
     2        1449
  2144.4        5100
       4        4284
                81600 ←
```

This is the most accurate of the method requiring no help from machines or tables. The greatest disadvantage is that with disuse the method is easily forgotten; whereas, the following procedure can be worked out again fairly easily if forgotten.

(c) By approximation (estimate, divide, average, etc.) Use if a calculator is available.

Example: Find $\sqrt{115}$

Estimate 10

Divide by 10 $\frac{115}{10} = 11.5$

Average of 10 and 11.5 = 10.75

Divide by 10.75 $\frac{115}{10.75} = 10.698$

Average of 10.75 and 10.698 = $\underline{10.724}$

(Correct to 3 decimal places, in tables)

(d) By logarithms (Later in the course)

Example: $\sqrt{115}$ $\quad \text{Log}_{10}\ 115 = 2.0607$

$$\frac{2.0607}{2} = 1.03035 = 1.0303$$

Antilogarithm of 1.0303 = 10.72

The usual logarithm table gives a result correct to only a few figures, 10.72 instead of 10.724, the correct answer to 5 figures. This is the chief disadvantage of using the logarithm method of finding square root.

(e) By slide rule (later in the course)

(f) By use of the binomial theorem (later in the course)

(g) By office machine such as the Monroe Calculator.

This is a good method, not difficult, correct to 9 or 10 figures. It is based on the fact that all squares are the sums of the odd integers.

1 = 1
1 and 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16 etc.

Put in 16 in the machine. Subtract 1 from it $\longrightarrow$ 15
Subtract 3 from 15 $\longrightarrow$ 12
Subtract 5 from 12 $\longrightarrow$ 7
Subtract 7 from 7 $\longrightarrow$ 0

Since the final remainder is 0, 16 is a perfect square. The number of subtractions is 4 (Crank handle revolved 4 times, registers 4). The Answer is 4. This method appeals to some student.

(h) By Computer (not in this course)

3. Cube roots, etc.

There are 3 cube roots of any number.

Example: Find the 3 cube roots of 8.

$$x^3 - 8 = 0, (x-2)(x^2+2x+4) = 0$$
$$x - 2 = 0, x = 2$$
$$x^2 + 2x + 4 = 0 , x = \frac{-2 + \sqrt{4 - 16}}{2} \text{ or}$$
$$x = -1 \pm \sqrt{-3}, \ 2$$

The three cube roots of 8 are $\left\{2, -1+i\sqrt{3}, -1-i\sqrt{3}\right\}$

In the same manner, it can be shown that there are 4-fourth roots of a number, etc., or n-nth roots. Plenty of time should be allowed for fractional and negative exponents (changing from fractional exponents to radical form, and vice versa).

C. Scientific or Standard Notation: $a \times 10^2$

Standard notation: $a \times 10^n$; where the absolute value of a, a rational number in finite decimal form, is from 1 to 10, and n is an integer. The decimal for a contains as many digits as the precision of the measurement justified.

4.256×10^{-3} = .004256, with four significant digits.
3.20×10^6 = 3,200,000, with three significant digits.

Since 3.20 has 3 digits, this means that the number was closer to 3.20, not 3.19 or 3.21.

D. Approximations

(1) To round a decimal, add 1 to the last digit kept if the first digit dropped is 5 or more; otherwise, leave the retained digits unchanged.

1.325 = 133 to the nearest hundredth.
.03047 = .030 to the nearest thousandth.

(2) **Precision of a measurement**

"The expression 'precise to the nearest thousandth of an inch' means, for example, that a measurement of 2.142 inches lies between 2.1415 and 2.1425 or the length ℓ satisfies the inequality $2.1415 < \ell < 2.1425$."

(3) **Maximum possible error**

The maximum possible error is half the given statement of precision (one-thousandth) or $\frac{1}{2}(.001) = .0005$.

(4) **Finding a one-significant figure estimate by using standard notation.**

Example: Give a one-significant figure estimate of

$$\frac{(1045)(0.0432)}{(1.85)(.00089)} = \frac{(1000)(.04)}{(2)(.001)} = \frac{(10^3)(4 \times 10^{-2})}{2(10)^{-3}} =$$

$$\frac{4(10)}{2(10)^{-3}} = 2(10)^4 = 20{,}000$$

Round each number to one significant figure, convert to standard notation, and complete the computation, rounding the result to one significant figure.

(5) **Accuracy of measurement, expressed as relative error.**

Relative error is the ratio of the maximum possible error to the measurement itself. In this case,

$$\text{Relative error} = \frac{.0005}{2.142} = \frac{5}{21420} = .02\%$$

(6) **Using standard notation in computation.**

The method is the same as in (4), except that numbers are not rounded off until the final result.

F. **Properties of Radicals**

1. **Properties of Radicals**

These properties can easily be demonstrated by changing to the fractional exponent form, and back to the radical form.

(a) $\sqrt[n]{a^n} = (\sqrt[n]{a})^n$ Example: $\sqrt[3]{a^3} = (a^3)^{1/3} = a^{3/3} = a$

$(\sqrt[3]{a})^3 = (a\ 1/3) = a^{3/3} = a$

(b) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Example: $\sqrt[3]{6} = (2 \cdot 3)^{1/3} \cdot 3^{1/3} = \sqrt[3]{2} \cdot \sqrt[3]{3}$

(c) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Example: $\sqrt[3]{\frac{2}{5}} = \frac{(2)}{5}^{1/3} = \frac{2^{1/3}}{5^{1/3}} = \frac{\sqrt[3]{2}}{\sqrt[3]{5}}$

(d) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ Example: $\sqrt[3]{\sqrt[4]{a}} = (a^{1/4})^{1/3} = a^{1/12} = \sqrt[12]{a}$

(e) $\sqrt[mp]{a^{np}} = \sqrt[m]{a^n}$ Example: $\sqrt[6]{a^{10}} = \sqrt[3\cdot 2]{a^{5\cdot 2}} = (a^{5\cdot 2})^{\frac{1}{3\cdot 2}} =$

$a^{\frac{5\cdot 2}{3\cdot 2}} = a^{\frac{5}{3}} = \sqrt[3]{a^5}$

2. **Simplification of Radicals.** $\sqrt[n]{b}$ is simplified form if:

(a) b is an integer having no integral factor which is the nth power of any integer.

The expression of radicals in simplest form is of great importance in streamlining all operations with irrational numbers.

Example: $\sqrt{80} = \sqrt{4 \cdot 20} = \sqrt{4}\ \sqrt{20} = 2\sqrt{20}$

But 20 is not in simplified form, since it $= \sqrt{4}\ \sqrt{5} = 2\sqrt{5}$, $2\sqrt{20} =$ $2(2\sqrt{5}) = 4\sqrt{5}$

This difficulty, which led to a double operation of removing square factors twice, would have been eliminated if the largest square factor of 80, 16, had been used at the beginning. $\sqrt{80} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$ directly.

For large numbers, it often pays to find the prime factors and group any pairs of factors to form squares.

$$\begin{array}{l|l} \left\{\begin{array}{l}2\\2\end{array}\right. & \begin{array}{l}80\\40\end{array} \\ \left\{\begin{array}{l}2\\2\end{array}\right. & \begin{array}{l}20\\10\end{array} \\ & 5 \end{array}$$

It is not easily seen that 4 x 4 or 16, a perfect square, is a factor of 80.

(b) b is a polynomial, having no factor which is the nth power of any polynomial

Example: Simplify $\sqrt{8x^2 + 16x + 8} \rightarrow \sqrt{4(2x^2 + 4x + 2)}$

$$\sqrt{4}\,\sqrt{2x^2 + 4x + 2} \rightarrow 2\sqrt{2x^2 + 4x + 2}$$

This expression must be factored again:

$$2\sqrt{2(x^2 + 2x + 1)} \rightarrow 2\sqrt{2(x+1)^2} = 2\,|x+1|\sqrt{2} \quad \underline{\text{Answer}}$$

This problem shows the kind of errors (of incompleteness) often made by students. The original expression should have been completely factored at the start:

$$\sqrt{8x^2 + 16x + 8} \rightarrow \sqrt{4 \cdot 2(x^2 + 2x + 1)} \rightarrow 2\,|x+1|\,\sqrt{2}$$

(c) The radicand, b, is not a fraction, nor does it have any powers involving negative exponents.

Example: $\sqrt{\frac{5}{8}}\ \sqrt{\frac{5}{8} \cdot \frac{2}{2}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4}$ Answer

---

The aim here is to change the denominator 8, into a perfect square, 16, through multiplication by 2.

Suggestion: For large denominators, find the prime factors and group them in pairs, etc.

Example: $\sqrt{\frac{5}{108} \times \frac{3}{3}} = \frac{\sqrt{15}}{\sqrt{108 \times 3}} = \frac{\sqrt{15}}{18}$, Answer

$$\begin{array}{l|l} 2 & 108 \\ 2 & 54 \\ 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

Since the factor, 3, is the only unpaired one, multiply by $\frac{3}{3}$.

All radical denominators are eliminated by multiplying the fraction by some expression that will make the denominator a perfect square or rationalize the denominator.

(d) The radicand, b, does not appear in any denominator.

Example: $\frac{3}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{36}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

Here students have a tendency to multiply by $\frac{\sqrt{12}}{\sqrt{12}}$, thereby obtaining an unwieldy answer which must be reduced:

$$\frac{3}{\sqrt{12}} \cdot \frac{\sqrt{12}}{12} = \frac{3\sqrt{12}}{12} = \frac{3\sqrt{4}\sqrt{3}}{12} = \frac{1}{4}(2)\sqrt{3} = \frac{\sqrt{3}}{2}$$

Example: $\frac{2}{3\sqrt{x + 3}} \cdot \frac{\sqrt{x} - 3}{\sqrt{x} - 3} = \frac{2\sqrt{x} - 6}{x - 9}$

The denominator $(\sqrt{x} + 3)$ is rationalized by multiplying by its conjugate, $\sqrt{x} - 3$. We are here relying on the fact that $(a + b) \cdot (a-b) = a^2-b^2$ an expression containing only squared terms. Thus $\sqrt{x}$ becomes x, a rational quantity.

(e) The order of radicals is in simplest form.

Using the property of radicals given above in Unit VIII, F, 1-(e),

$\sqrt[mp]{a^{np}} = \sqrt[m]{a^n}$:

Example: $\sqrt[10]{a^{15}} = a^{15/10} = a^{3/2} = \sqrt{a^3}$

Students will notice that 10 and 15 can legally each be divided by 5, but they should be cautioned against using rules without understanding the basis for them.

A great amount of practice in changing to fractional exponents is indicated on the simplification of radicals.

3. Factoring radicals (Use distributive property): $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$, etc.

One good reason for always writing radicals in simplified form is the need for combining them.

Example: The addition of $\sqrt{24}$ and $\sqrt{54}$ cannot be performed unless we simplify:

$2\sqrt{6} + 3\sqrt{6}$, which = $5\sqrt{6}$.

4. Multiplication of radicals

(a) of like order: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Example $\sqrt[3]{6a^2b} \cdot \sqrt{18ab^3} = \sqrt[3]{2 \cdot 3\, a^2b} \cdot \sqrt[3]{2 \cdot 3ab^3} = \sqrt[3]{27 \cdot 4a^3b^4} = 3ab\sqrt[3]{4b}$

Example 2: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$

Note: here are 2 binomial factors of one!

Example 3: $(\sqrt{2} + \sqrt{5})(\sqrt{2} - 3) = \sqrt{4} + \sqrt{10} - 3\sqrt{2} - 3\sqrt{5} =$

$2 + \sqrt{10} - 3\sqrt{2} - 3\sqrt{5}$

(b) of unlike order: $\sqrt[n]{a} \cdot \sqrt[m]{b}$

Here we must obtain radicals of the same order, by changing to fractional exponents.

Example: $\sqrt{2} \cdot \sqrt[3]{5} = 2\tfrac{1}{2} \cdot 5\ 1/3 = 2\ 3/6 \cdot 5\ 2/6 = \sqrt[6]{2^3} \cdot \sqrt[6]{5^2} = \sqrt[6]{200}$

5. Division of radicals

(a) of like orders: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Example: $\dfrac{\sqrt[3]{12a^2}}{\sqrt[3]{4a}} = \sqrt[3]{3a}$

(b) of unlike order: $\dfrac{\sqrt[n]{a}}{\sqrt[m]{b}}$

Here we could change to radicals of the same order, as on page 70, but it is better to rationalize the denominator first, then change the radicals in the numerator to the same order, if necessary.

Example: $\sqrt{3} \div \sqrt[3]{4}$

Example: $\dfrac{\sqrt{3}}{\sqrt[3]{4}} \cdot \dfrac{\sqrt[3]{2}}{\sqrt[3]{2}} = \dfrac{\sqrt{3}\,\sqrt[3]{2}}{\sqrt[3]{8}} = \dfrac{\sqrt{3} \cdot \sqrt[3]{2}}{2} =$

$$\frac{3^{3/6} \cdot 2^{2/6}}{2} = \frac{\sqrt[6]{27} \cdot \sqrt[6]{4}}{2} = \frac{\sqrt[6]{108}}{2}$$

6. **Rationalizing denominators**

This process was illustrated above under Page 69

7. **Equations containing radicals**

(a) If there is only one radical term, set it alone in one member of the equation; if more than one, place the most complicated one along in one member.

Example: Solve for x:

$x - 5 = 10$

$x = 15$

$x = 225$

Check:

$225 - 5 = 10$

$15 - 5 = 10$

$10 = 10$

(b) Raise each member to the power shown by the order of the radical.

(c) If there is still a radical, proceed as in (a) above.

(d) Solve the resulting equation.

(e) Check all apparent solutions in the original equation.

Example: Solve for x:

$\sqrt{x} + 3 = \sqrt{x + 21}$ Squaring both members

$x + 6\sqrt{x} + 9 = x + 21$

$6\sqrt{x} = 12$

$\sqrt{x} = 2$

$x = 4$

Check:

$\sqrt{x} + 3 = \sqrt{x + 21}$

$2 + 3 = \sqrt{4 + 21}$

$5 = 5$

F. Complex numbers:

If $i = \sqrt{-1}$ and a and b are real, then (a + bi) is the general form of a complex number (a can be 0 and b can be 0).

1. Changing to the i-form: $\sqrt{-a} = i\sqrt{a}$

2. The powers of i:

$i = \sqrt{-1}$	$i^5 = i$
$i^2 = -1$	$i^6 = -1$
$i^3 = -\sqrt{-1} = -i$	$i^7 = -1$
$i^4 = +1$	$i^8 = +1$

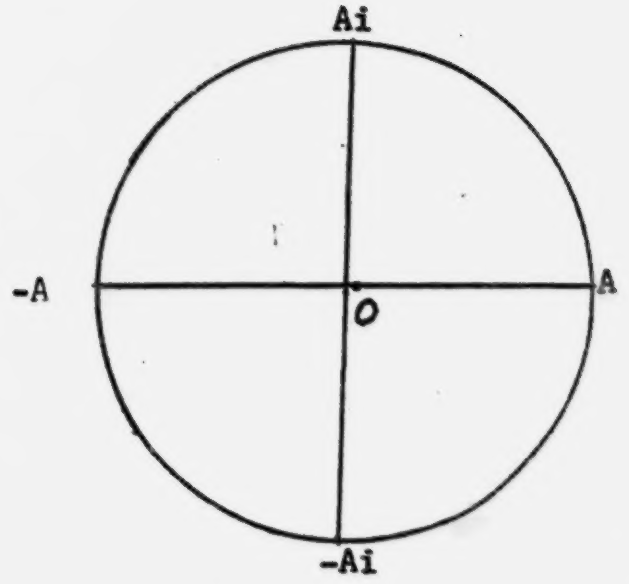
Note: This sequence on the graphing of complex numbers has a more detailed discussion than most topics because many texts do not have adequate explanation of this important subject.

3. Graphing Complex Numbers

If the line segment 0A is rotated counterclockwise about 0, through 180^o, we obtain -A. We can say that this motion produces a multiplication by -1. If we consider this to be a multiplication twice by i (once up to the vertical position, Ai, again over to -A) we would obtain -A, since A(i)(i) = -A.

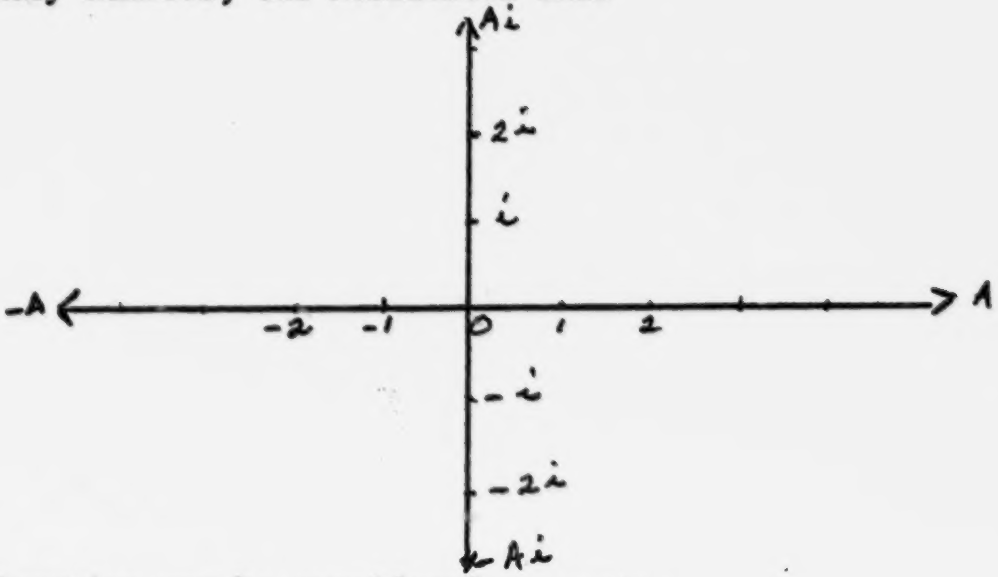
Therefore, we consider one rotation through 90^o to be equivalent to a multiplication by i. This easily explains the periodicity of the above table, where $i^5 = i$, etc.

Note: Use plastic with center fastener at the origin for demonstration on an overhead projector.



The vertical axis (Ai, -Ai) is the axis of imaginary numbers, the horizontal axis is that of real numbers.

Example: Plot: P: 4 + 2i
Q: -2 + i
R: -1 - 3i



(a) Complex numbers as vectors

(1) Definition of a vector

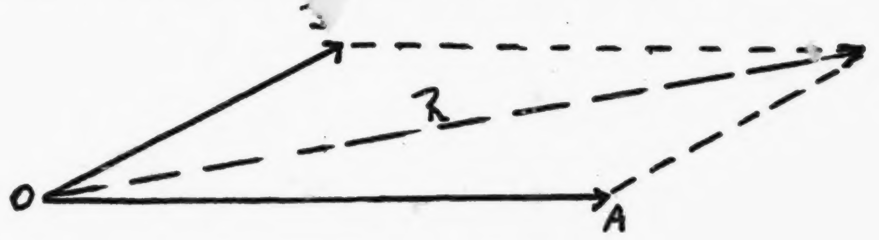
Quantities having both magnitude and direction may be considered as vector quantities and may be represented by directed line aegments or arrows called vectors. The length of the arrow shows the magnitude of the vector quantity and the direction of the arrow shows the direction of the vector quantity.

(2) The reason for relating complex numbers to vectors

Since there is a one-to-one correspondence between complex numbers and the points of the plane, and a one-to-one correspondence between the points of the plane, there is also a one-to-one correspondence between complex numbers and vectors drawn from the origin.

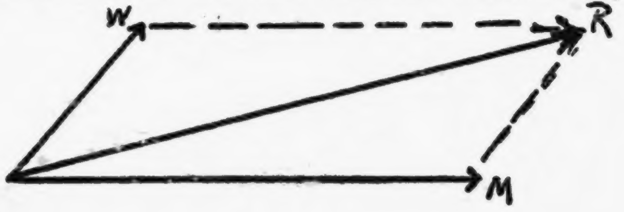
A vector quantity can be related to a complex number. From the complex number we can determine both the size and direction of the vector quantity. Thus we see that while real numbers are useful for expressing size, complex numbers are useful for expressing size in conjunction with direction.

(3) In the study of physics we learn that whenever two or more forces act on an object there is one force which could replace them and have the same effect; this force is called their resultant.



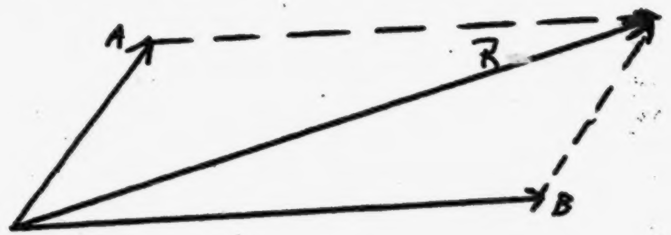
Thus if the force A acts upon object O, and at the same time force B acts upon it, the net effect of the two forces, or their resultant, R, is usually different in amount and direction from either of them. It is, in fact, the diagonal of the parallelogram having the two given forces as adjacent sides.

For example, if an airplane is pushed by the force of its motor, M, and at the same time it is acted upon by a wind, W, the plane will actually move along the path of the resultant, R, and be at a predictable point at a given time (provided, of course, the wind and motor forces remain constant) in amount and direction.



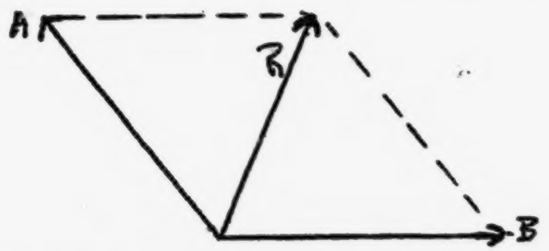
To determine R, a parallelogram is constructed on sides W and M, and R will be the diagonal.

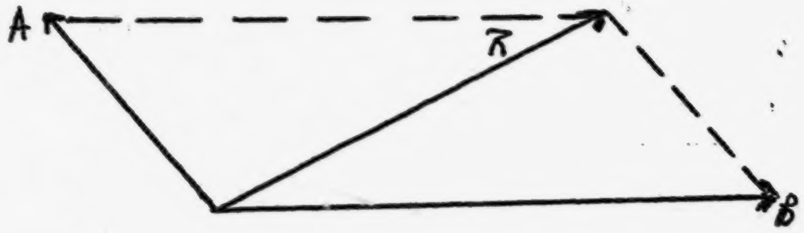
(4) Value of resultant, in comparison to values of components.



R is the resultant of the components A and B. In this particular example, R is greater than either A or B. However, this is not always the case, and the student should become aware of this fact.

In the following examples R is smaller than one or both components A and B.

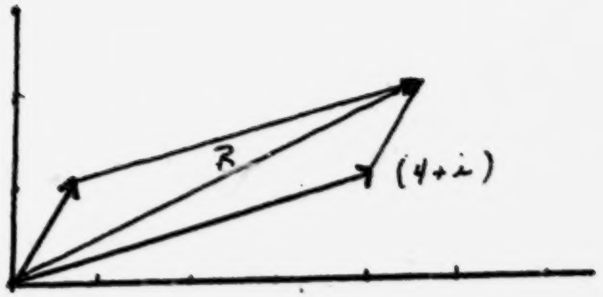




If A and B are equal, as A
rotates toward a direction
exactly opposite from B,
R approaches zero as a limit.

(5) Graphing the sum of two complex numbers.

Example: Graph the sum of vectors 4 + i and 1 + 2i



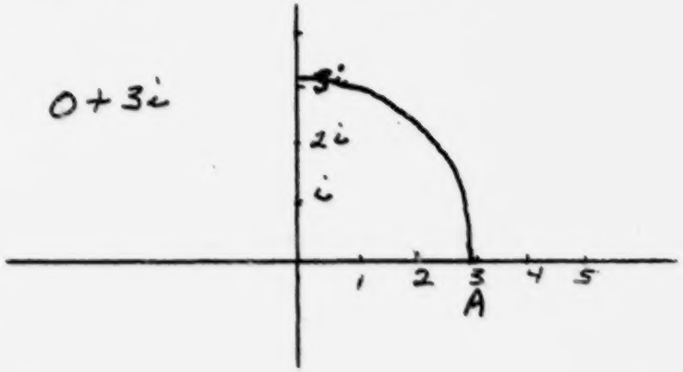
The resultant R is seen to be 5 + 3i, both on the graph and by addition.

Students should be given exercises in finding the sums of pairs of complex numbers, both by graph and by algebraic combination.

Example: x + bi + c + di

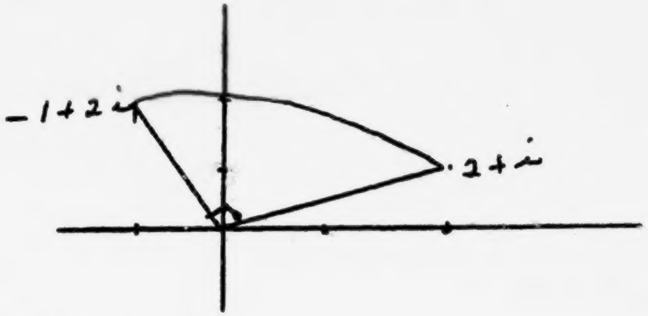
(6) Multiplying real and pure imaginary numbers

Since multiplication by i rotates 0A counterclockwise through 90 degrees, 3i or 3.i is represented by the point B (0 + 3i) on the vertical i axis.



(7) Multiplying complex numbers

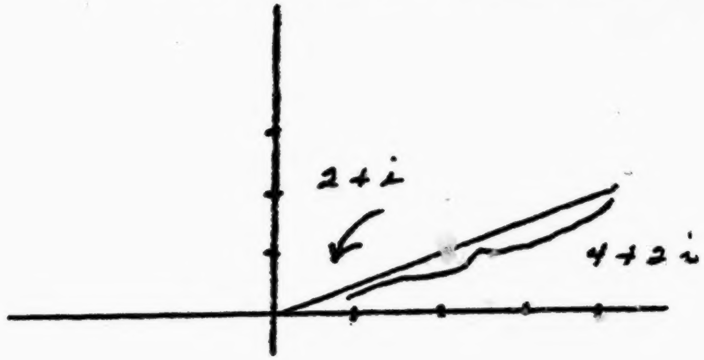
Likewise, multiplying (2 + i) times i rotates the vector of (2 + i) through 90 degrees to form the vector (-1 + 2i). Also, $(2 + i)i = 2i + i^2 = -1 + 2i$.



Multiplying by a constant, such as two, doubles the length of the vector multiplied.

Problems should be given on the multiplication and division of pure imaginary numbers and complex numbers, to be solved by the same rules as for any irrational numbers, with one difference: the various powers of i should be simplified. For example: $3i^2 = -3$, $4i^3 = 4i$, etc. Some exercises could be given on graphing the multiplication of complex numbers.

Example: Muliplying $(2 + i) \cdot 2$ doubles its length, forming $4 + 2i$.



IX. Quadratic Functions and Equations - Time: 20 days

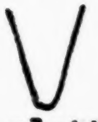
A. Quadratic Functions: $f(x) = ax^2 + bx + c$ (where $a \neq 0$)

1. Graphing quadratic functions

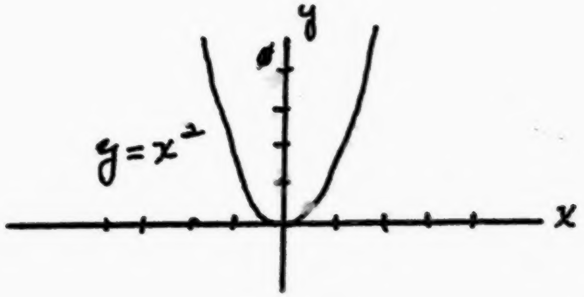
(a) Shape of the curve: a parabola

Students should become familiar with the shape of a parabola--the name of the curve formed by graphing the quadratic equation at the left.

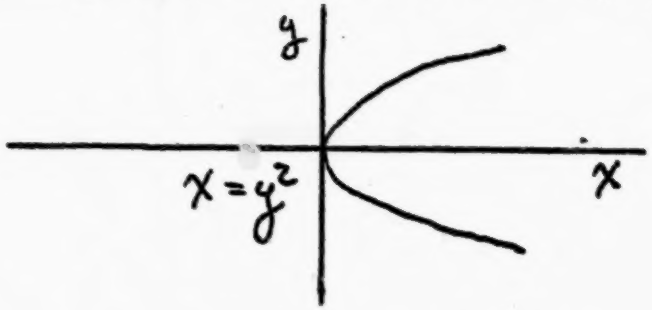
Some examples of the parabola are the path of a projectile in a vacuum (air resistance modifies the curve somewhat), and the cross section of some headlight reflectors and of some antennae used for radio and television. The hyperbolic paraboloid form in architecture has a parabolic cross section.

The exact proportions of a parabola may vary from  to , but if the ends are graphed to a great distance by plotting additional points, the lines become almost parallel.

(b) For every parabola there is a line, called the axis of symmetry, about which the curve is symmetric. The discussion can well begin with a simple example, $y = x^2$. The axis of symmetry is the y-axis, with the vertex at the origin.



In the equation, $y = ax^2 + bx + c$, if $b \neq 0$, the parabola will have a different axis of symmetry, not the y-axis but a line parallel to it. A discussion of what happens to the curve as a, b, and c vary is valuable at this point. Some mention may be made of the equation $x = y^2$, and of $x = ay^2 + by + c$.



The concept of symmetry should receive attention: the fact that there are matching points, positive and negative, which cause the curve to be symmetric with respect to a line. For example, in the equation $y = x^2$, when x is either plus 2 or minus 2, y equals 4.

(c) The turning point or vertex

For $y = ax^2 + bx + c$, if $a > 0$, when the absolute value of x is allowed to become very large, the value of y will be large and positive; as x increases x^2 will increase still more rapidly and ax^2 will eventually become much greater than bx. Even if b is negative, $ax^2 + bx > 0$.

(1) If when $|x|$ is large, y also becomes large and positive, the curve will be opened upward, $\cup$. In this case, the vertex or turning point will have a maximum value.

In like manner, if $a < 0$, the curve will open downward and will rise to a maximum point at its vertex.

A discussion of the general characteristics of the parabola, followed by student experience in sketching, freehand, the various curves of quadratic equations, will be time well spent.

Examples: graph free-hand curves of:

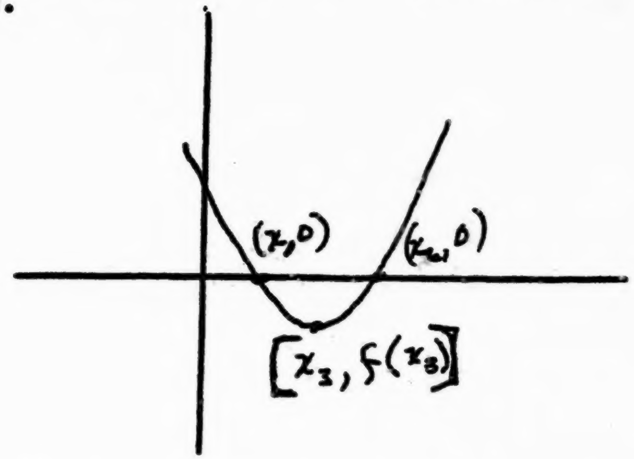
$y = x^2$; $y = 2x^2$; $y = x^2 + 1$; $y = -x^2$; $x = y^2$; $y = x^2 + x$;
$y = -x^2 + x$, etc.

(2) Finding the coordinates of the turning point: $x = \frac{-b}{2a}$, $y = f\left(\frac{-b}{2a}\right)$.

It should be emphasized that there are several ways to determine the coordinates of the turning point, but that since all of those available at this stage are more or less time-consuming, this is one time when memorization is advisable! (In calculus, the derivative will provide an easy solution).

However, the student should be able to find the coordinates of the turning point by some other method, if he cannot recall the formula.

(2-a) By the average of the roots, if the solutions of the equation, given by the values x_1 and x_2, are known, they can be averaged to find the abscissa, x_3, of the turning point. After finding x_3, substitution of it in the equation will give the corresponding value of y: $y = f(x_3)$.



Example: find the coordinates of the turning point of the curve of $y = 2x^2 + 5x - 3$.

$(2x - 1)(x + 3) = 0; \; x = \frac{1}{2}, -3.$

The average of the roots $= \dfrac{\frac{1}{2} - 3}{2} = \dfrac{-5}{4}$

$y = 2\left(\dfrac{25}{16}\right) - \dfrac{25}{4} - 3 = \dfrac{25}{8} - \dfrac{50}{8} - \dfrac{24}{8} = \dfrac{-49}{8}$ Ans: $\left(\dfrac{-5}{4}, \dfrac{-49}{8}\right)$

(2-b) To obtain the formula, $x = \dfrac{-b}{2a}$, one can proceed as above with the formulas for the roots.

$$x_1 = \frac{-b + b^2 - 4ac}{2a} \qquad x_2 = \frac{-b - b^2 - 4ac}{2a}$$

$$x_3 = \frac{x_1 + x_2}{2} = \frac{\frac{-2b}{2a}}{2} = \frac{-b}{2a}$$

(2-c) By transforming the equation into the form, $y = a(x - h)^2 + k$, one can set $x = h$, thereby making $a(x - h)^2$ become 0, and y a minimum. $x = h$, then, is the equation of the axis of symmetry and is the abscissa of the turning point of the graph.

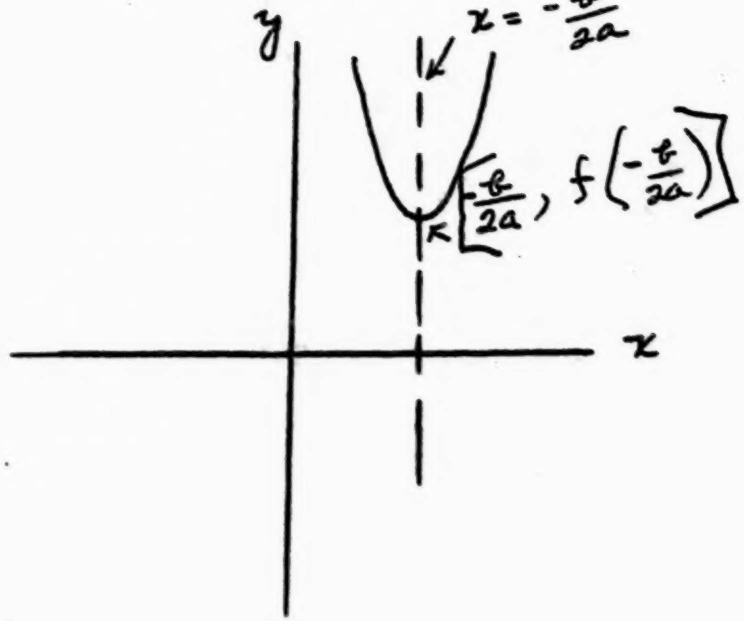
Example: $y = 2x^2 - 4x + 5$; find the coordinates of the turning point.

$$y - 5 = 2(x^2 - 2x + ?)$$
$$y - 5 + 2 = 2(x^2 - 2x + 1)$$
$$y - 3 = 2(x - 1)^2$$

When $x = 1$, the right member becomes 0, and y has a minimum value. To find y: $y = 2(1)^2 - 4(1) + 5 = 3$.

(3) Finding the equation of the axis of symmetry: $x = \frac{-b}{2a}$.

Since the axis of symmetry passes through the turning point or vertex of the parabola, and is parallel to the y-axis, the abscissa of the turning point, $x = -b/2a$, also gives the equation of the axis of symmetry.



B. Algebraic solution of quadratic equations

1. Incomplete quadratic equations

Incomplete quadratic equations ($ax^2 + bx + c = 0$, where $b = 0$ and/or $c = 0$) can be solved by finding the square root of both members of the equation after solving for x^2.

Example:

$$2x^2 = 5$$
$$x^2 = 5/2$$
$$x = \pm\sqrt{5/2} = \pm\frac{\sqrt{10}}{2} \quad \text{(like roots of equals are equal)}$$

Students sometimes lose the negative root, having been trained to think in terms of the positive principal root only. For this reason some instructors prefer to use the factoring method:

$$x^2 = a,\ x^2 - a = 0,\ (x + \sqrt{a})(x - \sqrt{a}) = 0;\quad x = \sqrt{a},\ -\sqrt{a}$$

Example:

$$2x^2 = 5$$
$$2x^2 - 5 = 0$$
$$x^2 - \frac{5}{2} = 0$$
$$x^2 - (\sqrt{5/2})^2 = 0$$
$$(x + \sqrt{\tfrac{5}{2}})(x - \sqrt{\tfrac{5}{2}}) = 0$$
$$x = \frac{\sqrt{10}}{2},\ \frac{-\sqrt{10}}{2}$$

2. Complete quadratic equations: $ax^2 + bx + c = 0;\ a,b,c \neq 0$

(a) By factoring

This topic was included in the union in factoring. Additional practice should be given here, and students should know that factoring is generally the preferred method, if the equation can be factored easily.

Sometimes, however, it is inadvisable to attempt to factor, if the coefficient a and the constant c are composite numbers with several factors. Before wasting time trying possible arrangements of factors, one can discover quickly whether there are rational factors by evaluating the discriminant $b^2 - 4ac$. If the discriminant is both positive and a perfect square, there are rational factors.

Example: Solve for x; $12x^2 + 10x + 5 = 0$

$a = 12$
$b = 10$
$c = 5$

$b^2 - 4ac = 100 - 4(12)(5) = 100 - 240 = -140$

Since the discriminant is not a perfect square, the equation has no rational factors.

Although the quadratic formula has not yet been introduced in this syllabus, the student should already be familiar with it from his first-year course in algebra. It may be necessary to review the quadratic formula at this time,

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, showing that the radical $\sqrt{b^2 - 4ac}$ determines the nature of the roots. If the discriminant $b^2 - 4ac$ is a positive perfect square, there are rational roots, and if so, the equation is factorable into rational factors. $b^2 - 4ac = 0$ can be factored as a trinomial square, $b^2 - 4ac < 0$. Involve i factors.

(b) By completing the square: $ax^2 + bx + c = 0$

Following are two examples of solution by completing the square. It is obvious that when $a \neq 1$, or b is an odd number, (or both) as in Example 2, the operation is laborious, whereas Example 1 can be solved easily by completing the square. Example 2 should be done by factoring, or by the quadratic formula.

$ax^2 + bx + c$

Divide both members by a. Clear c from left member. Square $\frac{1}{2}b$, add to both members. Find the square root of both members. Complete the solution by setting the binomial in the left member equal to each term (+ and -) in the right member.

Example 1:

$x^2 + 2x - 2 = 0$
$x^2 + 2x = 2$

$x^2 + 2x + 1 = 2 + 1$
$(x + 1)^2 = 3$
$x + 1 = \pm \sqrt{3}$

$x + 1 = \sqrt{3}, \quad x = -1 + \sqrt{3}$
$x + 1 = -\sqrt{3}, \quad x = -1 - \sqrt{3}$

Ans: $x = -1 + \sqrt{3}, \quad -1 - \sqrt{3}$

Example 2:

$3x^2 - 5x + 2 = 0$
$x^2 - \frac{5x}{3} = \frac{-2}{3}$

$(x - ?)^2 =$

$x^2 - \frac{5}{3} x + \frac{25}{36} = \frac{-2}{3} + \frac{25}{36}$

$(x - 5/6)^2 = \frac{-24}{36} + \frac{25}{36} = \frac{1}{36}$

$x - 5/6 = \pm 1/6$

$x - 5/6 = 1/6, \quad x = 6/6$ or 1

$x - 5/6 = -1/6, \quad x = 4/6$ or 2/3

Ans: $x = 1, 2/3$

Students should be urged to use the method of completing the square on any quadratic equation having $a = 1$ and b an even number. They tend to use the quadratic formula on all unfactorable equations, but will usually learn to turn to the completing-the-square method when they have had enough experience. For example, the equation, $x^2 + 6x + 4^1 = 0$, can be more easily solved by completing the square than by formula.

(c) By the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Some students will enjoy deriving the quadratic formula by completing the square. The derivation should at least be demonstrated to all students.

Any quadratic equation can be solved by the quadratic formula, especially equations that are difficult to factor. The fact that the expression 2a is written as a divisor under the entire rest of the formula will probably need emphasis; otherwise (-b) "call this the opposite of b" may be left by itself, without a denominator.

3. Forming a quadratic equation when the roots x_1 and x_2 are given.

(a) By factoring (in reverse)
(b) By the sum and product of the roots

Example: Given the roots -3 and +2, write the equation.

If -3 is a root, $(x + 3)$ must have been a factor, etc., or $(x + 3)(x - 2) = 0$, or $x^2 + x - 6 = 0$

(Students should be reminded that $a \cdot b = 0$ if and only if $a = 0$ and/or $b = 0$)

This problem can also be solved by the sum and product of the roots. In the general form of the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots equals $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $\dfrac{-2b}{2a}$ or $\dfrac{-b}{a}$

Product of Roots: $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{c}{a}$

To write an equation of the roots is r_1r_2, where $(x - r_1)(x - r_2) = 0$, or

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0 \qquad (1)$$

Writing the general form of the quadratic equation $ax^2 + bx + c = 0$ in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (2)$$

and equating (1) and (2) we see that $r_1r_2 = \frac{c}{a}$, or the product of the roots $= \frac{c}{a}$.

For the above example, where the roots are -3 and 2, the sum $= \frac{-b}{a} = -1$, and the product = -6. Therefore, the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ becomes $x^2 + 1x - 6 = 0$.

Students should have experience writing equations when the roots are given, by both methods. They might be told that the second method, by the sum and product of the roots, can be used for equations of higher degree, in which the product of the roots is equal to c/a, where c is the constant and a is the coefficient of the highest degree term. The sum of the roots can also be found by a simple formula.

X. Progressions and the Binomial Theorem - Time: 10 days

A. Arithmetic Progression nth term: $z + (n - 1)d$

1. Writing the nth term, t_n, given a, n, and d

The student should be able to develop the formula for the nth term:

1st	2nd	3rd	4th	...nth
a	$a+d$	$a + 2d$	$a + 3d$	$\underline{t_1 + (n-1)d}$

Example: Find the 21st term of the arithmetic progression 10, 13, 16, 19. . .

$$t_n = t_1 + (n - 1)d$$

$$t_{21} = 10 + (20)3 = 10 + 60 = \underline{\underline{70}}$$

2. Given any 3 of the 4 variables, a, d, n, t_n, find the 4th one.

Example: Find the number of terms of an arithmetic progression, if the first term is 3, the difference is -2, and the last term is -19.

$$t_n = t_1 + (n - 1)d$$
$$-19 = 3 + (n - 1)(-2)$$
$$-19 = 3 - 2n + 2$$
$$2n = 5 + 19 = 24$$
$$n = 12$$

or solve for n:

$$t_n = t_1 + dn - d$$
$$t_n - t_1 + d = dn$$
$$\frac{t_n - t_1 + d}{d} = n$$
$$\frac{-19 - 3 - 2}{-2} = n$$

The second method of solving first for n is more difficult for most students but provides better practice in simple algebraic techniques.

3. Inserting arithmetic means

Example: Insert 4 arithmetic means between 2 and 4. $a = 2$, $n = 6$, $t_n = 4$, $d = ?$

$$t_n = t_1 + (n - 1)d$$
$$4 = 2 + 5d,\ d = 2/5$$

Ans: 2, $\underline{2.4,\ 2.8,\ 3.2,\ 3.6}$, 4

4. Finding the sum of n terms of an arithmetic progression: $S = \frac{n}{2}(t_1 + t_n)$, $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$

Students can derive the sum formula for an arithmetic series by thinking that the average term is $\frac{t_1 + t_n}{2}$ and there are n terms, therefore the sum equals $\frac{n(t_1 + t_n)}{2}$ usually written $\frac{n}{2}(t_1 + t_n)$.

Another way of deriving the formula is to rewrite the series in reverse order and add:

a	$t_1 + d$	$t_1 + 2d$	$\ldots t_1 + (n-2)d$	$t_1 + (n-1)d$
$s = t_1 + (n-1)d$	$t_1 + (n-2)d$	$t_1 + (n-3)d$	$\ldots t_1 + d$	t_1
$s = 2t_1 + (n-1)d$	$2t_1 + (n-1)d$	$2t_1 = (n-1)d$	$\ldots 2t_1 + (n-1)d$	$2t_1 + (n-1)d$

Since n is the number of terms, we have $2s = n\left[2t_1 + (n-1)d\right]$ or $2 = \frac{n}{2}\left[2t_1 + (n-1)d\right]$

Example: Evaluate

$$\begin{array}{c} \left|\begin{matrix} 6 & 2 \\ -1 & 3 \end{matrix}\right| \\ \swarrow \quad \searrow \\ -2 \qquad 18 \end{array} \qquad 18 - -2 = \underline{\underline{20}}$$

B. Solution of linear systems in two variables

(1) $a_1x + b_1y = c_1$

(2) $a_2x + b_2y = c_2$

$$\left.\begin{array}{ll} (1) & a_1a_2x + a_2b_16 = a_2c_1 \\ (2) & \underline{a_1a_2x + a_1b_2y = a_1c_2} \end{array}\right\rangle \text{Subtract}$$

$$(a_2b_1 - a_1b_2)y = a_2c_1 - a_1c_2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} \qquad (3)$$

Equal

$$(\text{if } a_2b_1 - a_1b_2 \neq 0)$$

$$\text{or } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (4)$$

1. Demonstration of the determinant formula for solving linear systems in 2 variables

Since the solution for y (marked (3) above) resulting from the usual subtraction method of solving linear systems, is the same as the determinant in (4), we have here a new (determinent) method of solving linear systems in two variables.

It is interesting that for the series of consecutive integers, where $a = 1$ and $d = 1$, we have $2s = n(n + 1)$ or $s = \frac{n(n + 1)}{2}$.

$$1 + 2 + 3 \;\ldots\ldots\ldots\ldots\; n - 1 + n$$

$$\frac{n}{n+1} \quad \frac{n - 1}{n + 1} \quad \frac{n - 2}{n + 1} \quad \frac{2}{n + 1} \quad \frac{1}{n + 1}$$

5. The use of the symbol $\sum$ in writing a series: $\sum_{n=b}^{a} kn$

"The summation from b to a of kn"

$\sum$: summation sign, sigma
kn: summand
n: index

Example 1: Write $\sum_{n=1}^{4} 5n$, "the summation from 1 to 4 of 5n"

$$= 5(1) + 5(2) + 5(3) + 5(4)$$

$$= 5 + 10 + 15 + 20$$

$$= \frac{4}{2}\left[10 + 3(5)\right] + 2(25) = \underline{\underline{50}}$$

Example 2: Write as a summation: $3 + 7 + 11 + 15 + 19$

$$n = 1 \overset{1}{\frown} 2 \overset{1}{\frown} 3 \overset{1}{\frown} 4 \overset{1}{\frown} 5$$

$$\text{series:} \quad 3 \underset{\underset{4}{\smile}}{+} 7 \underset{\underset{4}{\smile}}{+} 11 \underset{\underset{4}{\smile}}{+} 15 \underset{\underset{4}{\smile}}{+} 19$$

Since the terms of the series are increasing by 4 at the same time that n increases by one, the series is increasing 4 times as fast, or 4n. 4n - 1 can be seen to be one suitable formula, or

$$\sum_{n=1}^{5} (4n - 1) \quad \text{or if } n = 0,\ 4n + 3 \text{ can be used: } \sum_{n=0}^{4} (4n + 3).$$

Read: "the summation from 0 to 4 of (4n + 3)"

B. Geometric Progression nth term: $t_n = ar^{n-1}$

The student should be able to develop the formula:

1st	2nd	3rd	4th	....nth
a	ar	ar^2	ar^3	ar^{n-1}

1. Writing the nth term given a, r, n.

   (All that is needed is to write a few terms of the progression, as above, and the formula for n terms becomes apparent)

   Example: Find the seventh term of the geometric progression:

$$2, \frac{4}{3}, \frac{8}{9} \ldots\ldots$$

$$a = 2$$

$$r = \frac{2}{3}$$

$$n = 7$$

$$t_n = t_1 r^{n-1} \text{ or } t_7 = 2\left(\frac{2}{3}\right)^6$$

$$t_n = \frac{2^7}{3^6} = \frac{128}{729}$$

2. Given any 3 of the variables t_n, a, r, n, find the 4th one.

   Example: Given the geometric progression 5, -25, 125 .....3125, find the number of terms.

$t_n = ar^{n-1}$

$3125 = 5(-5)^{n-1}$

$625 = (-5)^{n-1}$

$625 = (-5)^4$ or $5 = n - 1$

$n = 5$

Check: $5, -25, 125, \underline{\underline{-625}}, 3125$

Note: the equation $625 = (-5)^{n-1}$ is actually an exponential one, suitable for logarithmic solution, but solved here simply by developing the powers of 5.

3. Inserting geometric means

Example: Insert 5 geometric means between 2 and 4

$t_n = ar^{n-1}$

$4 = 2r^6$

$r^6 = 2$

Check: $r = {}^{6}\,2$

$$2, \underbrace{2\sqrt[6]{2}, 2\sqrt[6]{4}, 2\sqrt[6]{8}, 2\sqrt[6]{16}, 2\sqrt[6]{32}}, 4$$

4. Finding the mean proportional (one geometric mean)

Students might be reminded of the construction of the mean proportional from geometry:

$\frac{a}{m} = \frac{m}{b}$

$m^2 = ab$

$m = \pm\sqrt{ab}$

RT is a diameter.

$\triangle$ RST is a right $\triangle$, since

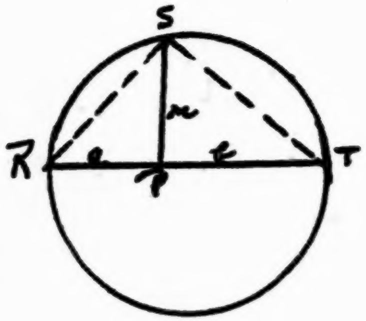
$\angle$RST is inscribed in a semicircle.

SP is $\perp$ RT.

$\therefore$ m is the mean proportional

between a and b

or $\frac{a}{m} = \frac{m}{b}$



5. Finding the sum of a geometric progression: a geometric series $s = \frac{a - ar^n}{1 - 4}$ $(r \neq 1)$

The formula can be derived as follows:

$s = a + ar + ar^2 + ar^3 \dots ar^{n-1}$

$$\begin{array}{rcl} * \ rs & = & ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n \\ \hline s - 4s = a & & -ar^n \end{array}$$

* Multiply both members by r, and subtract.

$s(1 - r) = a - ar^n$

$s = \frac{a - ar^n}{1 - r}$

Example: Find the sum of the geometric series $\sum_{r=1}^{5} 9\left(\frac{1}{3}\right)^{r-1}$

$$9\left(\frac{1}{3}\right)^0 + 9\left(\frac{1}{3}\right)^1 + 9\left(\frac{1}{3}\right)^2 + 9\left(\frac{1}{3}\right)^3 + 9\left(\frac{1}{3}\right)^4$$

$$9 \quad + \quad 3 \quad + \quad 1 \quad + \quad \frac{1}{3} \quad + \quad \frac{1}{9}$$

$$s = \frac{a - ar^n}{1 - r} = \frac{9 - 9\left(\frac{1}{3}\right)^5}{1 - \frac{1}{3}} = \frac{9 - 3^2\left(\frac{1}{3}\right)^5}{\frac{2}{3}}$$

$$= \frac{9 - \frac{1}{27} \cdot 27}{\frac{2}{3} \cdot 27} = \frac{243 - 1}{18} = \frac{242}{18} = \frac{121}{9}$$

$$= 13\,\frac{4}{9}$$

6. Infinite geometric series: Sum $= \frac{a}{1 - r}$ as n increases without bound. $|r| < 1$.

Consider the series, $1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16}$ .... to infinitely many terms.

$s = \frac{a - ar^n}{1 - r}$; $r = \frac{1}{2}$. As n grows larger and larger, the value of

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ .... becomes smaller and smaller, differing from zero by as little as desired, if n be large enough. Therefore, the limit of s as n increases without bound is $\frac{a-0}{1-r}$ or $\frac{a}{1-r}$.

This formula can be derived by the student without much difficulty.

7. Finding the value of a repeating decimal

Example: change .363636... to an equivalent common fraction:

$$.363636\ldots = .36 + .0036 + .000036 \text{ etc.} \qquad \text{where } a = .36,\ r = .01$$

$$s = \frac{a}{1 - r} = \frac{.36}{1 - .01} = \frac{.36}{.99} = \frac{4}{11}$$

C. Recognition of arithmetic and geometric progressions

As a culmination of the unit, there should be problems having miscellaneous kinds of progressions, for the student to tell whether they are arithmetic, geometric, neither, or both.

Example 1: $24, 8, 2\frac{2}{3}, \frac{8}{9}$ Geometric: $r = \frac{1}{3}$

Example 2: $\sqrt{5}, 3\sqrt{5} + 2, 5\sqrt{5} + 4 \ldots.$ Arithmetic: $d = 2\sqrt{5} + 2$

Example 3: 1, 4, 12, 36 .... neither

1. Relationship between the geometric mean and the arithmetic mean: $\frac{a + b}{2}$, ab

The arithmetic mean is always equal to more than the geometric mean:

Proof

$(a - b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + 2ab + b^2 \geq 4ab$

$\frac{a^2 + 2ab + b^2}{4} \geq ab$

$\frac{a + b}{2} \geq \sqrt{ab}$

Example 1: 3, 8

arithmetic mean $= \frac{11}{2} = 5.5$

geometric mean $= \sqrt{24} = 4.9$

Example 2: 3, 3

arithmetic mean $= \frac{6}{2} = 3$

geometric mean $= \sqrt{9} = 3$

2. Rules for expansion of $(a + b)^n$

a. Number of terms in expansion of $(a \pm b)^n$ is $n + 1$.

b. If the binomial is a sum, all terms of the expansion are positive; if the binomial is a difference, every other term is negative, starting with the second term.

c. The coefficient of the first term is one.

d. The coefficient of any other term is found by multiplying the coefficient of the preceding term by the exponent of a and dividing by the number of the term in the expansion.

e. The exponents of a decrease by one for every term, while the exponents of b increase by one for every term. (b^0 or 1 for the first term, b^1 for the second, b^2 for the third, etc.)

f. The sum of the exponents in any one term is n. (NOTE: Pascal's triangle may be introduced here.)

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \end{array}$$

Example 1: Expand $(a - b)^7$:

First write the literal portions of the terms:

$$a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7$$

$a^7 - (\ )a^6b + (\ )a^5b^2 - (\ )a^4b^3$ etc. then the alternating minus signs.

Then follow the rules for coefficients:

$$a^7 - (7 \cdot 1)a^6b \ ^*$$

$$+ \frac{7 \cdot 6}{2} a^5b^2 \ ^{**}$$

$$- \frac{7 \cdot 6 \cdot 5}{2 \cdot 3} a^4b^3$$

$$+ \frac{7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4} a^3b^4 \quad \text{etc.}$$

$$a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 70a^3b^4 \text{ etc.}$$

* At this point multiply the coefficient,7, by the exponent of a, which is 6 and divide by 2 (since this is the second term).

** Here multiply the coefficient 21 of a^5b^2 by the exponent of a, which is 5, and divide by 3 (since this is the third term), etc. Sometimes it is easier to divide first by the exponent of b (increased by one) before multiplying by the exponent of a.

When expanding a binomial containing coefficients other than one, it is especially important to write the exponents of all terms before finding the coefficients.

Example: Expand $(2a^3 + 3b)^5$

$$(2a^3)^5 + (\quad)(2a^3)^4(3b) + (\quad)(2a^3)^3(3b)^2$$
$$+ (\quad)(2a^3)^2(3b)^3 + (\quad)(2a^3)(3b)^4$$
$$+ (\quad)(3b)^5$$

Then the coefficients:

$$^* (2a^3)^5 + \underline{\underline{5}}\,(2a^3)^4(3b) + \underline{\underline{10}}\,(2a^3)^3(3b)^2$$
$$+ \underline{\underline{10}}\,(2a^3)^2(3b)^3 + \underline{\underline{5}}\,(2a^3)(3b)^4$$
$$+ (3b)^5$$

Then: simplify the expression given for each term:

$32a^{15} + 5 \cdot 16 \cdot 3a^{12}b + 10 \cdot 8 \cdot 9a^9b^2$ etc. or

$32a^{15} + 240a^{12}b + 720a^9b^2$ etc.

* Notice that the rule "the sum of the exponent of a and b in each term = n" was illustrated here at this step, not in the final form.

3. The use of the binomial theorem in finding powers of numbers.

Example 1: Find $(29)^4$ or $(30-1)^4$

$$(30-1)^4 = 30^4 - 4(30)^3(1) + 6(30^2(1)^2$$

$$- 4(30)(1)^3 + 1^4$$

$$= 810000 - 108000 + 5400 - 120 + 1$$

$$\begin{array}{r} 810{,}000 \\ 5{,}400 \\ \underline{1} \\ 815{,}401 \end{array} \qquad \begin{array}{r} -\ 108{,}000 \\ -\ \underline{120} \\ -\ 108{,}120 \end{array} \qquad \begin{array}{r} 815{,}401 \\ -\ \underline{108{,}120} \\ 707{,}181 \end{array}$$

Example 2: Find $(1.04)^8$ correct to the nearest thousandth

$$(1 + .04)^8 = 1^8 + 8(1)^7(.04) + 28(1)^6(.04)^2$$

$$= 1 + .32 + 28(.0016)$$

$$\downarrow$$

$$.0448$$

$$+ 56(1)^5(.04)^3 + 70(1)^4(.04)^4 \text{ etc.}$$

$$+ 56(.000064) + 70(.00000256)$$

$$\searrow \qquad \swarrow$$

$$.003584 \quad + \quad .0001792$$

$$\begin{array}{rl} = & 1.0000 \\ + & .32 \\ + & .0448 \\ & .003584 \\ & \underline{.0001792} \\ & 1.3685632 = 1.369 \end{array}$$

It is obvious that the next term of this expansion, $(1.04)^8$ would be too small to influence the accuracy of our result, which is required to be correct only to thousandths.

4. Use of the binomial series to compute approximate roots of numbers $(1 + x)^n$

The binomial series, $(1 + x)^n$ (where n is = any number)

$$1^n + n(1)^{n-1}(x) + \frac{n(n-1)(1)^{n-2}x^2}{2} + \frac{n(n-1)(n-2)(1)^{n-3}x^3 \ldots.}{2\cdot 3}$$

Example: Find $\sqrt[4]{17}$ correct to thousandths:

$$17^{\frac{1}{4}} = (16 + 1)^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(1 + \frac{1}{16}\right)^{\frac{1}{4}} = 2\left[1 + \frac{1}{16}\right]^{\frac{1}{4}} =$$

$$2\left[1^{\frac{1}{4}} + \frac{1}{4}(1)^{-\frac{3}{4}}\left(\frac{1}{16}\right) + \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)(1)^{-\frac{7}{4}}\left(\frac{1}{16}\right)^2}{2} + \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)(1)^{-\frac{11}{4}}\left(\frac{1}{16}\right)^3}{2\cdot 3}\right]$$

$$= 2\left[1 + \frac{1}{64} + \frac{-3}{32}\left(\frac{1}{256}\right) + \frac{21}{64\cdot 6}\left(\frac{1}{4096}\right) \ldots\right] =$$

$$2\left[1 + .015625 - .00037 \ldots. + \frac{21}{10^6} \ldots.\right] \text{(approximately)}$$

Since the 4th term, even when doubled, will be too small to influence the accuracy to thousandths, it can be discarded.

$$2(1.015625 - .00037) = 2.030510 \text{ or } \underline{2.031}$$

This method of approximating roots becomes laborious when carried out far beyond the decimal point, but it has the advantage of being able to produce accuracy to any required degree. The logarithmic method, for example, is limited in accuracy to the number of figures in the logarithmic table used. To reduce the tedium of the arithmetic, the use of a calculator is recommended, a manually operated Monroe or similar, one capable of multiplying or dividing. The manual one is preferred for the high school student, so that he can control the various operations, "see them work."

This type of machine can be used for square root, and the more advanced students will enjoy performing this operation.

This particular problem $\sqrt[4]{17}$, can be checked on a Monroe by finding the square root twice successively $\sqrt{\sqrt{17}}$, and the answer will be correct to 8 or 10 figures, depending upon the machine used.

5. Finding any one term of the expansion: $(a + b)^n$

In the binomial expansion $(a + b)^n$:

$$\underset{}{\overset{\text{1st}}{a^n}} + \overset{\text{2nd}}{na^{n-1}b} + \overset{\text{3rd}}{\frac{n(n-1)a^{n-2}b^2}{2}} + \overset{\text{4th}}{\frac{n(n-1)(n-2)a^{n-3}b^3\ldots.}{2\cdot 3}}$$

It is easy to see the rule governing the selection of coefficients and exponents for any given term.

For the rth term: $$\frac{n(n-1)(n-2)\ldots\overset{\overset{*}{\downarrow}}{[n-(r-2)]}\; a^{n-(r-1)}b^{r-1}}{(r-1)!**}\quad\overset{**}{\swarrow\ \searrow}$$ or

for the (n + 1)st term: $\binom{n}{r}a^{n-r}b^r$.

The crucial points here are * that $n(n-1)(n-2)\ldots$.etc. goes as far as $r-2$, whereas the exponent of a is ** $n-(r-1)$, and $(r-1)$ is also the exponent of b, and the factorial in the denominator.

This is a difficult formula to remember, for students and teachers alike. Students should be urged to practice writing all the terms, up to the 3rd or 4th term (just as done at the beginning of the above discussion). This will take only a minute or so, and the formula is derived!

Example: Find the sixth term of the expansion $(a - 2y)^{15}$

$$\frac{n(n-1)(n-2)(n-3)(n-4)a^{n-5}(-2y)^5}{(n-5)!}$$

$$= \frac{\overset{5}{\cancel{15}}\cdot\overset{7}{\cancel{14}}\cdot 13\cdot\overset{3}{\cancel{12}}\cdot 11(a)^{10}(-2y)^5}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}} = 3003a^{10}(-2y)^5$$

$$= 3003\cdot(-32)a^{10}y^5 = -96096a^{10}y^5$$

Students should be reminded that with $(a - b)^n$ all even terms are negative. The truth of this fact is demonstrated above when $(-2y)^5$ is seen to be negative. (The even-numbered terms will have odd powers of the -b term.)

XI. Determinants - Time: 10 days

Although Gaussian elimination techniques are now extensively used in computer solution of higher order linear systems, determinants are of great value in this area as well as others.

A. Definitions:

1. Matrices: $\begin{Vmatrix} 2 & 3 & 5 \\ 1 & 2 & 1 \end{Vmatrix}$ $\begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 5 & 7 \end{bmatrix}$ $\begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$

   A matrix is a rectangular array of numbers (or other elements) exhibited between brackets, double lines, or parentheses. Horizontal sets of elements are called rows and vertical sets are called columns.

2. Determinants: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

B. **Evaluation of determinants**

Picking the second row,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left\{ \begin{array}{l} -a_{21}\left(-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}\right) \\ +a_{22}\left(+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right) \\ +a_{23}\left(-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\right) \end{array} \right.$$

Any determinant's value is the sum of the products of any row or column's elements and their cofactors. It should be noted that in the second order, this is simply the difference of the products of the diagonals. Usually the third order is also evaluated by using diagonals with repeated rows or columns, or by the "horseshoe" method, but since this method is tried by students on higher orders, it may be inadvisable to bring it up until later as a manipulative trick.

C. Properties of determinants (of any order)

1. A determinant's sign is changed by interchanging any two rows or two columns.

2. The value of a determinant is zero if any two rows or two columns are identical.

3. The value of a determinant is unchanged if the elements of any row (or column) are multiplied by a constant and added to another row (or column) correspondingly

D. Solution of simultaneous linear equations using determinants

Given the set of equations in n unknowns, we may multiply each by the cofactor of the cofactor of the coefficient of x_1 and add them, the totals being here indicated

$$a_{11}x_1a_{12}x_2+\cdots+a_{1n}x_n = b_1$$

$$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n = b_n$$

$$a_{11}A_{11}x_1+a_{12}A_{11}x_2+\ldots+a_{1n}A_{11}x_n = b_1A_{11}$$

$$a_{22}A_{22}x_1+a_{22}A_{22}x_2+\ldots+a_{2n}A_{21}x_n = b_2A_{21}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\underline{a_{n1}A_{n1}x_1+a_{n2}A_{n1}x_2+\ldots+a_{nn}A_{n1}x_n = b_nA_n1}$$

$$(a_{11}A_{11}+a_{21}A_{21}+\ldots+a_{n1}A_{n1})x_1$$

$$+(a_{12}A_{11}+a_{22}A_{21}+\ldots+a_{2n}A_{21})x_2$$

$$+(\ldots+(a_{n1}A_{n1}+a_{n2}A_{n1}+\ldots+a_{nn}A_{n1})x_n =$$

$$(b_1A_{11}+b_2A_{21}+\ldots+b_nA_{n1})$$

Now we may consider the coefficients of the x_i unknowns. The coefficient of x_1 is seen to be the determinant of the coefficients of our system, developed by the first column. The coefficients of all the other x_i are seen to be this same quantity, but with the elements of the first column replaced by the elements of the second column. This is then seen as equivalent to a determinant in which the second column would appear twice, which is then equal to zero. Similarly, all the other coefficients of the x_i are seen to vanish.

The right hand side of the equation is seen to be this same determinant, but with the column of coefficients of x_i (i.e., the first column) replaced by the b_i constants.

Discarding the other variables (with zero coefficients), then, and dividing both sides by the coefficient of x_1, x_2 is seen to equal the determinant but with the column of coefficients of x_1 replaced by the column of constants. This is known as Cramer's rule and is the main motivation for the study of determinants at this level.

Similar results are seen to hold for any of the x_i by first multiplying each equation by the cofactor of the coefficient of that x_i.

XII. Permutations and Combinations - Time: 10 days

A. Fundamental principles of counting

1. "If one act can be performed in any one of m different ways, and after it has been done a second act can be done in n different ways, then the number of ways of performing the two acts in succession is mn."

   or

   "If a finite set A contains r elements and a finite set B contains s elements, then there are rs different ordered pairs (a,b) where $a \in A$ and $b \in B$ (that is, A x B contains rs elements)." This set of ordered pairs is called a Cartesian product, denoted by A x B.

2. If a finite set A contains r elements, a finite set B contains s elements, and their intersection $(A \cap B)$ contains t elements, then the union of A and B $(A \cup B)$ contains r + s - t elements.

   Example of Rule 1: A building has 3 entrances and 2 exits. How many different routes can a man take in going into the building and coming out of it? Answer: $3 \cdot 2 = 6$.

   The student should think: "For each of the 3 ways of entering, there are two ways of leaving; therefore, $3 \cdot 2$ or 6 ways altogether, each one different in in some manner from all the others.

Example of Rule 2:

How many odd integers less than 1,000 can be written, using the digits 1, 4, 7?

one-digit integers: 2
two-digit integers: $3 \cdot 2 = 6$
three-digit integers: $3 \cdot 3 \cdot 2 = 18$

$2 + 6 + 18 = 26$, answer

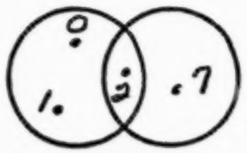
For the two-digit integers, the first integer can be a 1, 4, or 7; therefore, there are 3 choices. The second (last) integer must be a 1 or a 7; therefore, 2 choices. $3 \cdot 2 = 6$ ways of writing the two-digit integers.

For the three-digit ones, any one of the 3 can be selected for the first integer, and the same for the second, but the third or last integer may be only a 1 or a 7, 2 choices.

$3 \cdot 3 \cdot 2 = 18$ ways of writing the 3-digit integers.

$2 + 6 + 18 = 26$ total ways

Example 2:



$\{2\} = \{0,1,2\} \cap \{2,7\}$

Rule 2 is used here: $a \cap B = \emptyset$, $B \cap C = \emptyset$, $A \cap C = \emptyset$, or there are no intersections to subtract from the total, 26.

B. Permutations

1. Finding the number of permutations of n things taken n at a time: $n^{P}n$ or $n!$

Example: In how many ways can 4 books, A, B, C, and D, be arranged on a shelf?

Answer, $4 \cdot 3 \cdot 2 . 1 = 24$ ways.

There are 4 choices for the first book, 3 for the second, etc. or $4!$ $n^{P}n = n!$

2. Finding the number of permutations of n things taken r at a time: ${}_n P_n$ or $n(n-1)(n-2) \ldots \ldots [n-(r-1)]$

Example: In how many ways can 4 books be arranged on a shelf, 2 at a time?

There are 4 choices for the first, 3 for the second, 4 · 3 ways. Rather than memorizing the formula ${}_n P_r = n(n-1)(n-2) \ldots\ldots n-(r-1)$, the student should think of 2 spaces to be filled:

$$\boxed{4ch}\ \boxed{3ch} = 12.$$

Example: Find ${}_{10}P_6$: "10 things taken 6 at a time." There are 6 spaces, then:

$$\boxed{10ch}\quad \boxed{9ch}\quad \boxed{8ch}\quad \boxed{7ch}\quad \boxed{6ch}\quad \boxed{5ch}$$

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 90 \cdot 56 \cdot 30 = \underline{151200}$$

${}_n P_n$ for circular permutations = $(n-1)!$ instead of $n!$ come from the fact that there is no order of beginning. nothing is in first place. If 5 people are seated around a table, the student should, in his thinking, start with any one person, wherever he is sitting.

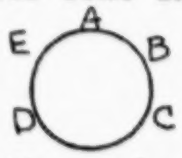
3. Finding the number of circular permutations: ${}_n P_n$ (circular) = $(n-1)!$

There will be only 4 choices for, say, the person to his left, 3 choices for the next one, etc., or 4! ways of arranging the people.

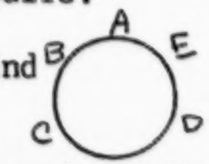
4. Circular permutations on a ring: ${}_n P_n$ (circular ring) $\frac{(n-1)}{2}!$

If, instead of seats at a table, we think of 5 keys on a ring, the ring can be flipped over so that the arrangements come in pairs.

For example, take the arrangements



and



These are different if they represent people around a table, but as keys on a ring that can be flipped over, they are the same. In other words, circular permutations come in pairs, each pair being the same if flipping is possible, or $n^{P}n$ of a circular ring = $\frac{(n-1)}{2}$! It is suggested that a physical model be made of a ring of keys.

5. Finding the number of permutations of things not all different: $\frac{n!}{r!\ s!\ \ldots.}$

The letters of the word PEAR have 4! permutations, but the letters of the word PEER have only 3! The reasoning is somewhat the same as for the circular ring; the E's look alike, but if they were different, PE_1E_2R and PE_2E_1R would be counted separately.

Thus the permutations of PE_1E_2R come in pairs, and we divide by 2. If there were 3 e's, there would be six permutations for E_1,E_2, and E_3, and the list would be six times as great as it should be. Therefore, we would divide by 6, or 3! If there are additional sets of "alikes" we would divide by the appropriate factorials of each.

For example, in the word Mississippi there are 4 s's, 4 i's and 2 p's, 11 letters in all. The number of different-appearing permutations is $\frac{11!}{4!\ 4!\ 2!}$

G. Combinations

1. Finding the number of combinations of n elements taken r at a time: $n^{C}r = \frac{n^{P}r}{r!}$

With permutations, order is considered.

With combinations, order is not considered; a combination is simply a set of things. The six permutations ABC, ACB, BCA, BAC, CAB, CBA, are all the same combination. (The student's locker does not have a combination lock. It is a permutation lock.)

If $n^{C}r$ = the number of combinations of n elements taken r at a time and if $n^{P}r$ = the number of permutations of n elements taken r at a time, since from each combination of r elements there can be formed r! permutations, then $(r!)n^{C}r$, or $n^{C}r = \frac{n^{P}r}{r!}$

2. Finding the number of combinations when r is large

Whenever r elements are selected from a set of n elements, (n-r) elements are left behind. Therefore, the number of sets of r elements equals the number of sets of (n-r) elements and ${}_{n}C_{r} = {}_{n}C_{n-r}$.

When finding ${}_{20}C_{15}$, for example, it is much simpler to find ${}_{20}C_{20-15}$ or ${}_{20}C_{5}$

$$
\begin{aligned}
{}_{20}C_{15} &= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot \cancel{15}\cdot \cancel{14}\cdot \cancel{13}\cdot \cancel{12}\cdot \cancel{11}\cdot \cancel{10}\cdot \cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}}{\cancel{15}\cdot \cancel{14}\cdot \cancel{13}\cdot \cancel{12}\cdot \cancel{11}\cdot \cancel{10}\cdot \cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}\cdot 5\cdot 4\cdot 3\cdot 2} \\
&= \frac{\cancel{20}\cdot 19\cdot \overset{3}{\cancel{18}}\cdot 17\cdot 16}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} = 15504 \\
\text{While } {}_{20}C_{5} &= \frac{\cancel{20}\cdot 19\cdot \overset{3}{\cancel{18}}\cdot 17\cdot 16}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} = 15504
\end{aligned}
$$

3. Finding the total number of combinations

When combinations are formed from two or more groups, the product of the combinations gives the total number, according to fundamental principle number 1. Sometimes both permutations and combinations are involved in the same problem.

Example: Suppose there are 10 boys and 8 girls in a mathematics club. In how many ways can a committee of 3 boys and 2 girls be seated around a circular table?

There are ${}_{10}C_{3}$ or $\frac{10\cdot 9\cdot 8}{3\cdot 2}$ or 120 ways of choosing 3 boys from a set of 10 boys. There are ${}_{8}C_{2}$ or $\frac{8\cdot 7}{2}$ or 28 ways of choosing 2 girls from a set of 8 girls. There are 120·28 ways of choosing the committee, and for every one of these ways, there are 4! ways of seating the 5 members about a circular table. Ans: 120·28·24 ways.

II. Polynomial Functions, Higher Degree Equations - Time: 15 days

A. Theorems

1. Fundamental theorem of algebra

The fundamental theorem of algebra: "Every polynomial equation of degree $n \geq 1$ with complex coefficients has at least one root." Although the fundamental theorem covers the entire field of complex numbers, in this course only those polynomials with real coefficients will be considered.

Corollary: If the degree of the equation is n, there are n roots.

2. Remainder theorem

If a polynomial in x, P(x), is divided by x-c, the remainder is P(c).

$$(x^2 - 5x + 13) + (x-2) = x - 3 + \frac{7}{x-2}$$, where 7 is the remainder.

But $P(2) = 4 - 10 + 13 = 7$

3. The factor theorem

The factor theorem follows from the remainder theorem: "An algebraic polynomial in x is exactly divisible by (x-2) if it reduces to zero when $p(a) = 0$

4. Theorem on rational roots of a polynomial equation

If p and q denote integers ($q \neq 0$) such that $\frac{p}{q}$ is in lowest terms and represents a rational root of a polynomial equation in simple form with integral coefficients, then p must be an integral factor of the constant term and q an integral factor of the leading coefficient of the equation.

Example: In the equation $3x^2 + 5x - 4 = 0$, the possible roots are :

$$P \in \{\pm 1, \pm 2, \pm 4\}$$

$$Q \in \{\pm 1, \pm 3\}$$

$$\frac{P}{Q} \in \{\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3\}$$

5. Theorem on conjugate complex roots

If a polynomial with real coefficients has a + bi as a root (a and b real, $b \neq 0$) then a - bi is also a root.

6. Descartes' rule of signs

The number of positive real roots of P(x) = 0 where P(x) is a polynomial with real coefficients is equal to the number of variations in sign cccurring in P(x) or else is fewer than this number by a positive even integer. The number of negative real roots of the P(x) is equal to the number of variations in sign occurring in P(-x), or else is fewer than this number by a positive even integer.

7. The location principle

If P is a function whose values are given by a polynomial with real coefficients, and if a and b are real numbers such that one of the values P(a) and P(b) is positive and the other is negative, then P has an odd number of zeros between a and b.

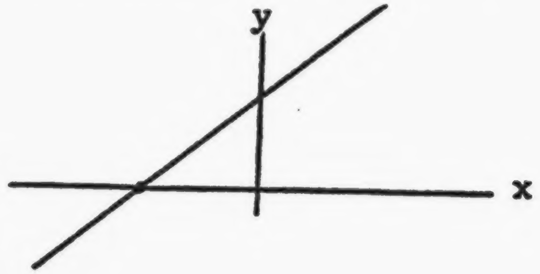
8. Theorem on the upper and lower limits for roots

When P(x) is divided by x-a by synthetic division, a being positive or zero, if all the numbers in the third line of the synthetic division are zero, or positive, then a is an upper limit of the positive roots of P(x) = 0 (For lower limit, use the P(-x) etc.

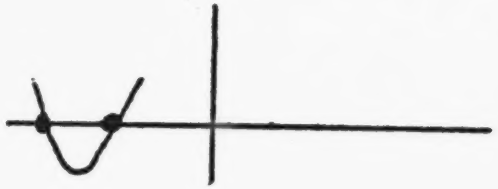
B. The number of roots of an equation of nth degree

On the graph of a equation $f(x) = 0$, graphed as $f(x) = y$, the solutions will show as the point or points at which the curve crosses the x-axis, since these are the points at which the value of the function equals zero (given), or $y = 0$.

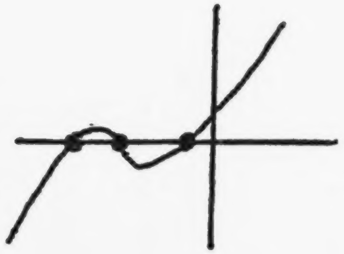
1. First-degree equations: one solution



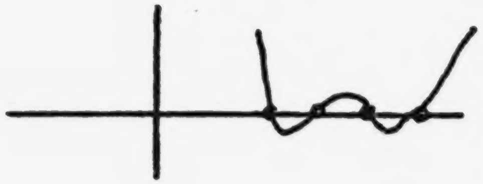
2. Second-degree equations: two solutions (sometimes imaginary, sometimes double)



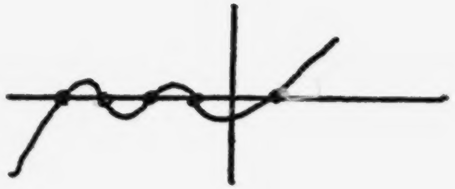
3. Third-degree equations: three solutions (always at least one real solution; sometimes 2 imaginary ones)



4. Fourth-degree equations: four solutions (sometimes 2 or 4 imaginary ones.)



5. Fifth-degree equations: five solutions (Always at least one real solution; sometimes 2 or 4 imaginary ones)



C. Synthetic division (or synthetic substitution)

Dividing $(x^2 - 5x + 8)$ by $(x - 2)$ by long division we have:

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & x - 3 + \frac{2}{x-2} \\
\cline{2-2}
x - 2 & x^2 - 5x + 8 \\
 & \underline{x^2 - 2x} \\
 & \quad - 3x + 8 \\
 & \quad \underline{- 3x + 6} \\
 & \qquad\qquad 2
\end{array}
$$

Shortening the process by synthetic division we have:

$$\begin{array}{r|rrl} 2 & 1 & -5 & +8 \\ & & 2 & -6 \\ \hline & 1 & -3 & \textcircled{+2} \text{ remainder} \\ & \downarrow & \downarrow & \\ & x & -3 & +\frac{2}{x-2}\text{: answer} \end{array}$$

The remainder, 2, equals the f(2), or the value of the function $x^2 - 5x + 8$ when 2 is substituted for x.

1. Uses of synthetic division

Synthetic division is useful to shorten the processes of

(a) Division of a polynomial by a monomial
(b) Finding the factors of a given polynomial (searching for a divisor that will have a zero remainder)
(c) Finding the value of a polynomial when a given number is substituted for the variable.

2. Finding one root of an equation by synthetic division

Example: Solve the equation $3x^3 + x^2 - 15x - 5 = 0$; $p = -5$, $q = 3$.
Possible values for roots are the factors of p over the factors of q:
$1, -1, 5, -5, \frac{1}{3}, -\frac{1}{3}, \frac{5}{3}, -\frac{5}{3}$

$$\begin{array}{r|rrrr} 1 & 3 & 1 & -15 & -5 \\ \hline & & 3 & 4 & -11 \\ \hline & 3 & 4 & -11 & \cancel{-16} \end{array} \qquad \begin{array}{r|rrrr} -1 & 3 & 1 & -15 & -5 \\ \hline & & -3 & +2 & +13 \\ \hline & 3 & -2 & -13 & \cancel{+8} \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 1 & -15 & -5 \\ \hline & & 1 & \frac{2}{3} & \\ \hline & 3 & 2 & -14\frac{1}{3} & X \end{array} \qquad \begin{array}{r|rrrr} -\frac{1}{3} & 3 & 1 & -15 & -5 \\ \hline & & -1 & 0 & +5 \\ \hline & 3 & 0 & -15 & \textcircled{0} \end{array}$$

$$\begin{array}{r|rrrr} 5 & 3 & 1 & -15 & -5 \\ & & 15 & 80 & 325 \\ \hline & 3 & 16 & 65 & X \end{array} \qquad \begin{array}{r|rrrr} -5 & 3 & 1 & -15 & -5 \\ & & -15 & 70 & -275 \\ \hline & 3 & -14 & 55 & X \end{array}$$

Since division by $(x + 1/3)$ produces a zero remainder, $x + 1/3)$ is a factor, and the depressed, or reduced equation is $3x^2 + 0x - 15$.

The factors are $(x + 1/3)(3)(x^2-5)$ or $(3x + 1)(x^2 - 5) = 0$.
The solutions are therefore $x = -1/3$, $\sqrt{5}$, and $-\sqrt{5}$.

The student should be reminded to look for n roots, including multiple roots, to an nth degree equation; here, for 3 roots. All 3 solutions must, of course, be checked in the given equation.

D. Using Descartes' Rule of Signs as an aid in solving equations. See page 112

Example: Determine the possible number of real positive and negative roots of the equation $x^5 + 2x^4 - 3x^3 + x - 4 = 0$.

Since there are 3 changes of sign, there can be no more than 3 positive real roots.

Substituting $(-x)$ for x in the equation $p(-x) = -x^5 + 2x^4 - 3x^3 - x - 4 = 0$.

Since there are 2 changes of sign in $p(-x)$, there can be no more than 2 negative real roots.

E. Using the theorem on upper and lower limits to locate the roots of an equation (see page 112)

Example: Find the upper and lower limits of the roots of the equation, $x^5 + 2x^4 - 3x^3 + x - 4 = 0$.

$$\begin{array}{r|rrrrrr} 1 & 1 & 2 & -3 & 0 & 1 & -4 \\ & & 1 & 3 & 0 & 0 & 1 \\ \hline & 1 & 3 & 0 & 0 & 1 & -3 \end{array}$$

$$\begin{array}{r|rrrrrr} 2 & 1 & 2 & -3 & 0 & 1 & -4 \\ & & 2 & 8 & 10 & 20 & 42 \\ \hline & 1 & 4 & 5 & 10 & 21 & 38 \end{array}$$

Since the coefficients of the third line are all positive, 2 is an upper limit of the positive roots.

To investigate the lower limit of the roots, p (-x) must be found:

$$p(-x) = -x^5 + 2x^4 + 3x^3 - x - 4 = 0 \quad \text{or}$$
$$x^5 - 2x^4 - 3x^3 + x + 4 = 0$$

$$\begin{array}{r|rrrrrr} & 1 & -2 & -3 & 0 & +1 & +4 \\ 1 & & 1 & -1 & -4 & -4 & -3 \\ \hline & 1 & -1 & -4 & -4 & -3 & +1 \end{array}$$

$$\begin{array}{r|rrrrrr} & 1 & -2 & -3 & 0 & +1 & +4 \\ 2 & & 2 & 0 & -6 & 12 & 22 \\ \hline & 1 & 0 & -3 & -6 & 11 & -18 \end{array}$$

$$\begin{array}{r|rrrrrr} & 1 & -2 & -3 & 0 & +1 & +4 \\ 3 & & 3 & 3 & 0 & 0 & 3 \\ \hline & 1 & 1 & 0 & 0 & 1 & 7 \end{array}$$

Since the coefficients of the third line are all positive, -3 is a lower limit of the negative roots.

Answer: the roots lie between -3 and +2.

F. Using the location principle as an aid in finding the roots of an equation. (See pg. 116)

Example: In the interval $-4 < x < 4$, locate consecutive integers between which are found zeros of the equation $y = 2x^3 + 7x^3 + 7x + 6$.

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ -4 & & -8 & +4 & -44 \\ \hline & 2 & -1 & +11 & \boxed{-38} \end{array}$$

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ -3 & & -6 & -3 & -12 \\ \hline & 2 & +1 & +4 & \boxed{-6} \end{array}$$

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ 1 & & 2 & 9 & +16 \\ \hline & 2 & +9 & +16 & \boxed{+22} \end{array}$$

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ -2 & & -4 & -6 & -2 \\ \hline & 2 & +3 & +1 & \boxed{+4} \end{array}$$

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ -1 & & -2 & -5 & -2 \\ \hline & 2 & +5 & +2 & \boxed{+4} \end{array}$$

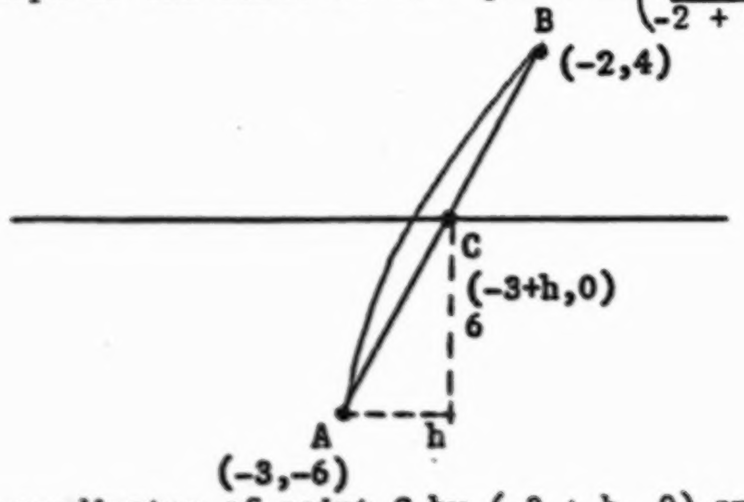
At this point it is easy to see that all positive divisors will give positive remainder.

x = -4 -3 -2 -1 0 1 etc.
y = -38 -6 +4 +4 +6 +22

There are one (or three) roots between -2 and -3.

G. Using linear interpolation to approximate the roots of an equation

In the preceding equation, a portion of the graph in the interval of the root shows that the approximate slope of the curve at this point is $\left(\frac{4 + 6}{-2 + 3}\right)$ or 10.



Representing the coordinates of point C by (-3 + h, 0) and considering the slope of AC to be apprixinately equal to the slope of AB, --

$$\text{slope } AC = \text{slope } AB$$

$$\frac{0 + 6}{h} = \frac{10}{1}$$

$$h = .6$$
$$-3 + h = -2.4$$

trying -2.4, -2.5, etc. by synthetic division:

$$\begin{array}{r|rrrr} & 2 & +7 & +7 & +6 \\ -2.4 & & -4.8 & -5.28 & -4.128 \\ \hline & 2 & +2.2 & +1.72 & \boxed{+1.872} \\ & & & & \text{remainder} \end{array}$$

$$\begin{array}{r|rrrl} -2.5 & 2 & +7 & +7 & +6 \\ & & -5 & -5 & -5 \\ \hline & 2 & +2 & +2 & \boxed{+1} \text{ remainder} \end{array}$$

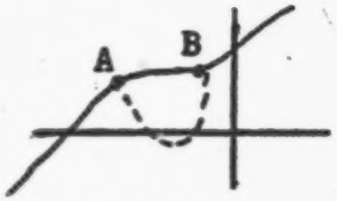
$$\begin{array}{r|rrrl} -2.6 & 2 & +7 & +7 & +6 \\ & & -5.2 & -4.68 & -6.032 \\ \hline & 2 & +1.8 & +2.32 & \boxed{-.032} \quad \text{remainder} \end{array}$$

Since the value, -2.6 gives the remainder closest to zero, the solution correct to tenths is -2.6. This process can be repeated as many times as desired, for a higher degree of accuracy. This procedure is somewhat tedious, unless a Monroe or other calculator is used. There are methods in calculus and other college courses that are preferable.

H. Summary of method of procedure in solving equations of third degree or higher

1. Arrange the polynomial in descending powers of the variable.
2. Use Descartes' rule of signs to determine the possible numbers of positive and negative roots.
3. Find the upper and lower limits of the roots.
4. Determine the possible integral or fractional values of the roots by inspection of the constant term and the coefficient of the first term.
5. Use synthetic division to investigate these possible roots. When a root is discovered, repeat the process on the depressed equation until a quadratic equation remains; solve this quadratic equation by factoring, completing the square, or formula.

6. If there were no integral or fractional roots, use the location principle, with synthetic division, to find the interval or intervals within which the roots lie. Concurrently, a graph of all points found by synthetic division will be helpful to the student in completing the solution. For example, in this equation of the third degree, a graph will disclose the possibility of roots lying between A and B:



7. Use linear interpolations to carry out the solutions to the degree of accuracy required.

8. Checking the solutions should be unnecessary; synthetic division is an automatic check if correctly performed.

XIV. Logarithms - Trigonometry

A. Logarithms

Students have studied some trigonometry in geometry, but a review of logarithms is usually necessary at the beginning of this unit. Explain a logarithm as an exponent. The equation $10^x = n$ can be written to show the relation between a number n and its logarithm x. $\log_{10}x = n$ expresses the same relationship and is read "the log of x to the base 10 is n."

The laws of exponents will apply to logarithms. $(x^2)(x^3) = x^5$. In multiplication, the exponents are added. If the same two numbers A and B are to be multiplied, find the logarithm of each member and add. If the same two numbers A and B are to be divided, find the logarithm of each number, subtract and find the antilog:

If a number, x, is to be raised to the n^{th} power, find the log of x, multiply by n, and find the antilog.

If the n^{th} root of a number, x, is to be found, find the log of x, divide by n, and find the antilog.

1. Two parts of a logarithm characteristic, mantissa

   One method to find the characteristic is to convert the number to scientific notation:
   $1.56 \rightarrow 1.56 \times 10^0$; $27.6 \rightarrow 2.76 \times 10^1$; $3250 \rightarrow 3.25 \times 10^3$; $.0268 \rightarrow 2.68 \times 10^{-2}$

   The exponent of 10 is the characteristic of the log of the number. The mantissa can be found in the math tables book, and is usually expressed as a five place decimal.

   $\log 1.56 = 0.19312$

2. Using logs to solve equations

   Explain that an equation where the unknown quantity appears in an exponent is called an exponential equation.

   Ex: $16^x = 74$ Find the solution set.

$$x \log 16 = \log 74$$

$$x = \frac{\log 74}{\log 16} = \frac{1.86923}{1.20412}$$

$$\begin{array}{r} \log 1.869 = 0.27161 \\ -\log 1.204 = 0.08063 \\ \hline \log x = 0.19098 \\ x = 1.552 \end{array}$$

A logarithmic equation is one in which there appears the logarithm of some expression involving the unknown quantity.

Ex: $\log x + \log \frac{2x}{5} = 6$

$$\log x + (\log 2 + \log x \ - \log 5) = 6$$

$$2 \log x = 6 + \log 5 - \log 2$$

$$2 \log x = 6 + 0.69897 - 0.30103$$

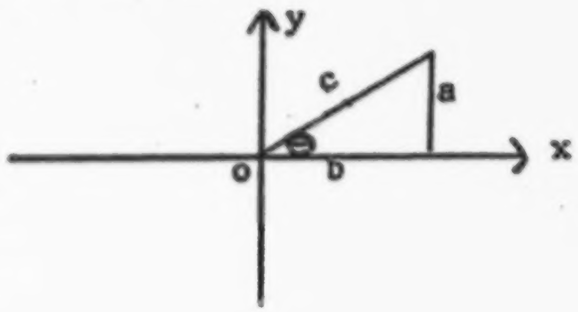
$$2 \log x = 6.39796$$

$$\log x = 3.19898 \qquad x = 1581$$

The students should work a sufficient number of problems of these two types to become efficient.

B. The fundamental identities

A brief review here of the trigonometric functions will probably be necessary.



$\text{Sin } \theta = \frac{a}{c} \qquad \text{Cos } \theta = \frac{b}{c} \qquad \text{Tan } \theta = \frac{a}{b}$

Some textbooks insist upon the use of abscissa (distance from the y-axis, ordinate (distance from the x-axis) and radius vector (the radius of the unit circle) rather than the use of a, b, c, but facility in working with the functions is speeded if a,b,c, (or x, y, r) are used.

An equation which involves at least one variable is an equation, and an equation that is true for all values of the variable for which both of its members are defined is called an identity. The trigonometric identities are identified under reciprocal relations.

$$\text{Sin } \theta = \frac{1}{\text{Csc}} \qquad \text{Cos } \theta = \frac{1}{\text{Sec}} \qquad \text{Tan } - = \frac{1}{\text{Cot}} \; \theta$$

Quotient relations

$$\text{Tan } \theta = \frac{\text{Sin } \theta}{\text{Cos } \theta} \qquad \text{Cot } \theta = \frac{\text{Cos } \theta}{\text{Sin } \theta}$$

Pythagorean relations

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

All three of these relations should be proved by the students.

Example: $\cot^2\theta + 1 = \csc^2\theta$

$$\frac{b^2}{a^2} + 1 = \frac{c^2}{a^2} \qquad \frac{b^2 + a^2}{a} = \frac{c^2}{a}$$

$a^2 + b^2 = c^2$ (Rule of Pythagoras)

In terms of these identities, problems of this type may be given. Ex: Express the following in terms of a single trigonometric function.

(1) $1 + \tan^2\theta \rightarrow 1 + \frac{\sin^2\theta}{\cos^2\theta} \rightarrow$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \rightarrow \frac{1}{\cos^2\theta} \rightarrow \sec^2\theta$$

(2) $\frac{\sqrt{\sec^2\theta - 1}}{\sqrt{\csc^2\theta - 1}} \quad \frac{\sqrt{\tan^2\theta}}{\sqrt{\cot^2\theta}} \rightarrow \frac{\tan\theta}{\cot\theta} \rightarrow$

$$\frac{\tan\theta}{\frac{1}{\tan\theta}} \rightarrow \tan\theta \cdot \tan\theta \rightarrow \tan^2\theta$$

The student should note that we do not "say" in our expression

$$\sqrt{\sec^2\theta - 1} \rightarrow 1 + \tan^2\theta = \sec^2\theta \rightarrow$$

$$\sec^2\theta - 1 = \tan^2\theta, \therefore \sqrt{\sec^2\theta - 1} \rightarrow \sqrt{\tan^2\theta}$$

We simply express $\sqrt{\sec^2\theta - \theta\, 1}$ as $\sqrt{\tan^2}$

Many problems of this type should be assigned to the student so that he becomes adept at working them.

1. Expressing one function in terms of another function

Students should also have practice with problems of the type:

Express $\sin\theta$ and $\tan\theta$ in terms of $\cos\theta$

Ex.: $\sin\theta$ in terms of $\cos\theta$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1 - \cos^2\theta$$
$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

Ex. $\tan \theta$ in terms of $\cos \theta$

$$1 + \tan^2 \theta = \sec^2 \theta$$
$$\tan^2 \theta = \sec^2 \theta - 1$$
$$\tan^2 \theta = \frac{1}{\cos^2 \theta} - 1$$
$$\tan^2 \theta = \frac{1 - \cos^2 \theta}{\cos^2 \theta}$$
$$\tan \theta = \pm\sqrt{\frac{1 - \cos^2 \theta}{\cos^2 \theta}}$$
$$\tan \theta = \pm\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$$

2. Proving identities

The students should have a sufficient number of examples to prove so that they become "friendly" with them. Proving identities is a very difficult part of this unit.

Problems of the type: Prove the identity.

(1) $\text{Sin } \theta \text{ Cot } \theta = \text{Cos } \theta$

Choose either member to be transformed - say the left -

$$\text{Sin } \theta \text{ Cot } \theta \rightarrow \text{Sin } \theta \cdot \frac{\text{Cos } \theta}{\text{Sin } \theta} \rightarrow \text{Cos } \theta$$

$$\therefore \text{Cos } \theta = \text{Cos } \theta$$

(2) $\dfrac{\text{Sin } \theta}{1 - \text{Cos } \theta} = \dfrac{1 + \text{Cos } \theta}{\text{Sin } \theta}$

transform the right side
multiply the numerator and denominator by Sin θ which is the numerator of the left side

$$\frac{1 + \text{Cos } \theta}{\text{Sin } \theta} \cdot \frac{\text{Sin } \theta}{\text{Sin } \theta}$$

$$\frac{\text{Sin } \theta \; (1 + \text{Cos } \theta)}{\text{Sin}^2 \theta} \rightarrow \frac{\text{Sin } \theta \; (1 + \text{Cos } \theta)}{1 - \text{Cos}^2 \theta}$$

$$\frac{\text{Sin } \theta \; \cancel{(1 + \text{Cos } \theta)}}{(1 - \text{Cos } \theta)\cancel{(1 + \text{Cos } \theta)}} \rightarrow \frac{\text{Sin } \theta}{1 - \text{Cos } \theta}$$

$$\frac{\text{Sin } \theta}{1 - \text{Cos } \theta} = \frac{\text{Sin } \theta}{1 - \text{Cos } \theta}$$

If time permits, this unit could include a review of the numerical trigonometry covered in plane geometry.

A. Computation with logarithms

   1. Multiplying, dividing, raising to a power, finding the root

B. Logarithmic solutions of right and oblique triangles

   1. Law of sines
   2. Law of cosines
   3. Law of tangents
   4. Tangent half-angle law

With the use of instruments, and the advent of the computer, more and more textbooks are omitting the solutions of oblique triangles. Despite this, it is an excellent review of arithmetic for high school students.

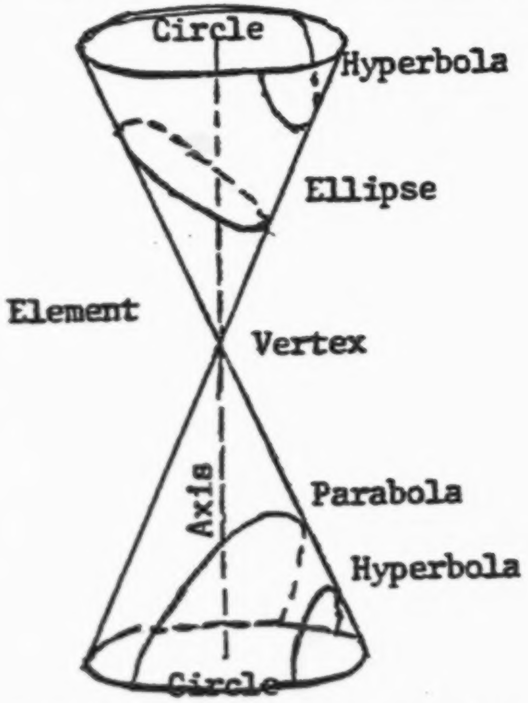
XV. Optional Quadratic Relations and Systems

A. Conic Sections

The student will be able to visualize the shapes of the various curves if he grasps the manner in which they are formed by intersection.

He should also know that there are "degenerate" curves formed by having the plane intersect the conical surface in special ways:

1. A point, formed by a plane passing through the vertex only (degenerate ellipse).
2. A line (an element of the conical surface) formed by a plane through the vertex tangent to the conical surface (degenerate parabola).
3. Two intersecting lines, formed by a plane through the vertex intersecting the conical surface in two elements (degenerate hyperbola).



Conic sections: the curves in which a plane intersects a conical surface whose right cross section is a circle.

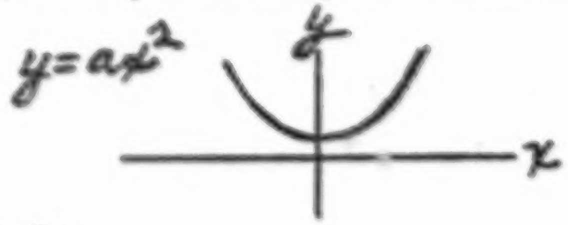
a. Ellipse: plane cutting all elements (straight line passing through vertex of surface and lying wholly in the surface.
b. Circle: special case of ellipse; plane is perpendicular to the axis.

c. Parabola: plane parallel to one element

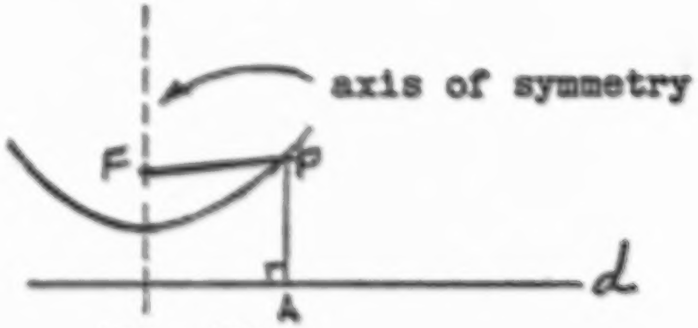
The student is already familiar with the parabola as the curve of the general quadratic, $y = ax^2 + bx + c$.

d. Hyperbola: plane parallel to the axis

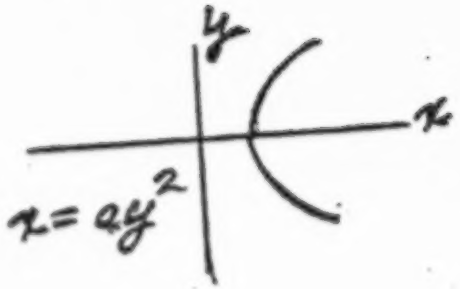
A parabola having its axis of symmetry parallel to the y-axis will have as its equation some form of the equation, $y = ax^2 + bx + c$. If $b = 0$ and $c = 0$ the equation becomes $y = ax^2$, and the y-axis will be the axis of symmetry.



B. The parabola



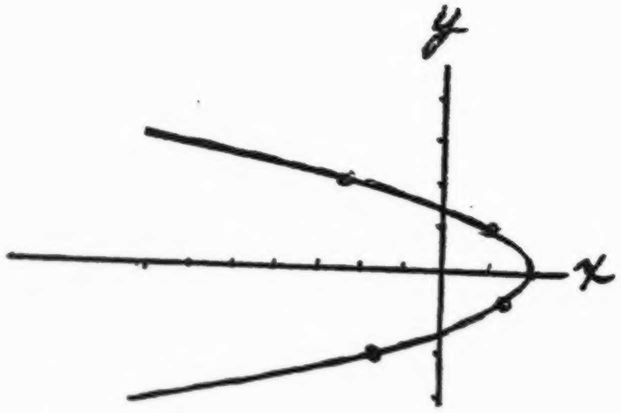
FP = PA



If the equation is of the form $x = ay^2 + by + c$, the curve will have its axis either parallel to or coinciding with the x-axis.

1. Definition of a parabola: a curve consisting of a set of points equidistant from a line (called the directrix) and a point (called the focus), or FP = PA
2. Equation of the parabola: $y = ax^2 + bx + c$, or $x = ay^2 + by + c$, where $a \neq 0$.
3. Graphing the parabola.

Example: Construct the graph of $x + y^2 = 2$



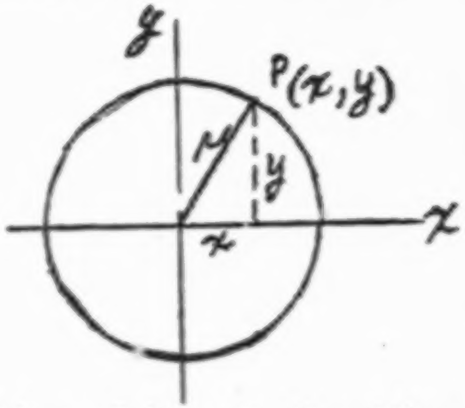
1. Solve the equation for x: $x = 2 - y^2$
2. Make a table of values:

x:	2	1	-2	-7	0
y:	0	± 1	± 2	± 3	± 2

3. Plot the points.

C. The circle

1. Definition: the set of points equidistant from a given point within called the center.



From the Pythagorean theorem,
$x^2 + y^2 = r^2$

2. Equation of the circle with center at the origin: $x^2 + y^2 = r^2$, where r is equal to the radius.

In this course we shall deal with only those circles having their center at the origin. It is a good exercise for the student to graph a few circles. Plotting the points as accurately as possible, he will have a good check on his calculations and method of plotting (if the figure actually turns out to be a circle!) The repetition of coordinates, as one moves around the circle, is difficult for some students.

3. Graphing the circle

Ex: Graph the equation $x^2 + y^2 = 25$

x	y
0	± 5
± 1	$\pm\sqrt{24} = \pm 5.8$
± 2	$\pm\sqrt{21} = \pm 4.4$
± 3	± 4
± 4	± 3
± 5	0
± 5.8	± 1
± 4.4	± 2

If the points, x = 0, 1, 2, 3, 4, 5 alone are plotted, there will be a gap in this region. To make the picture complete, these points should be added.

D. The Ellipse

1. Definition: a curve consisting of the set of points such that the sum of the distances from two given points (called the foci) is a given constant, or $PF_1 + PF_2 = K$.

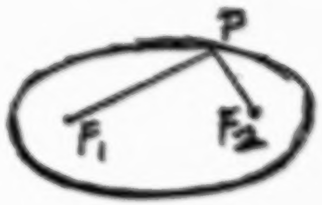
   The circle is actually a special case of the ellipse, in which the two foci are not separate but are one and the same point. However, in this course we shall call the figure an ellipse only if the foci are separate points, and the figure is consequently elongated either horizontally or vertically.

   The formula for the ellipse has the distinguishing characteristics:

   a. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$

   b. $a \neq b$.

If one and only one of the signs of a and b is negative, the curve is a hyperbola. If a and b have the same sign, and a = b, the curve is a special form of the ellipse, called the circle.



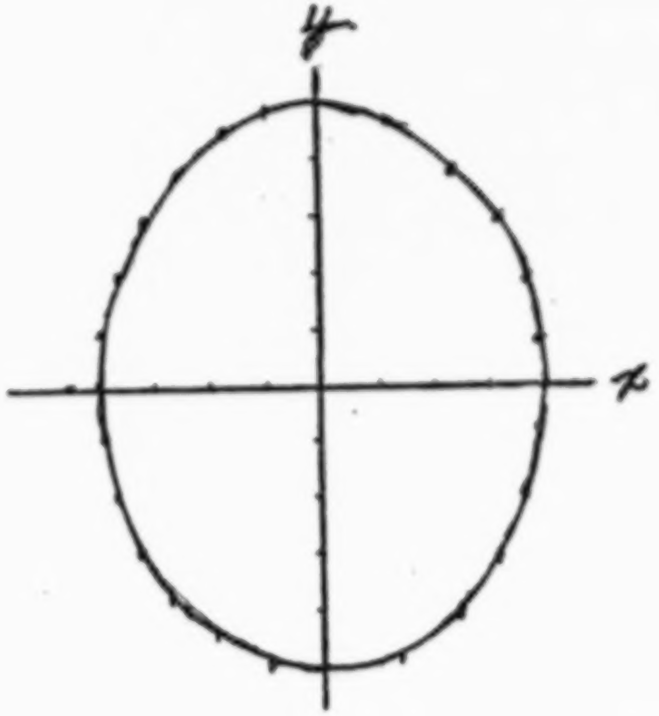
2. Equation of the ellipse with foci on one of the axes: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ex: graph the equation, $\frac{x^2}{16} + \frac{y^2}{25} = 1$

$25x^2 + 16y^2 = 400$

$x = \pm \frac{\sqrt{400 - 16y^2}}{5}$

x	y
±4	0
±3.9	±1
±3.7	±2
±2.4	±4
0	±5
±1	±0.97



3. Graphing the ellipse

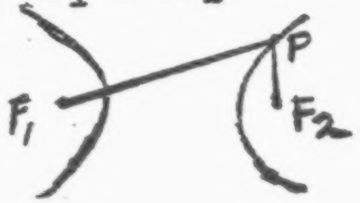
The x- and y- intercepts are useful[1] points, especially when making a sketch of an ellipse.

E. The Hyperbola

1. Definition: a curve consisting of the set of points such that for each point the absolute value of the difference of its distances from two given points, called the foci, is a given constant.

   The equation of the hyperbola is distinguished from that of the ellipse by the minus sign between the x^2 and y^2 terms. The equation of the hyperbola with center at the origin can be written $\frac{x^2}{a^2} - \frac{y^2}{b^2}$. As y approaches to infinity, so does x also; the curve is not limited in extent as is the ellipse. The curve approaches tangency with the asymptotes.

   $PF_1 - PF_2 = K$



2. Equation of the hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$

3. Graphing the hyperbola.

   Ex.: Graph the equation, $4x^2 - y^2 = 16$

   To get the equations of the asymptotes, replace the constant term by 0, or $4x^2 - y^2 = 0$ $(2x \pm y)(2x - y) = 0$, or $2x - y = 0$ and $2x + y = 0$

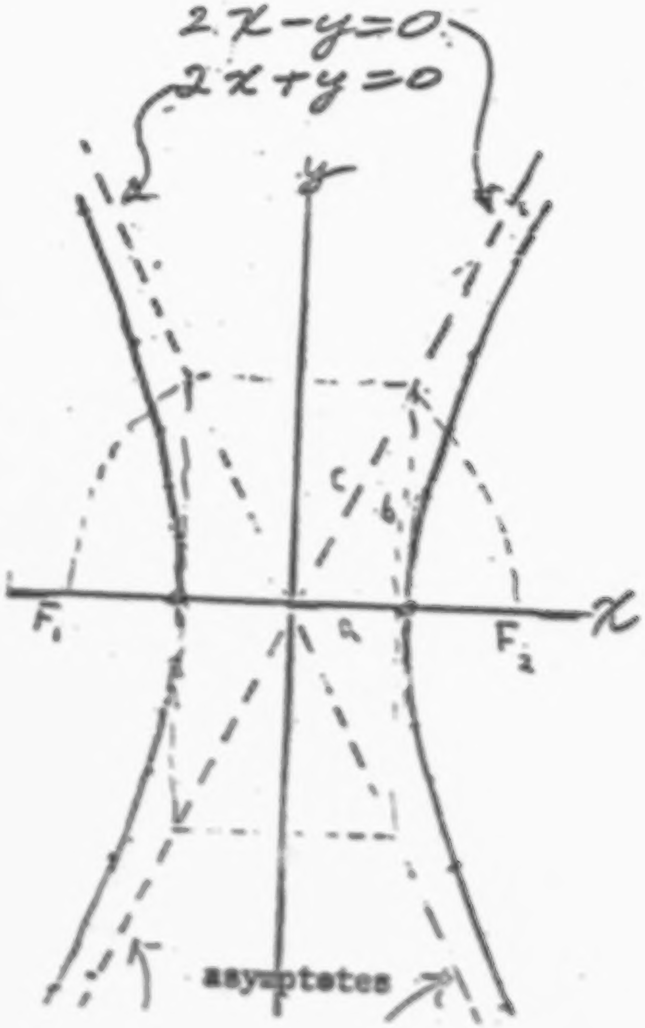
   $$4x^2 - y^2 = 16$$

   $$y = \pm\sqrt{4x^2 - 16}$$

   $$y = \pm 2\sqrt{x^2 - 4}$$

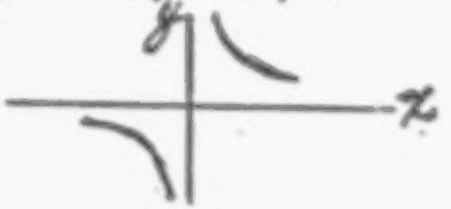
	x	y	
	0	-	
	1	-	
	±2	0	
	±3	$\pm 2\sqrt{5} = \pm 4.5$	
	±4	$\pm 2\sqrt{12} = \pm 6.9$	
n	$\pm 2\sqrt{5}$	±2	additional points
	±5	±3	

(For the additional points:
$x = \pm\sqrt{16 + y^2}$
needed for y = 2, 3)



4. The hyperbola of the form,

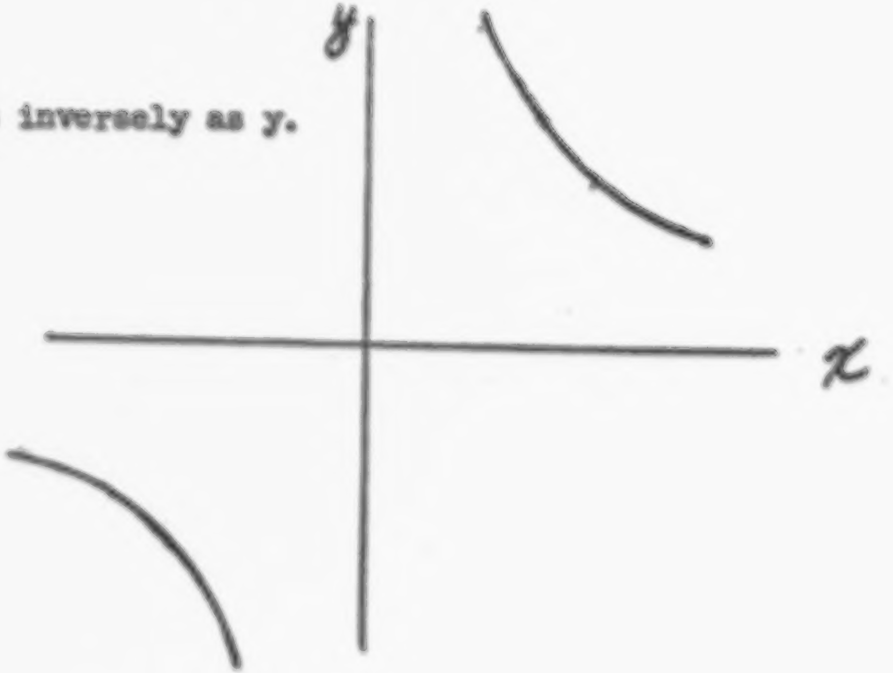
$$xy = k, \quad k \neq 0$$



K = the constant of proportionality. The x- and y- axes are asymptotes.

This is the curve of inverse variation. Here x varies inversely as y.
Ex: Graph the equation, $xy = 12$.

x	y
± 1	± 12
± 2	± 6
± 3	± 4
± 4	± 3
± 6	± 2
± 12	± 1



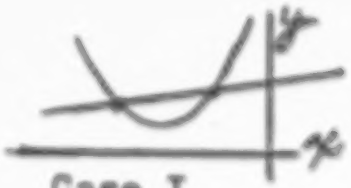
In the equation $xy = 12$, when x is positive, y is also positive; when x is negative, y is negative.

F. Quadratic systems

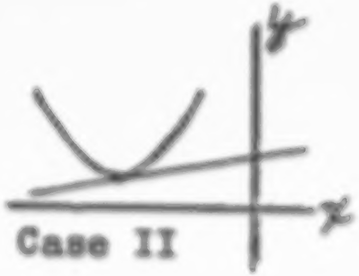
1. Systems consisting of one linear equation and one quadratic equation

   a. Number of solutions

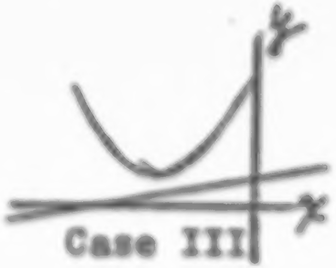
      The solutions of these systems are either 2, 1, or no points



Case I



Case II



Case III

Note that Case II has actually 2 solutions, both equal to each other. Case III has two imaginary solutions. In the case of the hyperbola, if the graph of the linear equation is a line parallel to an asymptote, there will be only one solution. (see example p. 143)

These systems can be solved graphically, but with only approximate solutions. The service performed by the graph is rather one of illustration or clarification.

b. Alegbraic solution of a system consisting of one linear and one quadratic equation, by substitution.

1. Solve the linear equation for x or y.
2. Substitute this value in the quadratic equation.
3. Solve the quadratic thus formed by factoring, completing the square, or formula.

Ex: $x^2 - y^2 = 16$ (1)

$x + y = 8$ (2)

or $x = 8 - y$

Solve for x and y.

Substituting (2) in (1),

$$(8 - y)^2 - y^2 = 16 \ (1)$$

$$64 - 16y + y^2 - y^2 = 16$$

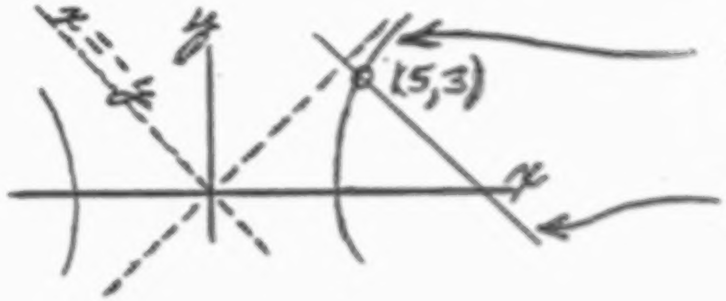
$$-16y = -48$$

$$y = 3$$

In (2), $x + 3 = 8$, $x = 5$ Solution $x = 5$

$y = 3$

A graph of the equations, or a rough sketch at least, is helpful in showing the number of solutions and their approximate value:



$x^2 - y^2 = 16$, a hyperbola.

One solution:

$x + y = 8$

$x = 5, y = 3$

Notice that if we had substituted $y = 3$ in equation (1) we would have obtained two solutions:

$$x^2 - y^2 = 16$$

$$x^2 - 9 = 16$$

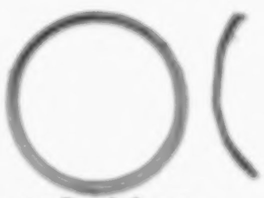
$$x = \pm 5, \text{ or } x = 5, y = 3 \text{ and } x = -5, y = 3$$

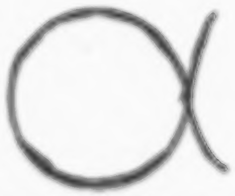
Nevertheless the graph indicates that the second solution is false. Checking the values $x = -5$, $y = 3$, in the original equation (2), $x + y = 8$, shows that this set is false.

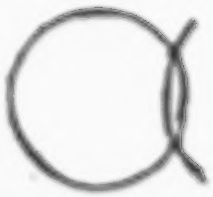
Although a linear equation solved with a quadratic equation will usually have two solutions (either real or imaginary), there will be only one when one equation represents a hyperbola and the other a line parallel to one asymptote of the hyperbola. In the above equation $x + y = 8$ is a line parallel to the asymptote, $x = -y$.

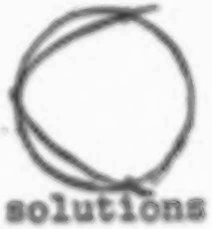
2. Systems of two quadratic equations

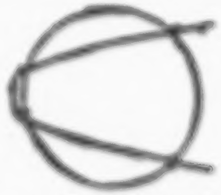
a. Number of real solutions: 0, 1, 2, 3, or 4


0 solution


1 solution


2 solutions


3 solutions


4 solutions

There are actually 4 solutions for each case, if the complex number solutions are included, and tangent points are counted as double solutions. The imaginary solutions are discovered through algebraic procedures.

b. Graphic method of solving systems of two quadratic equations

Students should have plenty of experience in graphing these systems. The best procedure is to ask for both the algebraic and the graphing methods of solution on the problems assigned. The two methods work together to present the complete solution.

c. Algebraic solution of systems of two quadratic equations

This is one section of the course in which the student needs to develop ingenuity in working out methods of procedure. Substitution and addition-subtraction are the methods commonly used. When one equation can be solved for one variable, in fairly simple form (usually without radicals or complex fractions), substitution should be tried.

1. Substitution

Ex: Solve the system, $xy = 6$ (1)

$$x^2 + y^2 = 13 \qquad (2)$$

(1) $x = 6/y$

(2) $\frac{36}{y^2} + y^2 = 13$, or $36 + y^4 = 13y^2$

$y^4 - 13y^2 + 36 = 0$, or $(y^2 - 4)(y^2 - 9) = 0$

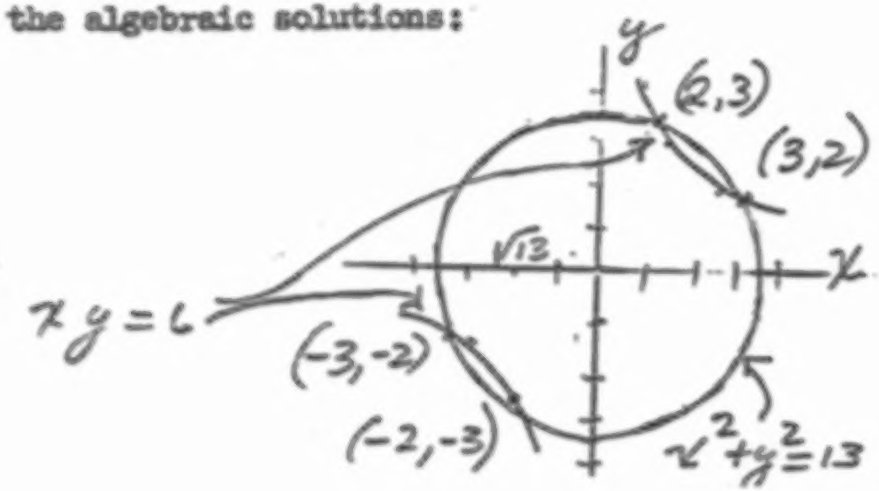
$$(y + 2)(y - 2)(y + 3)(y - 3) =$$

$$y = -2, +2, -3, +3$$

Substituting in (1), $xy = 6$,--

x:	-3	+3	-2	+2
y:	-2	+2	-3	+3

A graph of the above system of equations verifies the algebraic solutions:



2. Addition - subtraction

The addition-subtraction method is usually used to eliminate x^2 or y^2 terms. This method can also be used in some cases with xy terms.

Example: the problem

$$xy = 6 \quad (1)$$
$$x^2 + y^2 = 13 \quad (2)$$

To make a perfect square:

$$\begin{array}{lrcrl} & x^2 & + & y^2 & = 13 \\ \text{Add:} & & 2xy & & = 12 \\ \hline & & (x+y)^2 - & 25 & = 0 \\ & & (x+y+5)(x+y-5) & & = 0 \\ & & (3) \quad (4) & & \end{array}$$

$$\begin{array}{lrcrl} & x^2 & + & y^2 & = 13 \\ \text{Subtract:} & & 2xy & & = 12 \\ \hline & & (x-y)^2 - 1 = & & 0 \\ & & (x-y+1)\,(x-y-1) & = & 0 \\ & & (5) \quad (6) & & \end{array}$$

$$\begin{array}{ll} \text{Eq} & (3)\ x + y + 5 = 0 \\ \text{Eq} & \underline{(5)\ x - y + 1 = 0} \\ & 2x \qquad + 6 = 0 \\ & x = -3 \end{array}$$

$$\begin{array}{ll} \text{Eq} & (3)\ x + y + 5 = 0 \\ \text{Eq} & \underline{(6)\ x - y - 1 = 0} \\ & 2x \qquad + 4 = 0 \\ & x = -2 \end{array}$$

Completing the solution in this way, by adding equations (4) and (5), and by adding (4) and (6), we obtain all four sets of solutions:

x:	-3	+3	-2	+2
y:	-2	+2	-3	+3

Ex: Solve the system,

$$\begin{array}{lll} (1) & x^2 + y^2 = 5 & \text{---} \quad x^2 + y^2 = 5 \\ (2) & y = x^2 - 5 & \text{---} \quad \underline{x^2 - y = 5} \\ & & \qquad\ \ y^2 + y = 0 \\ & & \qquad\ \ y(y+1) = 0 \\ & & \qquad\ \ y = 0, -1 \end{array}$$

Substituting $y = 0, -1$ in equation (1):

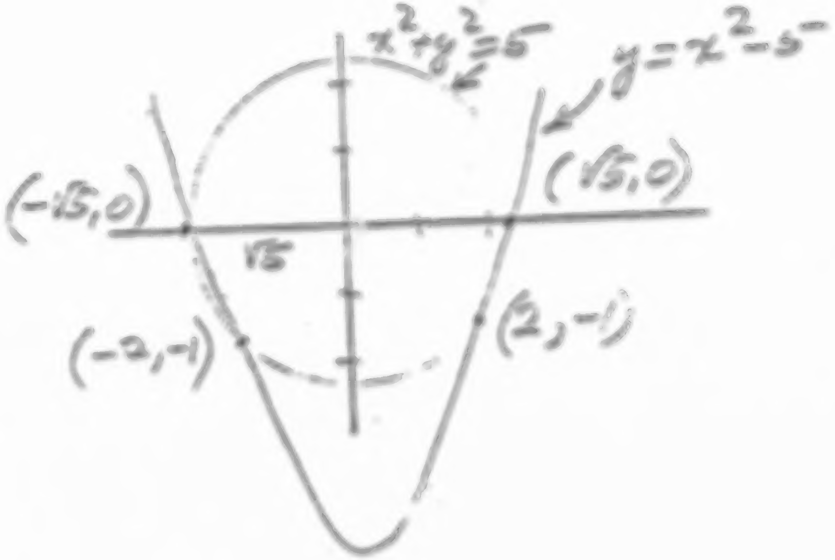
$$x^2 + 0 = 5 \qquad x^2 + 1 = 5$$

$$x = \pm \sqrt{5} \qquad x = \pm 2$$

The pairing of answers often gives trouble. When $y = 0$, x can equal either $+5$ or -5; when $y = -1$, x can be either $+2$ or -2.

x:	5	- 5	2	-2
y:	0	0	-1	-1

Students should be required to pair all sets of answers in table form, to avoid confusion in this regard. As usual, a graph of the system verifies the results:



A freehand graph such as this gives reassurance to the algebraic solutions.

SUMMARY

In the solution of quadratic systems to the following points are important:

1. The student should become adept at recognizing the various conic sections from their equations.
2. When graphing quadratics he should take special pains to make the drawings relatively neat, accurate, and complete.
3. During the graphing process he should make constant use of such special aids as slopes, x- and y- intercepts, asymptotes, foci, axes of symmetry, maxima and minima, etc.
4. When and if an exact graph is not required, he should habitually make at least a free-hand, approximate sketch, as an aid in visualizing the placement and probably number of solutions.
5. He should pair all solution sets in individual sets.
6. He should of course check every solution set by substitution in each equation.

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O.C.S.E.I.P. SYLLABUS

Geometry

SE006 070

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OFFICE OF EDUCATION

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PREFACE

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

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GEOMETRY

GEOMETRY

Motivation

Some students take geometry because it is required for college entrance, but how many have thought of all the vocational fields they could not enter if they hadn't studied geometry?

It is good to have a number of posters showing design engineers, detail draftsmen, electronic engineers, physicists, and atomic scientists at work. The students will discuss their interests and begin to think of possibilities for their life work.

INTRODUCTION

This syllabus is written for use by beginning teachers who would like to have a guide for the teaching of Geometry.

No two teachers will teach exactly the same way, nor will they want to take the various topics in the same order, but the topics included here have been found to be of the utmost importance to students who are going to continue in mathematics and science.

Topics which are well presented in most texts have been lightly covered here, and certain topics omitted in texts are rather thoroughly explained.

Approximate time has been suggested for each unit, but again, teachers will use their own judgment as they find how individual classes respond.

An attempt has been made to combine the best features of traditional and modern approaches to teaching, with the best interest of the student in mind at all times.

Cumulative tests given regularly will help to fix ideas in the minds of students, and if errors on these tests are discussed thoroughly, testing will become both a teaching and learning aid.

The syllabus is not intended for student use, nor is it geared to any particular textbook. It is hoped that it can be used with any textbook, and that teachers will find it to be a valuable reference.

MOTIVATION

↓

SEE SYLLABUS

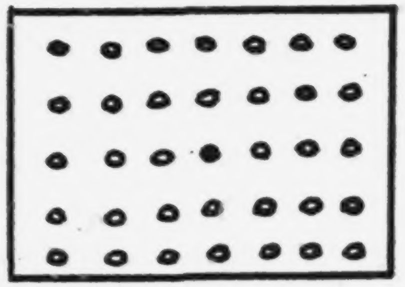
GEOMETRY

I. Elements of Geometry
    A. Basic Terms
        1. Point
        2. Line
        3. Plane
        4. Space

Students get a better idea of point, line, and plane if they spend some time drawing and constructing --- especially drawing planes (horizontal and vertical).

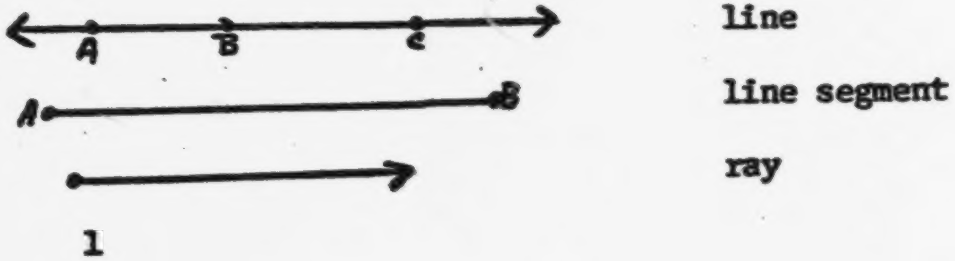
Discuss points in or on a plane; lines in or on a plane, line ||, lines ⊥ to a plane.

In demonstrating, use "pick up sticks" for lines, and a piece of masonite with drilled holes for planes.



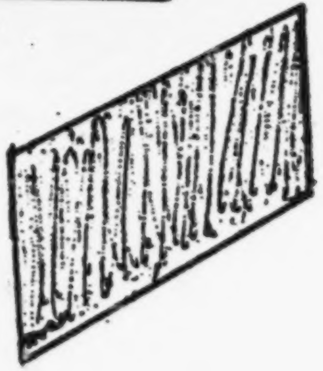
The pick up sticks will fit into the holes to demonstrate a line ⊥ to a plane, etc.

Points are described rather than defined. Represent a point by making dots of various sizes on the blackboard and explain that a dot in mathematics has position only. Use a capitol letter to designate a point. All geometric figures, whether on a plane or in space, are sets of points. Distinguish between a line and a line segment; and a line, line segment, and a ray.



A plane is usually represented by drawing a parallelogram.

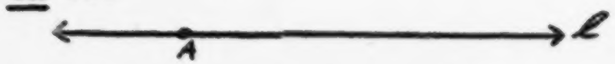
horizontal plane m



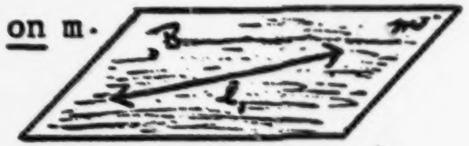
vertical plane m

Be sure that students are familiar with terms.

A lies in or on ℓ.



B lies in or on m.

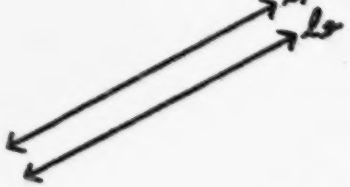


ℓ_1 lies in or on m (m contains ℓ_1)

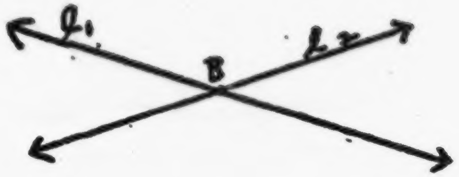
Explain how a plane is determined:

3 non-collinear points (points not in a straight line).

2 // lines.



2 intersecting lines.

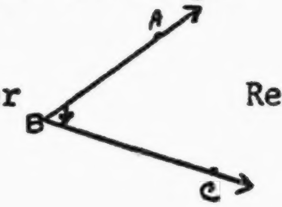


A line and a point outside the line

B. Angles and their Measurement
   1. The Angle
      a. rays or sides
      b. vertex

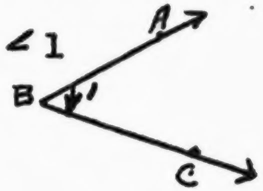
Students should have practice in constructing equal angles from given angles, and in reading angles. Students have some difficulty in seeing that the size of an angle does not depend upon the length of the sides on a ray.

They also need practice in reading angles with the vertex letter as the middle letter and as a numbered angle.

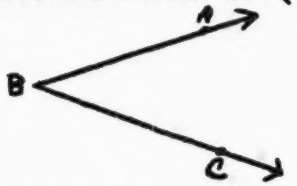


Read ∠ABC

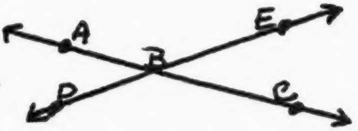
Read ∠1



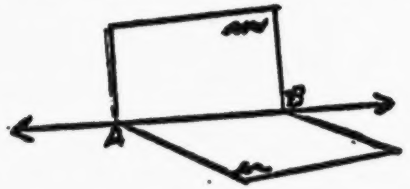
Explain and demonstrate that intersection of any two geometric figures is the set of points common to both figures. In the angle, point B is the intersection of (ray) $\overrightarrow{BA}$ and (ray) $\overrightarrow{BC}$.

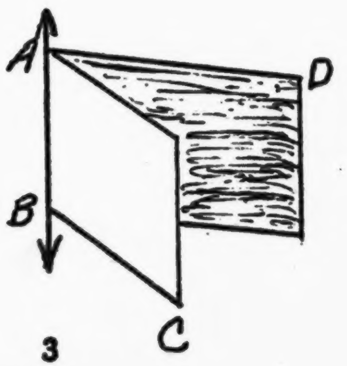


B is the only point $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$ have in common in the figures.



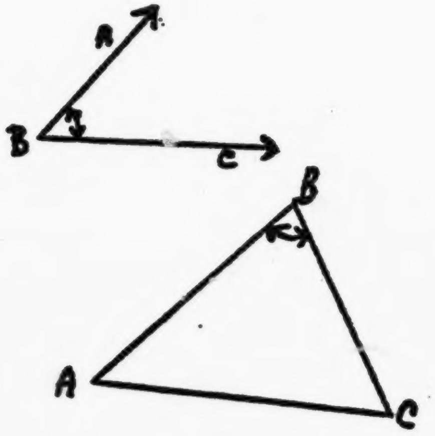
Planes m and n intersect in the line $\overleftrightarrow{AB}$. All points on $\overleftrightarrow{AB}$ are common to both plane m and plane n.

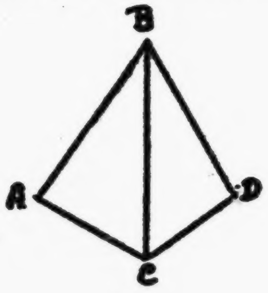


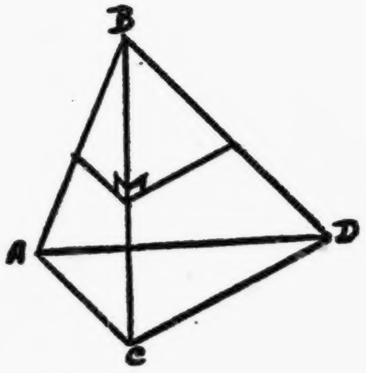


read dihedral ∠C-AB-D

Introduce the idea of the plane angle and the triangle in relationship to the dihedral angle and the tetrahedron. There are good commercial models to use to illustrate these relationships.

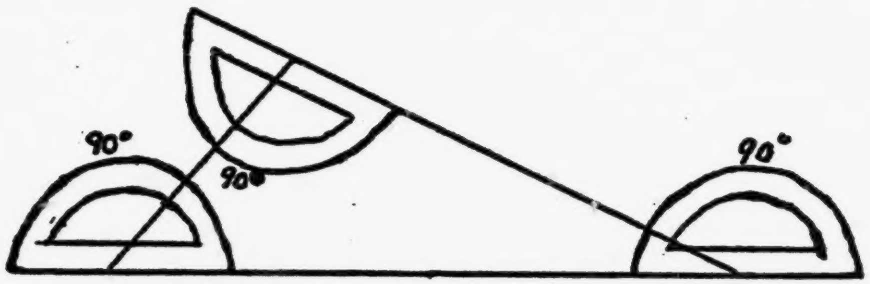






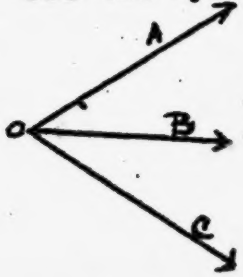
C. Measurement of Angles
    1. Use of protractor
    2. Degrees, minutes, seconds
    3. Adding and subtracting
    4. Bisection

The student must exercise care to see that a protractor is correctly placed in measuring an angle. All measurement is subject to error, but being careful with intruments helps accuracy.



The zero point on the protractor falls on the vertex of the angle.

Draw an example of adding and subtracing the measure of angles. Use the symbol $\overset{\circ}{=}$ (equal in degrees). Some textbooks use m∠AOB + m∠BOC = m∠AOC.



∠AOB + ∠BOC = ∠AOC
∠AOC - ∠BOC = ∠AOB

$(60'' = 1', 60' = 1^\circ)$

$180^\circ - 38^\circ\ 40' \rightarrow 179^\circ\ 60' - 38^\circ\ 40' = 141^\circ\ 20'$
$180^\circ - 52^\circ\ 45'\ 30'' \rightarrow 179^\circ\ 59'\ 60' - 52^\circ\ 45'\ 30'' =$
$127^\circ\ 14'\ 30''$

Have the students draw angles of various sizes, using the protractor, and then construct angles from the given angles. For practice in bisection, have the students construct the bisector of the angles they construct. Proof of these constructions follow later. Students can measure the angles here with a protractor.

II. Induction - Deduction
  A. Induction
    1. Observation
    2. Measurement
    3. Experimentation

Have students list geometric concepts they have observed (a ball is round, a book is rectangular, envelopes have triangular sealing flaps--spider webs--leaves).

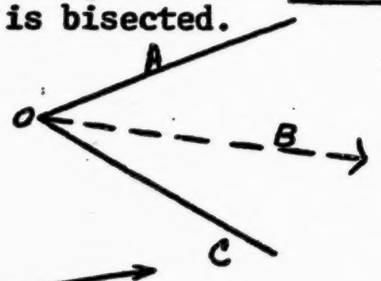
Have some optical illusion drawings posted around the room to show how students can arrive at false conclusions.

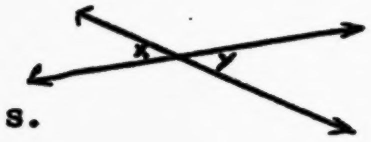
Have each student use a ruler to measure his desk. Have several students measure the length and width of the blackboard. Check measurements against each other for "sameness."

It should be pointed out that inaccurate measuring devices, human error, and the fact that not all cases can be examined, makes the inductive methods less successful than the deductive method.

  B. Deduction
    1. The General Statement
    2. The Specific Statement
    3. The Conclusion

In the analysis of a problem, the three steps of deduction may be used in any order. 1. General Statement: a ray bisects an angle when it forms with the sides of two equal angles. 2. Specific Statement: $\overrightarrow{OB}$ divides $\angle AOC$ so that $\angle AOB \cong \angle BOC$. 3. Conclusion: $\angle AOC$ is bisected.





1. General Statement: Any pair of vertical angles are =.
2. Specific Statement: $\angle x$ and $\angle y$ are a pair of vertical $\angle$ s.
3. Conclusion: $\angle x \cong \angle y$.

As much time as possible should be spent on the Analysis of problems. A good reference is Polya's "How to Solve It." (G. Polya, "How to Solve It," Doubleday)

Stress to the student that in geometric proof the general statement is usually expressed by an "if--then" sentence ("if" part is called the hypothesis, "then" part is called the Conclusion). "If two lines are $\parallel$ to a third line, they will not intersect."

III. Axioms and Postulates in Geometry
   A. Axioms
   B. Postulates (Assumption)
   C. Angle Relationships
   D. Perpendicular Lines

Many geometry books use algebraic and arithmetical examples of axioms, but it is a good idea to use geometric examples in explanations to students. If $\angle a \cong \angle b$ and $\angle c \cong \angle d$, then $\angle a + \angle c \cong \angle b + \angle d$.

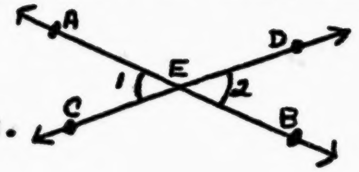
$$\begin{array}{lrcl} \text{If} & \angle a + \angle b & \cong & \angle c, \text{ and } \angle b \cong \angle d \\ & \angle a + \angle b & \cong & \angle c \\ - & \angle b & \cong & \angle d \\ \hline & \angle a & \cong & \angle c - \angle d \end{array}$$

A good oral exercise where the student reads a geometric statement and then states the axiom involved will fix the axioms in his mind and help him in formal proofs later on. (See Welchons and Krickenberger)

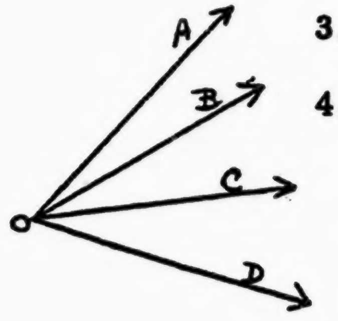
Have the students illustrate each of the postulates with a figure. Students may also be encouraged to try to draw a figure which will contradict a postulate.

Use the oral exercise again by drawing figures on the board, making statements, and having the students give the postulate or definition which justifies the statement. It should be stressed that a good definition is reversible. A ↔ B, and in a mathematical system, definitions are of prime importance. Discuss the difference between a postulate and a definition. Examples of illustrations to use in discussion:

1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines.



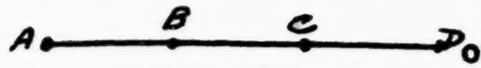
2. $\angle 1$ and $\angle 2$ are vertical $\angle$s. Why?



3. $\angle$AOC has only one bisector. Why?

4. Students practice adding and subtracting angles.

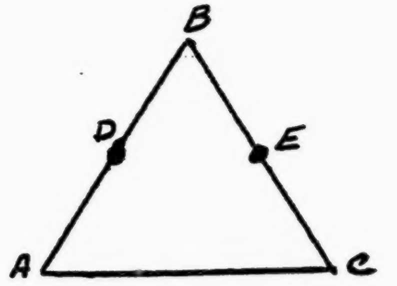
$\angle AOB + \angle BOC \overset{\circ}{=} \angle AOC$
$\angle AOD - \angle COD \overset{\circ}{=} \angle AOC$ Compare with adding and subtracting



on a line.

$\overline{AB} + \overline{BC} = \overline{AC}$

5. D and E are midpoints of $\overline{AB}$ and $\overline{BC}$.
If $\overline{AB} = \overline{BC}$, then $\overline{AD} = \overline{CE}$.
(If the measure of AB = the measure of BC, then the measure of AD = the measure of CE).



Some textbooks use $\overline{AB} \cong \overline{CD}$ or AB = CD to show equal lengths. At all times the student should understand that it is the measure of angles which are equal and the lengths of segments which are equal.

The idea of betweeness could be discussed here (See S.M.S.G.). Have the students construct or draw, using protractor and ruler, a $\perp$ to a line from a point outside the line; a $\perp$ to a point on the line; and the $\perp$ bisector of a line segment.

Discuss right angles, straight angles, supplementary $\angle$s, complementary $\angle$s and adjacent angles in connection with the above constructions.

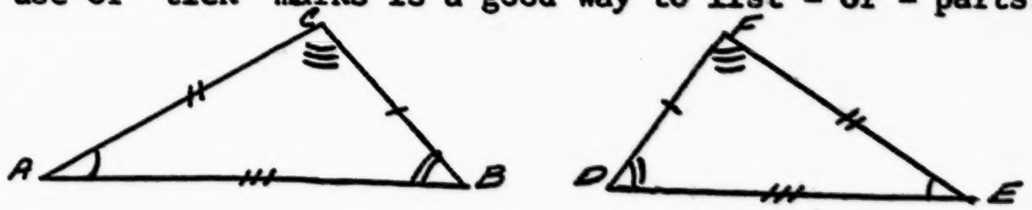
Have students draw other examples of supplementary angles and adjacent angles using several different polygons. Bring in a discussion of equal vertical angles in connection with adjacent and supplementary angles.

"Do angles have to be adjacent to be complementary or supplementary angles?" Discuss possibilities.

IV. Congruent Triangles - Formal Proofs
   A. Definitions
      1. Triangle
      2. Congruent
      3. Corresponding sides - angles
      4. Included $\angle$s, sides
      5. Coincide

Introducing "corresponding" parts of triangles early in the informal discussion of this unit will help students in proving triangles congruent. Have them draw two $\cong$ triangles and list all of the of the corresponding parts which would have to coincide in order that the two triangles would be congruent.

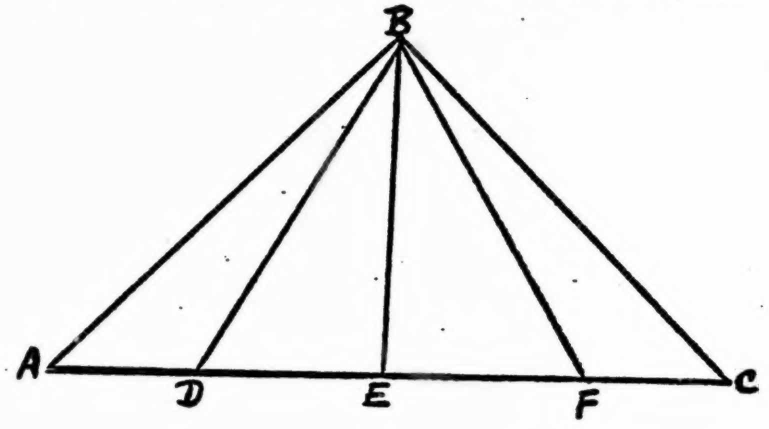
The use of "tick" marks is a good way to list = or $\cong$ parts.



Have students construct two $\triangle$s given two sides and the included angle, two angles and the included side, three sides, and two sides and the angle opposite one of them.

Cut out these figures to illustrate that the figures will coincide if they are $\cong$.

Draw a triangle on the board and have students locate all possible cases of S.A.S., A.S.A., S.S.S.



$\overline{AB}$

∠BAD SAS

$\overline{AD}$

etc.

Practice in locating two sides and the included ∠ will lead into a clearer understanding why two $\triangle$s are $\cong$ if two sides and the included ∠ of one are =, respectively, to two sides and the included of another.

B. Triangles Classified
   1. As to sides
   2. As to angles

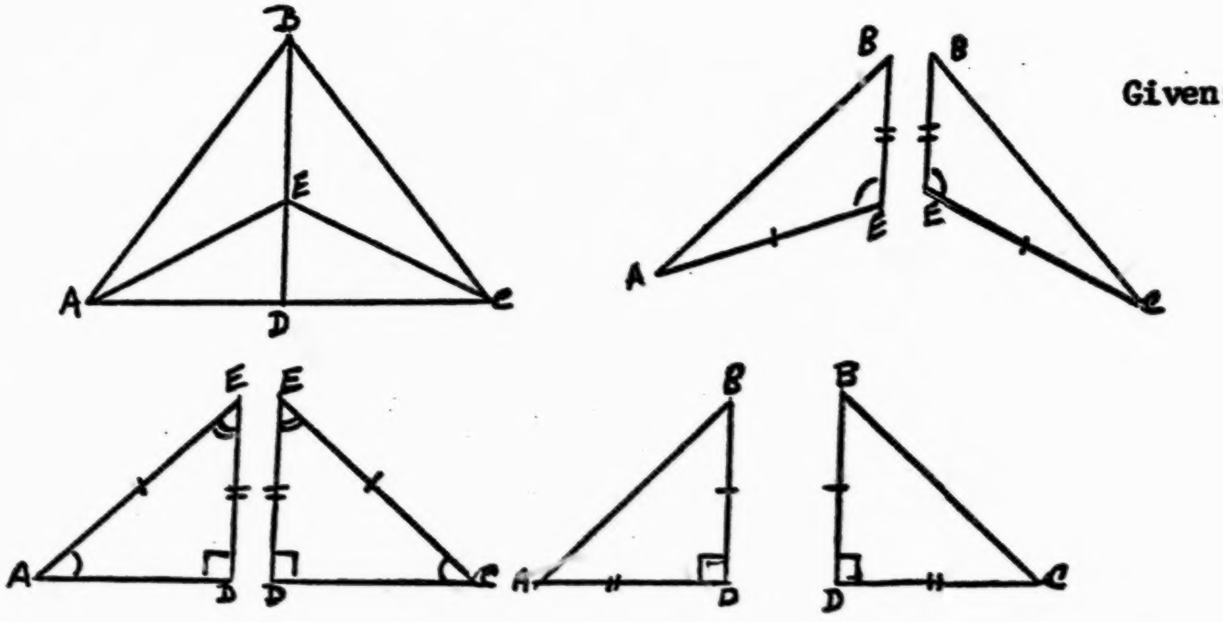
Students should be able to define and construct right, scalene, isosceles, and equilateral triangles, and be able to identify acute, obtuse, and equiangular triangles.

C. Formal Proofs
   1. S.A.S.
   2. A.S.A.
   3. S.S.S.

In beginning formal proofs, the student should be required to give the reason for the conclusion in complete sentences with geometric terms spelled correctly. (Cooperation with English teachers!)

One method of approach to setting up a formal proof is to have the student read the given parts (Hypothesis) and immediately mark the = sides and = $\angle$s (tick marks) on the figure to be used in proof. He then sees clearly what he has and what he needs for proof. He "talks to himself" until he has made his complete proof and has "ticked" off S.A.S., A.S.A., etc. The student then writes out his statements and reasons and completes the formal proof.

On more difficult proofs, the student should separate the figure into possible $\cong \triangle$s and then follow the above procedure.



Given: $\triangle AEC$ is isos.
$\overline{BD} \perp \overline{AC}$
To Prove: $\triangle ABC$ is isos.

The student should be given many simple proofs so that he becomes an "expert". Here is one of his best opportunities in high school to start with a premise and justify every statement he makes until he arrives at a logical conclusion.

Proofs are written as two column proofs, as one column proofs with marginal comments, and as paragraph proofs. Teachers and students can decide when to use each type.

The proofs can become progressively more thought provoking as the student becomes adept in proofs.

Some Suggested Proofs.

Definition: A one-to-one correspondence is said to have been set up between two sets of elements, C and D, when a pairing has been set up between them such that each element of D has been made to correspond to one and only one element of C, and each element of C has been made to correspond to one and only one element of D.

Alternate Definition: A correspondence (relation) between two sets of things such that pairs can be removed, one member from each group, until both groups have been simultaneously exhausted.

Examples: (1) A = {a, b, c, d}
B = {1, 2, 3, 4}

C_1 = {(a, 1), (b, 2), (c, 3), (d, 4)}
C_2 = {(d, 1), (c, 2), (b, 3), (a, 4)} A↔B
C_3 = {(d, 1), (c, 2), (a, 3), (b, 4)}

Alternate Notation:

A↔B

a↔1
b↔2
c↔3
d↔4

Further Examples:

(2) A = {students in classroom}
B = {occupied chairs in the room}

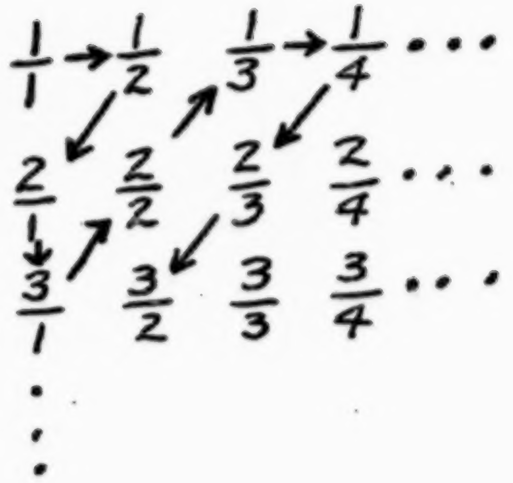
(3) A = {students in classroom}
B = {1, 2, 3, . . . n}

(4) any example of counting

Other Examples:

(1) A = {1, 2, 3 . . .}
B = {Rational Numbers}

Proof: (i.e. demonstration) ↗



(2) $A = \{\text{Natural Numbers}\}$
$B = \{\text{Even Natural Numbers}\}$

Demonstration:

$1 \leftrightarrow 2$
$2 \leftrightarrow 4$
$3 \leftrightarrow 6$
$4 \leftrightarrow 8$
$\vdots \quad \vdots$
$n \leftrightarrow 2n$
$\vdots \quad \vdots$

(3) $A = \{x \mid 0 < x < 1\}$
$B = \{x \mid -\infty < x < +\infty\}$

Demonstration:
(a) the line segment (i.e.: $[0, 1]$) may be tri-sected.
(b) the line segment $\cong$ to $[0, \frac{1}{3}]$ constructed at $x = \frac{1}{3}$, $x = \frac{2}{3}$

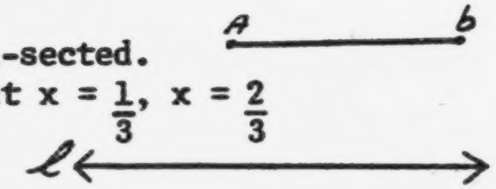
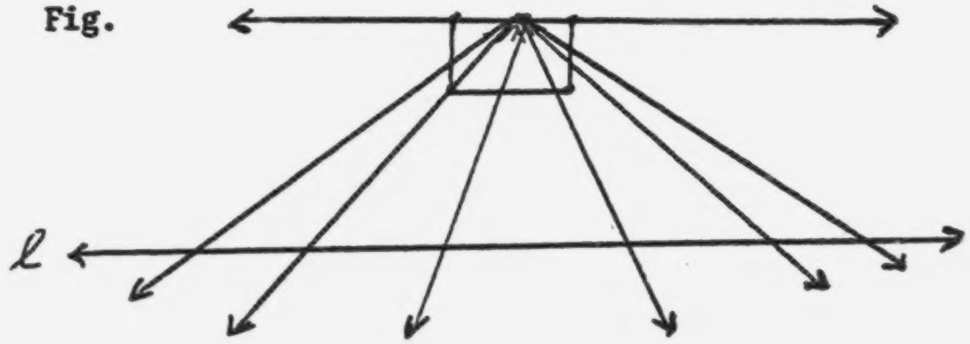
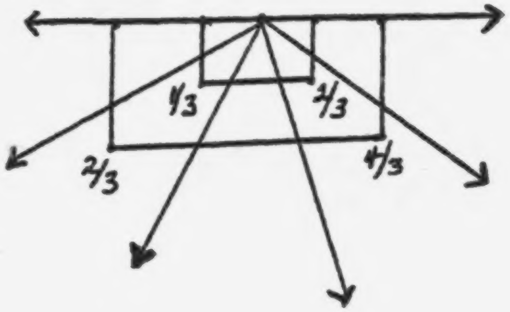


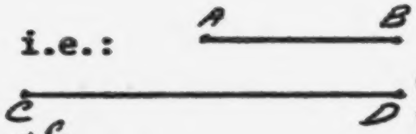
Fig.



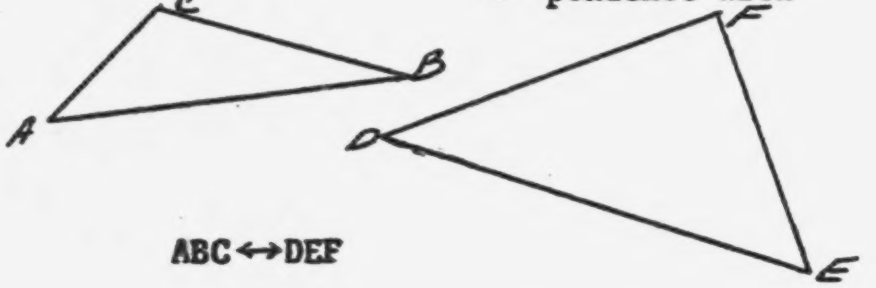
(4) $A = \{x \mid 0 \leq x \leq 1\}$
$B = \{x \mid 0 \leq x \leq 2\}$



i.e.:



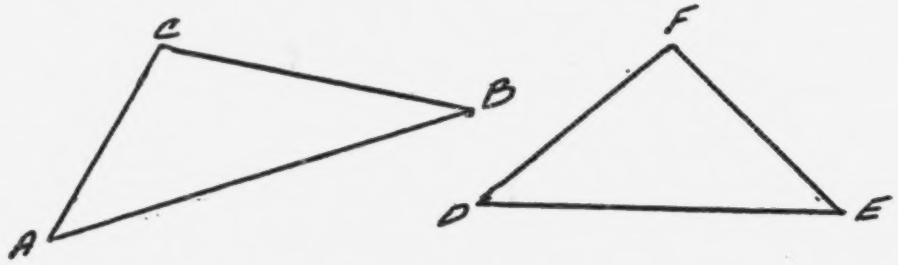
can be put in one-to-one correspondence with



$ABC \leftrightarrow DEF$

$\overline{AB} \leftrightarrow \overline{DE}$ $\angle A \leftrightarrow \angle D$
$\overline{BC} \leftrightarrow \overline{EF}$ $\angle B \leftrightarrow \angle E$
$\overline{AC} \leftrightarrow \overline{DF}$ $\angle C \leftrightarrow \angle F$

A Particular Example:



$$ABC \leftrightarrow DEF$$

$$\begin{array}{lcl} \overline{AB} \leftrightarrow \overline{DE} & & \angle A \leftrightarrow \angle D \\ \overline{BC} \leftrightarrow \overline{EF} & & \angle B \leftrightarrow \angle E \\ \overline{AC} \leftrightarrow \overline{DF} & & \angle C \leftrightarrow \angle F \\ & \text{and} & \\ \overline{AB} \cong \overline{DE} & & \angle A \cong \angle D \\ \overline{BC} \cong \overline{EF} & & \angle B \cong \angle E \\ \overline{AC} \cong \overline{DF} & & \angle C \cong \angle F \end{array}$$

This is a congruence correspondence denoted. $\triangle ABC \cong \triangle DEF$. With this definition and the usual ASA, SAS, SSS postulates, all the properties of congruent triangles can be derived.

e.g.: In $\triangle ABC$ and $\triangle DEF$ and $\triangle ABC \leftrightarrow \triangle DEF$
if $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\overline{AB} \cong \overline{DE}$
then $\triangle ABC \cong \triangle DEF$.

Identity Correspondence:

$$A = \{1, 2, 3\}$$
$$A \leftrightarrow A = \{(1, 1), (2, 2), (3,3)\} = \ldots \text{ etc.}$$
$$\triangle ABC \leftrightarrow \triangle BCA, \ldots \text{etc.}$$

Isosceles Triangle:

Given: $\triangle ABC$, $\overline{AB} \cong \overline{BC}$
Prove $\angle C = \angle A$

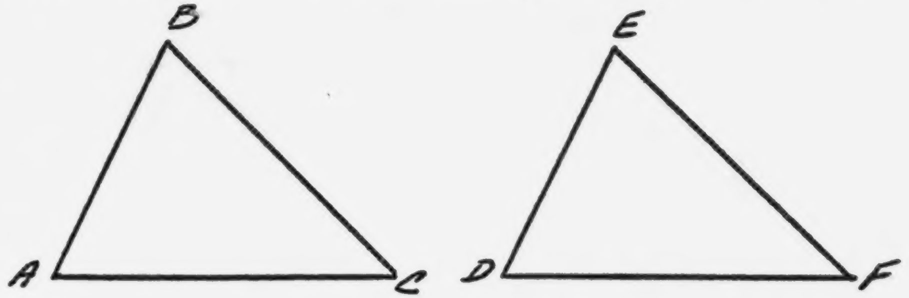
Consider $\triangle ABC \leftrightarrow \triangle BCA$

$$\left.\begin{array}{l} \overline{AB} \cong \overline{BC} \\ \overline{BC} \cong \overline{AB} \end{array}\right\} \text{given}$$
$$\overline{AC} \cong \overline{AC} \qquad \text{identity}$$

$$\triangle ABC \cong \triangle BCA \qquad \text{S.S.S}$$
$$\angle C \cong \angle A \qquad \text{CPCTC.}$$

A SSA Congruence Theorem Based on The Exterior Angle Theorem:

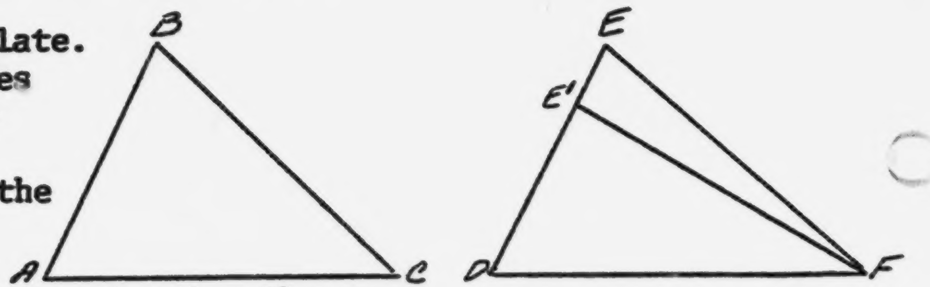
Given a correspondence $ABC \leftrightarrow DEF$ between two triangles such that $m\angle A = m\angle D$, $m\angle B = m\angle E$, and $AC = DF$; then the correspondence is called an SAA Correspondence. The SAA Theorem: Every SAA Correspondence is a Congruence.



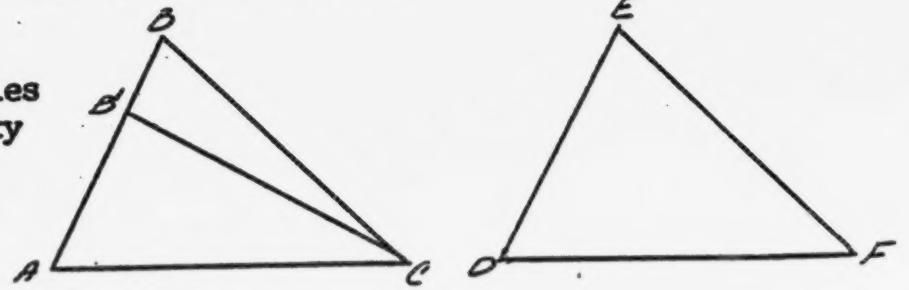
Given $\triangle ABC$ and $\triangle DEF$ such that $m\angle A = m\angle D$, $m\angle B = m\angle E$, and $AC = DF$; prove that $\triangle ABC \cong \triangle DEF$.

According to the Trichotomy Property of the Real Numbers there are three possibilities for AB and DE: (1) $AB < DE$, (2) $AB > DE$, and (3) $AB = DE$.

Suppose that (1) holds: $AB < DE$. Then by the Point-Plotting Theorem let E' be the point on $\overline{DE}$ such that $DE' = AB$. Then $\triangle ABC \cong \triangle DE'F$ by the SAS Congruence Postulate. Since Corresponding Angles of Congruent Triangles are congruent $\angle B \cong \angle DE'F$ and by the Transitivity Property of the Real Numbers $\angle DE'F \cong \angle E$. But this contradicts $\angle DE'F > \angle E$ which follows from the Exterior Angle Theorem. Thus $AB \nless DE$.



Suppose that (2) holds: $AB > DE$. Then by the Point-Plotting Theorem let B' be the point on $\overline{AB}$ such that $AB' = DE$. Then $\triangle AB'C \cong \triangle DEF$ by the Congruence Theorem. Since Corresponding Angles of Congruent Triangles are congruent $\angle E \cong \angle AB'C$ and by the Transitivity Property of the Real Numbers $\angle AB'C \cong \angle B$. But this contradicts $\angle AB'C > \angle B$ which follows from the Exterior Angle Theorem. Thus $AB \ngtr DE$.



Therefore by the Trichotomy Property AB = DE. $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Correspondence. Q.E.D.

Definitions, Postulates, and Theorems.

The Point-Plotting Theorem. Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that AP = x.

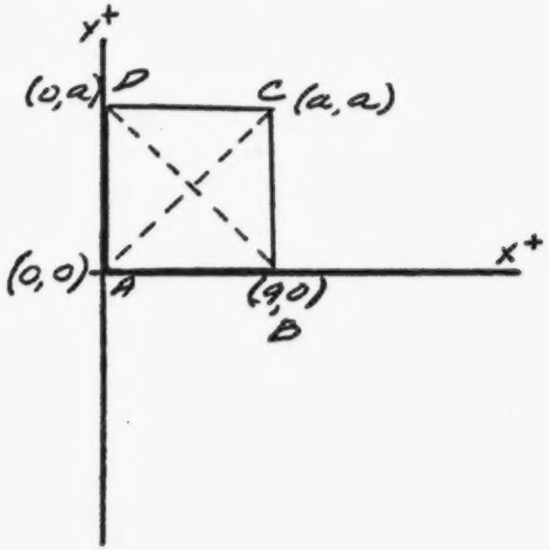
The Exterior Angle Theorem. An exterior angle of a triangle is greater than each of its remote interior angles.

Trichotomy Property for the Real Numbers. For every x and y, one and only one of the following conditions holds: $x < y$, $x = y$, $x > y$.

Transitivity Property of Inequality for the Real Numbers. If $x < y$ and $y < z$, then $x < z$.

Coordinate Geometry Proofs:

(1) Show diagonals of Square $\perp$.

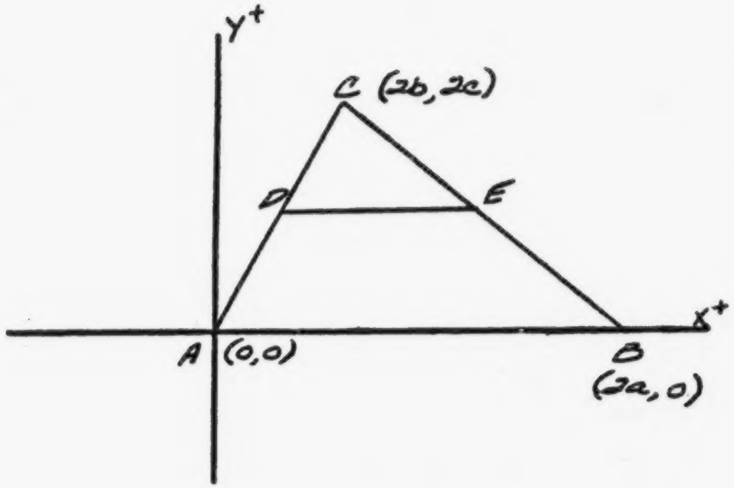


Show: The product of $m_1 \cdot m_2 = -1$ where m_1 is the slope of $\overline{AC}$ and m_2 is the slope of $\overline{BD}$.

Find: the slope of $\overline{AC}$ and $\overline{BD}$.

$$m_1 = \frac{a - 0}{a - 0} = 1 \qquad m_2 = \frac{0 - a}{a - 0} = -1$$

$$m_1 \cdot m_2 = (1)(-1) = -1$$



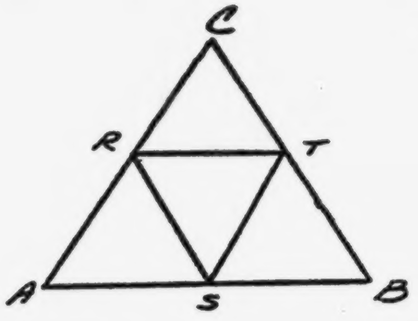
In ABC, Let D and E be midpoints of $\overline{AC}$ and $\overline{BC}$.
Show: $\overline{DE} \parallel \overline{AB}$ and DE = ½AB

Pet Problems on Congruent Triangles:

NOTE: (1) Problems require different stages of axiomatic assumption.
(2) My idea of a "pet" problem is one that emphasises symmetry with "obviousness"-- yet requires mathematical proof.

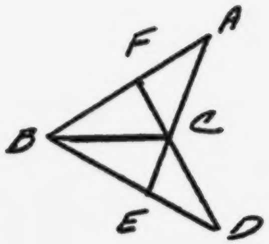
(1) Given: In △ABC
$\overline{AB} \cong \overline{BC} \cong AC$
$\angle A \cong \angle B \cong \angle C$
R, S, and T are midpoints of AC, AB, and BC respectively

Prove: $\triangle ARS \cong \triangle STB \cong \triangle RCT \cong \triangle RST$



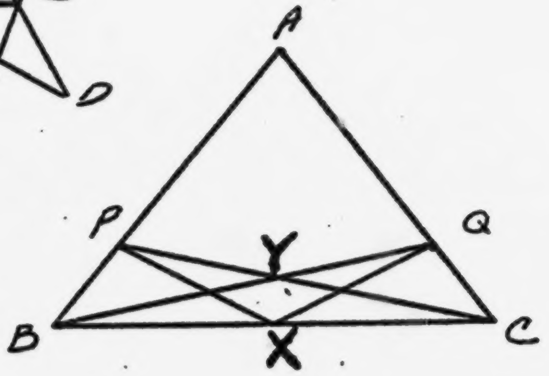
(2) Given: $\overline{AB} \cong \overline{BD}$
$\overline{AC} \cong \overline{CD}$
C is between A and E
C is between D and F

Prove: $\overline{AE} \cong \overline{DF}$



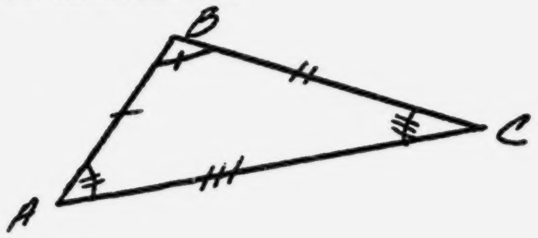
(3) Given: $\angle ABC \cong \angle ACB$;
BX = CX
$\angle PXB = \angle QXC$

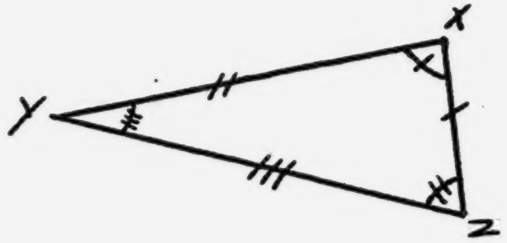
Prove: $\overline{BQ} \cong \overline{CP}$



Preliminary Work Before Postulates of Congruent Triangles.

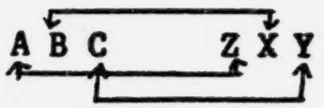
(1) Congruence Relations:





working with vertices or points only.

A ↔ Z
B ↔ X
C ↔ Y

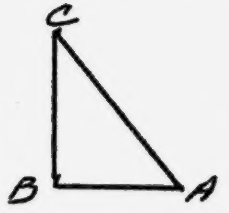


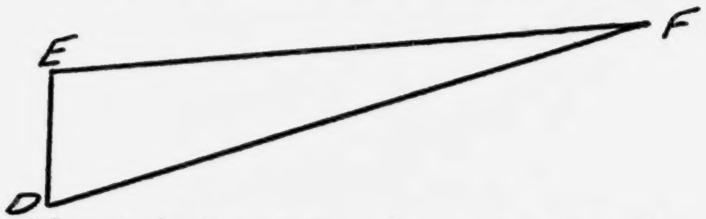
NOTE: One could have a relation, not necessarily a congruence relation.

(2) ABC ↔ XZY

We have a one to one correspondence, but the $\angle$s at the vertices don't "match up."

For example:





B → E A → D

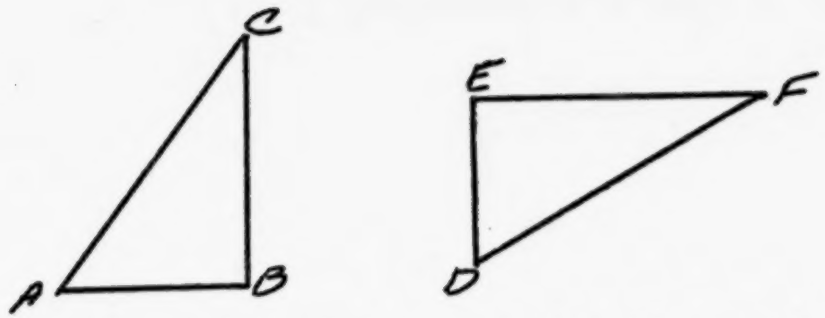
We could match C and F and surely the idea of ($\cong$) is out.

Now, back to (1) above to say $\triangle ABC \cong \triangle ZXY$ is to say the congruences

$\angle A \cong \angle Z$ $\overline{AB} \cong \overline{ZX}$
$\angle B \cong \angle X$ $\overline{AC} \cong \overline{ZY}$
$\angle C \cong \angle Y$ $\overline{BC} \cong \overline{XY}$

Here AB = ZX refers to distance--not segment AB = segment ZX because ABC ↔ ZXY.

(3)



Say $\triangle ABC \cong \triangle EDF$ which is a "big fat lie." This would mean

$\overline{AB} \cong \overline{DE}$ o.k.
$\overline{BC} \cong \overline{DF}$ lie
$\overline{AC} \cong \overline{EF}$ lie

$\angle A \cong \angle E$ lie
$\angle B \cong \angle D$ lie
$\angle C \cong \angle F$ o.k.

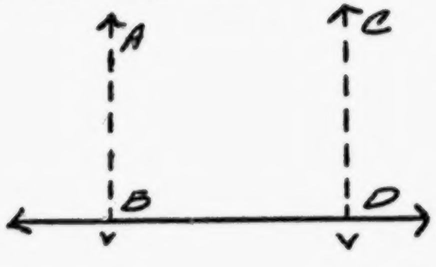
This type of thing has not been uncommon in the past.

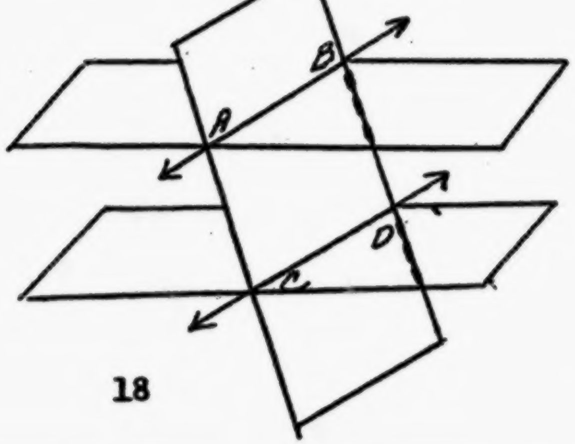
This is an important matter in regard to the idea of congruence.

V. Parallel Lines - Planes
   A. Definitions
      1. Skew lines
      2. Vertical lines and planes
      3. Horizontal lines and planes
      4. Transversal
      5. Special Angles

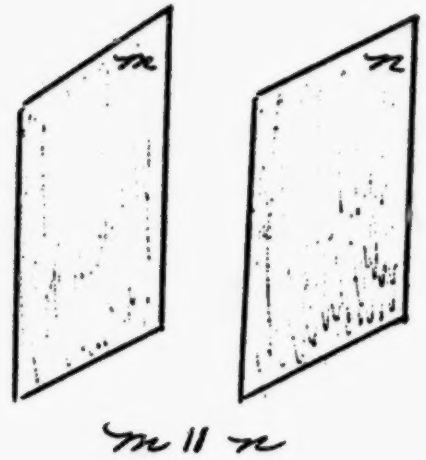
Have students construct parallel lines (vertical and horizontal), draw parallel planes (vertical and horizontal), and draw intersecting planes to see that two parallel planes cut by a third plane will have their lines of intersection parallel.

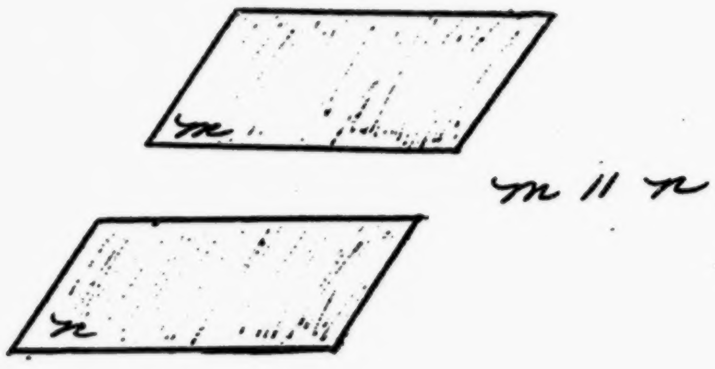
Two lines $\perp$ to the same line are $\parallel$





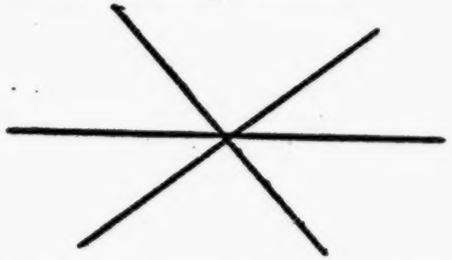
AB $\parallel$ CD



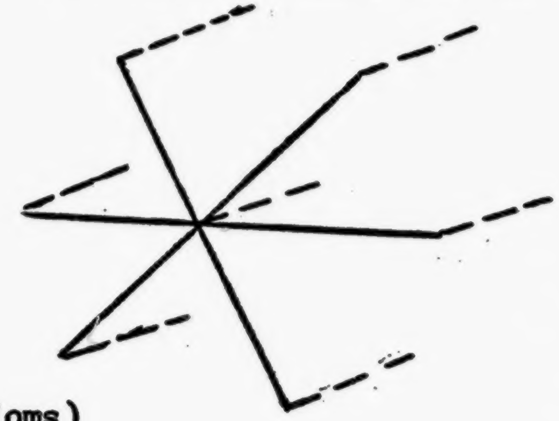


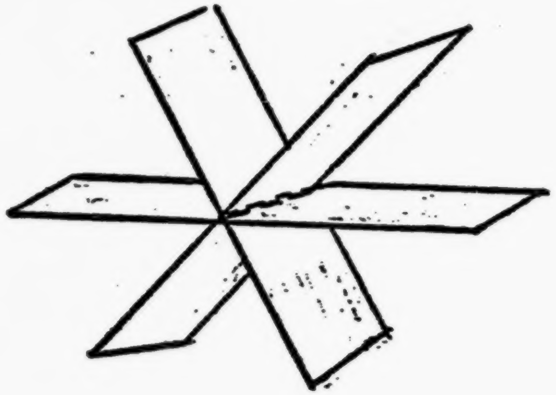
How to draw intersecting planes.

Draw two or more intersecting lines.



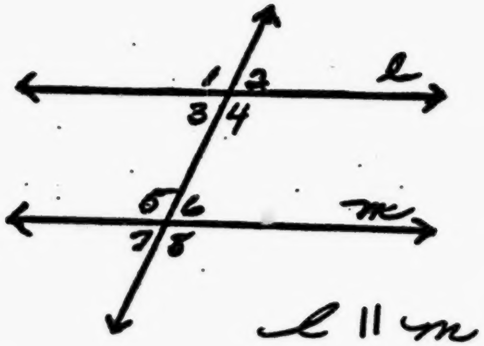
Draw lines of same length and at the same angle from the end points of the intersecting lines and at the point of intersection.



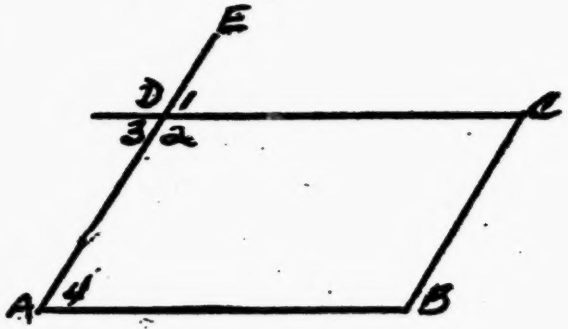


B. Basic Properties (postulates, Axioms)
   1. Parallel lines
   2. Parallel planes

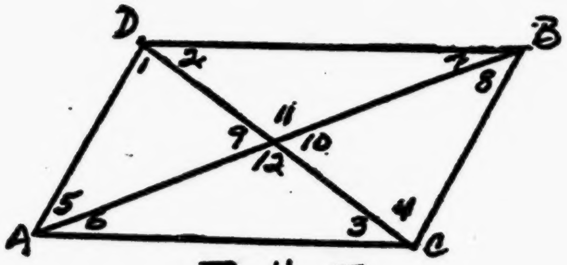
Students will retain the concepts presented in the assumptions if they draw figures to illustrate each of them. A variety of figures should be used in the illustrations.



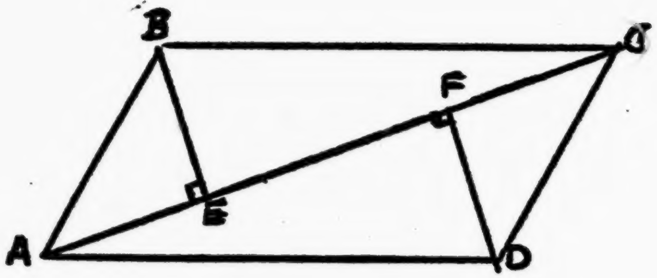
$l \parallel m$



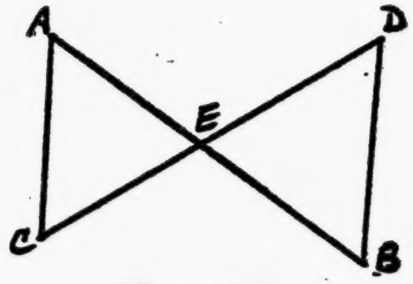
$\overline{AB} \parallel \overline{CD}$



$\overline{AD} \parallel \overline{BC}$
$\overline{AC} \parallel \overline{BD}$



$\overline{AB} \parallel \overline{CD}$
$\overline{BC} \parallel \overline{AD}$
$\overline{BE} \perp \overline{AC}$, $\overline{DF} \perp AC$



$\overline{AC} \parallel \overline{BD}$

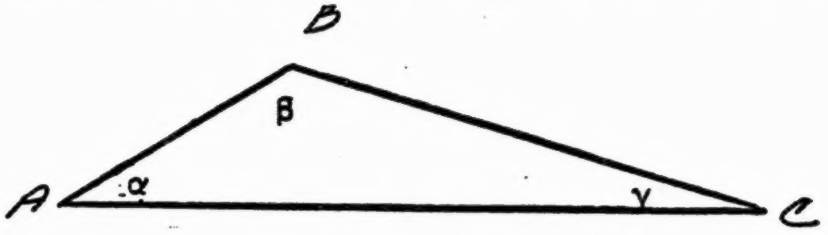
All of these figures may be used to illustrate "If two || lines are cut by TV, the alternate interior ∠s are ≅.

C. Formal Proofs

A sufficient number of formal proofs should be given to tie in the relationship between the concepts learned concerning || lines and those used in proving triangles ≅.

VI. Inequalities
    A. Assumptions
        1. Sides of a triangle
        2. Angles of a triangle

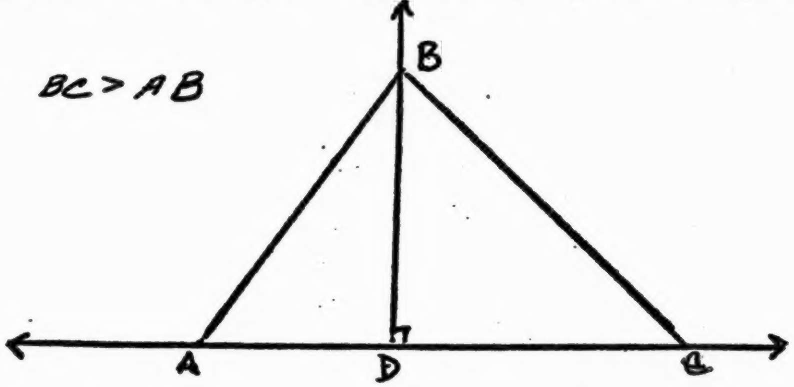
Have students draw a number of △s of different sizes and shapes, and use a ruler to show that each side of a △ is less than the sum of the other two sides. Have them measure (with a protractor) the angles in these triangles to "see" that the angle opposite the longer side is the greater angle.



$AC > AB \quad \therefore \quad \angle B > \angle \gamma$ etc.

Students should also construct a ⊥ to a line, draw two oblique line segments from a point on the ⊥ and measure to show that the oblique line having the greater projection upon the line is the greater.

$CD > AD \quad \therefore \quad BC > AB$



Several constructions such as the one above will fix this property of inequalities firmly in the minds of the students. In all units where they apply, inequalities should be introduced with equalities.

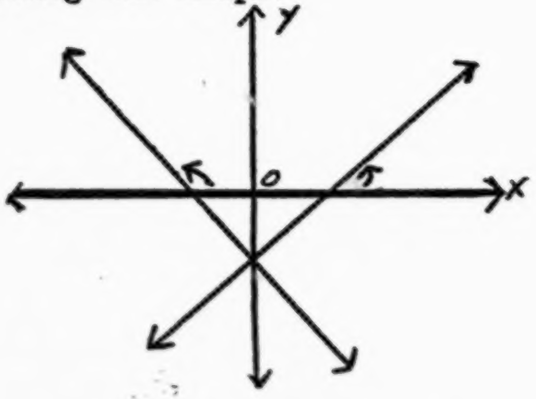
VII. Coordinate Geometry
    A. Cartesian or Rectangular
        1. Lines passing through pairs of points
        2. Drawing △s whose vertices are given.
        3. Drawing quadrilaterals whose vertices are given.

The students have had experience with the coordinate system when they graphed linear equations in Algebra I, but they tend to forget to label the origin and the x and y axes. It helps as a lead-in to have the students graph one or two linear equations when starting this unit.

Be sure that students recall the term abscissa and the term ordinate, which should have become familiar in Algebra I.

B. The Slope of a line
   1. Inclination
   2. Slope formula
   3. Parallel and ⊥ lines

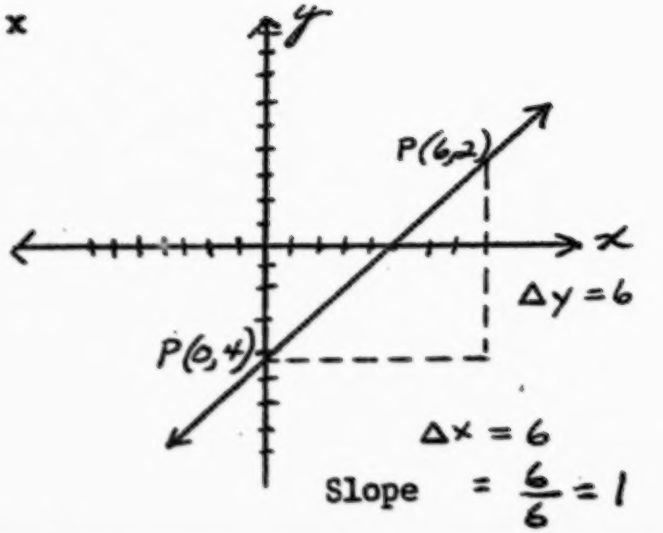
Illustrate that the inclination of a line is the smaller of the two angles that the line makes with the positive end of the x-axis, measured counterclockwise from the x-axis to the line, and that the inclination of the line can be described by giving its slope.



The slope of a line is the ratio of the change in y to the change in x between two points on the line.

Δ Delta (stands for "the change in")

$$\frac{\Delta y}{\Delta x} = \frac{\text{The change in } y}{\text{The change in } x}$$



The slope of a line is negative when its inclinations is $> 90^\circ$. Have the students draw sufficient examples so that they will understand that the slope is the same regardless of which two points on a straight line are used to compute it. The student should note that if the slope of a line is zero, the line is $\parallel$ to the x-axis, and if the line is $\parallel$ to the y-axis, there is no slope. (division by 0 is impossible)

The proportion for the slope may be written: $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 \quad x_1}$ or $y_2 - y_1 = m(x_2 - x_1)$

(m) is used to represent slope in some textbooks.

When the student draws two lines intersecting at right angles and finds the slope of each of the lines, he discovers that the slope of one of the lines is the negative reciprocal of the slope of the other and will relate this if questioned. "What did you discover about the slopes of these lines?'

C. Formulas to be understood and memorized
    1. Midpoint of a line
    2. Distance between two points
    3. Point-slope form
    4. Slope-Intercept form
    5. Two Point form
    6. The Intercept form

Students should have practice in finding the length of lines, proving that $\triangle$s are right, equilateral, isosceles, and that certain quadrilaterals are rectangles, parallelograms, rhombuses, etc.

Midpoint of a line: $x = \frac{x_1 + x_2}{2}$

given: (6,2) (4,4) $y = \frac{y_1 + y_2}{2}$ $y = \frac{2 + 4}{2} = 3$

$x = \frac{6 + 4}{2} = 5$ midpoint is (5,3)

Formulas listed here for easy reference.

Distance between two points:

$$d = \sqrt{(x_2 = x_1)^2 + (y_2 - y_1)^2}$$

Given: (6,2) (4,4)

$$d = \sqrt{(4-6)^2 + (4-2)^2} = \sqrt{(-2)^2 + (2)^2} \quad d = \sqrt{8} = \sqrt{2}$$

Point-slope form:

$$y - y_1 = m(x - x_1) \text{ or } \frac{y - y_1}{x - x_1} = m$$

Given: (-5,3); $m = 2/3$

$y - 3 = 2/3\ (x + 5)$
$3y - 9 = 2x + 10$
$2x - 3y = -19$

Slope-intercept form:

$y = mx + b$
Given: $m = \frac{1}{2}$, $b = 3$

$y = \frac{1}{2}x + 3$
$2y = x + 6$
$x - 2y = -6$

Two-point form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Given: (6,2); (4,4)

$$\frac{y - 2}{x - 6} = \frac{4 - 2}{4 - 6} \qquad x + y = 8$$

The intercept form. Given intercepts:

$\frac{x}{a} + \frac{y}{b} = 1$ x = 4, y = -1

$\frac{x}{4} + \frac{y}{-1} = 1$ x = 4 y = 4

The student should be required to memorize the abov e forms, and should work enough problems to become efficient in their use.

VIII. Similar Polygons - Ratio and Proportion

A. Similarity

Conclusions about a proposed product (car, airplanes, buildings) can be reached with a saving of time and money if experiments can be made with scale models. Similar polygons are geometric figures having the same shape but not necessarily the same size, and many of the properties of these figures are used regularly in industries where scale models play an important part.

B. Review of ratio and proportion
  1. Common unit of measure
  2. Terms of a ratio and of a proportion
  3. Means - Extremes product
  4. Equivalent forms

Ratio and proportion are studied in the junior high and in Algebra I, but a quick review at the beginning of the study of similar polygons will save time later.

Ratios must be expressed in the same unit of measure when used in a proportion. A ratio, fraction, and percent are three ways of saying the same thing. In the ratio a/b, a is the first term, b is the second or a is the numerator, b is the denominator.

In the proportion $a/b = c/d$, (a) and (d) are the extremes, (b) and (c) are the means.

A proportion is a statement of equality between two ratios. $2/5 = 12/30$ $a/b = c/d$

$(2)(30) = (5)(12)$ $ad = bc$ (cross-product)

The product of the means is equal to the product of the extremes.

Show that $a/b = c/d \rightarrow a/c = b/d \rightarrow b/a = d/c \rightarrow c/a = d/b$

C. Basic Properties

Very few geometry books list all of the important basic properties dealing with similar polygons. At the end of this unit these properties will be listed.

Have the students construct several examples of the properties to show that they are true.

Students who have experience with many problems dealing with each of the assumptions, will find future work in Algebra II and trigonometry much easier. Too many students have such limited experience working problems that they retain little knowledge of proportions and similar figures.

Assumptions:

1. If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.
2. The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.
3. If two triangles have their sides respectively proportional, they are similar.
4. The altitude on the hypotenuse of a right triangle forms two right triangles which are similar to the given triangle and to each other.
5. The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.
6. Either leg of a right triangle is the mean proportional between the whole hypotenuse and its projection on the hypotenuse.
7. If a line bisects two sides of a triangle, it is parallel to the third side and equal to half of it.
8. If a line bisects one side of a triangle and is parallel to another side, it bisects the third side.
9. If a line is parallel to one side of a triangle and intersects the other two sides, it divides them proportionally.
10. If a ray bisects one angle of a triangle, it divides the opposite side into segments which are proportional to the other two sides.

10a. If a ray bisects an exterior angle of a triangle, it divides the opposite side externally into segments which are proportional to the other two sides.

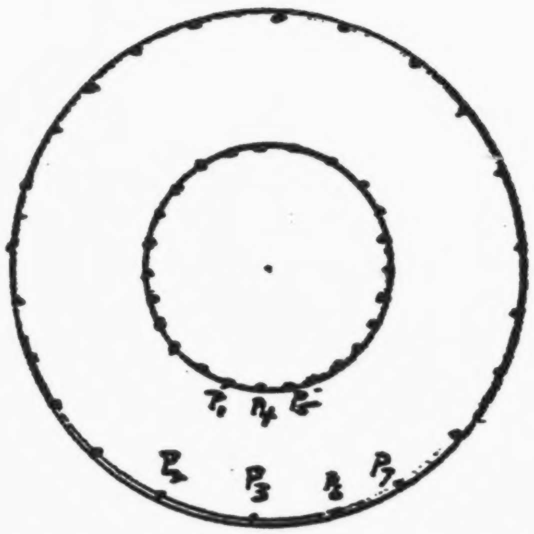
11. The rule of pythagoras.

12. In a 30° - 60° right triangle, the side opposite the 30° angle is equal to one half the hypotenuse, and the side opposite the 60° angle is equal to one half the hypotenuse times $\sqrt{3}$.
13. In a 45° right triangle, the side opposite the 45° angle is equal to one half the hypotenuse times $\sqrt{2}$.
14. The perimeters of two similar polygons have the same ratio as any two corresponding sides.
15. The areas of two similar polygons or polyhedrons have the same ratio as the squares of any two corresponding sides.
16. In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides and the projection of the other side on it. (A form of the Law of Cosines)
17. The volumes of two similar figures have the same ratio as cubes of any two corresponding sides.
18. If three or more parallel lines are cut by a transversal, the corresponding segments are proportional.
19. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides and the projection of the other side on it. (See 16 above)
20. If two lines are cut by three or more parallel planes, their corresponding segments are proportional.
21. If a pyramid is cut by a plane parallel to the base, the lateral edges and the altitudes are divided proportionally.
22. Two similar polyhedrons can be divided into the same number of tetrahedrons, similar each to each and similarly placed.

IX. Circles, Arcs, Angles
   A. Definitions
      1. Circle, radius, diameter
      2. Chord, tangent, secant
      3. Intercept - Intersect
      4. Concentric circles
      5. Inscribed and circumscribed polygons

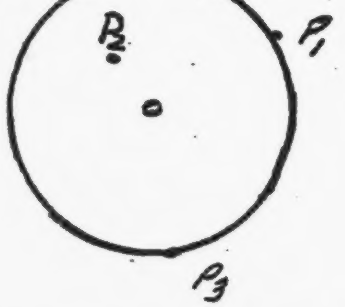
Have students discuss uses of the circle in daily life. Compare the use of triangular and rectangular figures with the use of a circle in all types of construction.

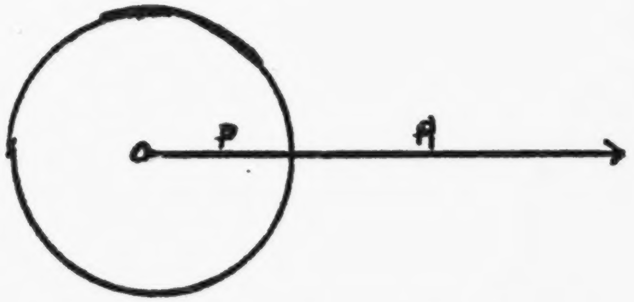
Concentric circles (in design) using straight lines only. Use three contrasting colors of thread. Figure to be drawn on cardboard.



Draw concentric circle. Using a fairly large needle, punch twice as many holes (evenly spaced) in the small circle as in the large circle. Thread the needle with any one of the three colors of thread, choose a starting point (p, above) push the needle in the hole and come out through P_2, go in P_3 and come out P_4, go in P_5 come out P_6. Continue in the same direction until all the holes in the smaller circle have been used once. (The holes in the larger circle will have been used twice) Any second and third starting place (with a different color of thread) will complete the design. On one side of the cardboard will be concentric circles and on the other side will be a geometric design. Students will think up many other geometric patterns.

Students should learn to distinguish between points of a circle and points in the interior of a circle. P_1 and P_3 are points of the circle. They lie on the circle. P_2 and 0 lie in the interior of the circle.





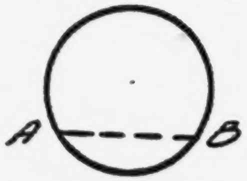
Have students draw circles given the center and radius, and given an arc , to locate the center of the circle.

To establish a one-to-one correspondence between the points in the interior of a circle and the points in the exterior of a circle: Define the inverse of a point. Given circle 0 with radius r. The inverse P_1 of P is that point P_1 on ray $\overrightarrow{OP}$ such that $(OP_1)(OP) = r^2$

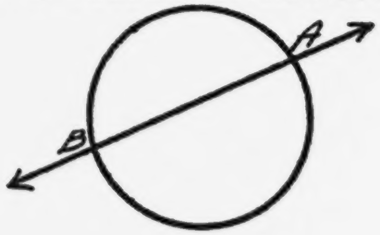
When P lies in the interior of the circle, $OP < r$, $OP_1 > r$, and P lies in the exterior of the circle.

Intercept - Intersect

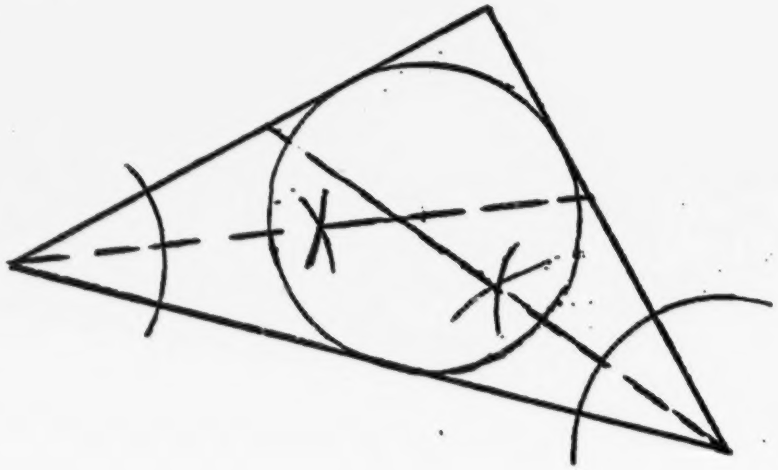
Students have some difficulty with the words intercept and intersect. Give several examples to explain the difference. Intercept--to cut off. A chord intercepts an arc on a circle.

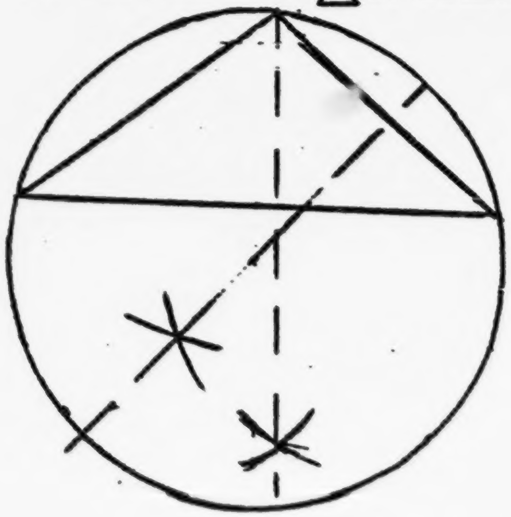


Intersect--to meet and pass on through. A secant intersects a circle.



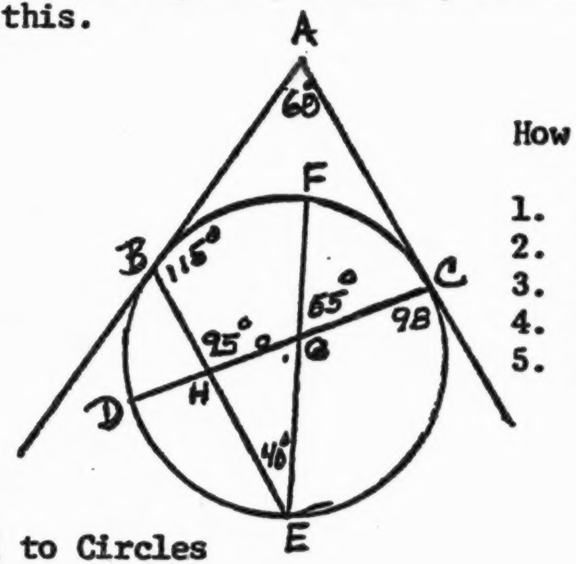
Have the students inscribe a circle in a triangle by bisecting the angles to locate the center, and circumscribe a circle about a $\triangle$ by constructing the $\perp$ bisect of the sides of the $\triangle$ to locate the center.





B. Measure of Arcs and Angles in a circle
   1. Central
   2. Minor arc, major arc, semicircle
   3. Inscribed angle
      a. corollaries
   4. An angle formed by a tangent and a chord
   5. An angle formed by two tangents, or two secants intersecting outside a triangle
   6. An angle formed by chords intersecting inside a triangle

Most geometry books have sufficient proofs and examples concerning these topics so that the students have little difficulty with them, but some examples should be given so that the student can find the number of degrees in the arcs cut off by tangents, secants, and chords when the degrees in the angles are given. Many textbooks neglect this.



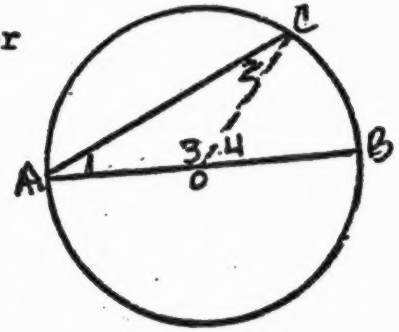
How many degrees in the following:

1. $\overset{\frown}{BF}$
2. $\overset{\frown}{CF}$
3. $\overset{\frown}{CE}$
4. $\overset{\frown}{DE}$
5. $\overset{\frown}{BD}$

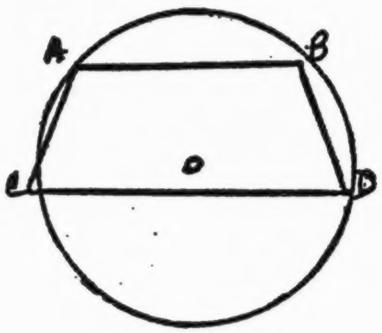
C. Lines and Segments Related to Circles
   1. Bisection of an arc
   2. A diameter ⊥ to a chord
   3. Equal chords and equal arcs in the same or equal circles
   4. Parallel chords intercept equal arcs on a circle

Students should construct these figures until they have a clear picture of them. Some formal proofs here can relate circles to other proofs previously done with triangles and parallel lines.

Example: Given: Circle o with AB a diameter
Prove: $\angle 4 \cong 2 \angle 1$



Example: Given: Trapezoid ABCD is inscribed in circle o.
Prove: ABCD is an isosceles trapezoid.



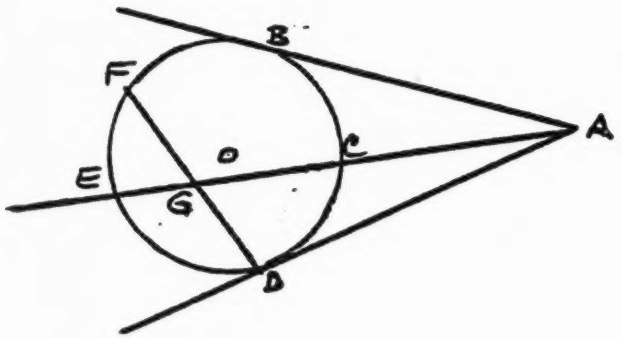
D. Proportions
   1. Chords intersecting within a circle
   2. Two tangents to a point from an outside point
   3. A tangent and a secant drawn to a circle from an outside point

These problems relate nicely to the problems worked in ratio and proportion, and the student can see the carry over of the assumptions dealing with proportions.

$$\frac{AE}{AB} = \frac{AB}{AC}$$

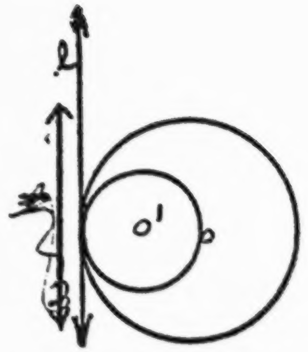
$$(EG)(GC) = (FG)(DG)$$

$$\overline{AB} = \overline{AD}$$

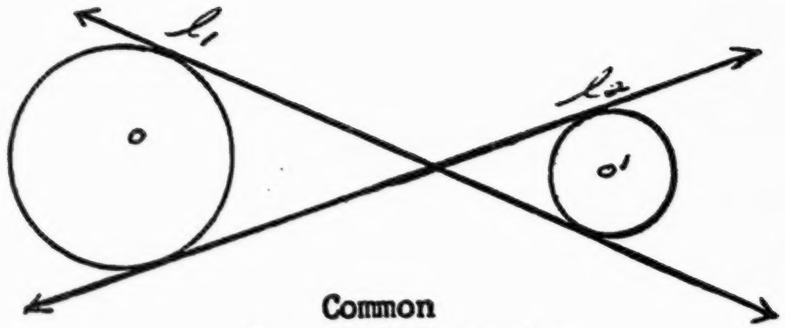


E. Tangent Circles
   1. Tangent internally and externally
   2. Common internal and external tangents

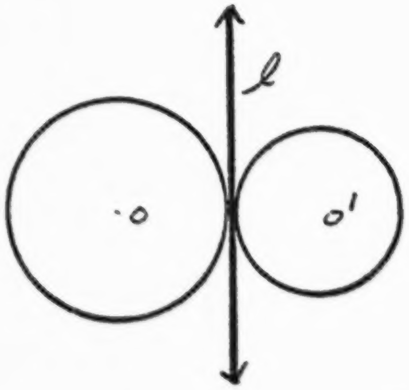
Students should spend some time drawing tangent circles, and circles with common internals and external tangents. They seem to confuse the terms.



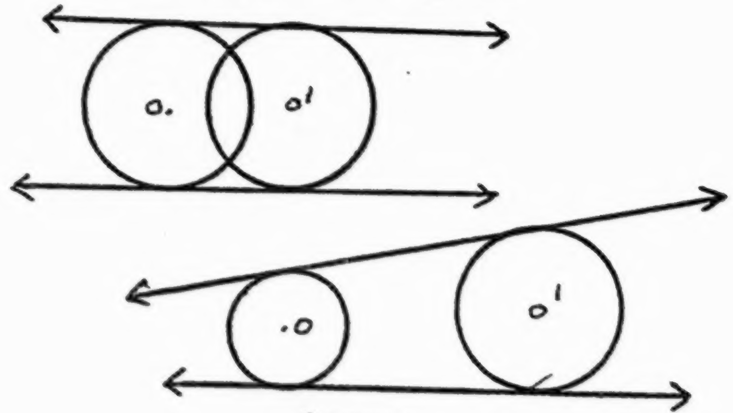
Tangent internally



Common
Internal Tangents



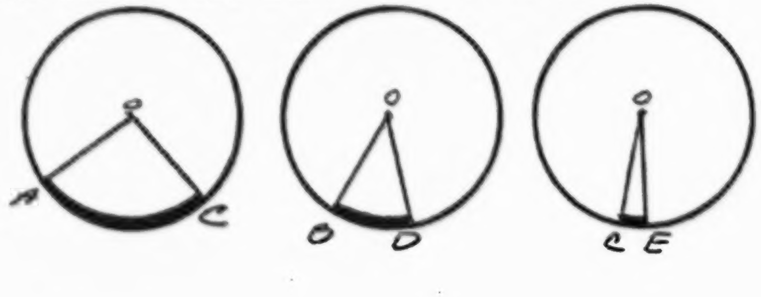
Tangent externally

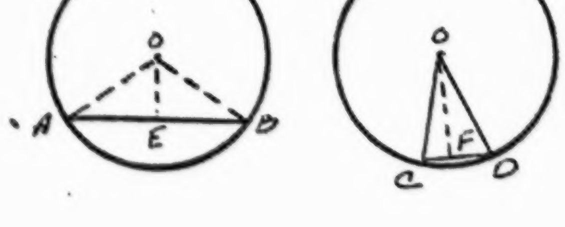


Common
External Tangents

F. Inequalities
   A. Basic Properties
      1. Arcs in a circle
      2. Chords in a circle

Have the students draw several circles to "see" that the greater of two unequal central angles has the greater arc and that in a circle or equal circles the greater of two unequal chords has the greater arc and is nearer the center of the circle.



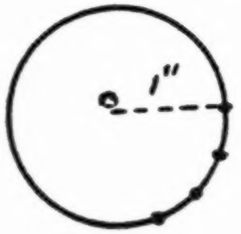


The better students will be able to give formal proof of these basic properties.

X. Loci - Locus of Points
   - A. Definition
   - B. Determining a locus

A locus may be defined as (1) the set of all points, and only those points, that satisfy a given set of conditions, (2) a geometric figure containing all the points, and only those points, that satisfy a given set of conditions, (3) fixing a position - which means that you locate an object or place with respect to other objects or places whose locations are known.

To illustrate: What is the locus of points one inch from a given point on a plane?



All of the points will lie on a circle with the green point as center and a radius of one inch.

Fixing a position: Addressing a letter:

Mr. S. M. Ruhl
2850 N. Jester Street
Ajam, California

California fixes the state, Ajam fixes the city in the state, Jester Street in Ajam, Mr. Ruhl at 2850 Jester Street.

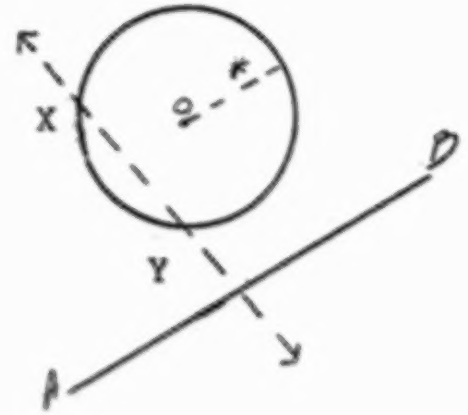
Explain the four important steps in determining a locus.
1. Locate several points which satisfy the given conditions.
2. Draw a smooth line through these points.
3. Describe the geometric figure you think is the locus.
4. Prove that the figure is a locus
   - a. every point of the locus is a point of the figure.
   - b. every point outside the figure is not a point of the locus.

C. Compound locus

Explain that there are times when more than one condition is specified for a set of points. In such cases the compound locus contains only those points common to the loci for the separate conditions.

To illustrate: In a plane, find the locus of points which are equidistant from two points A and B, and at a given distance (d) from point c.

①

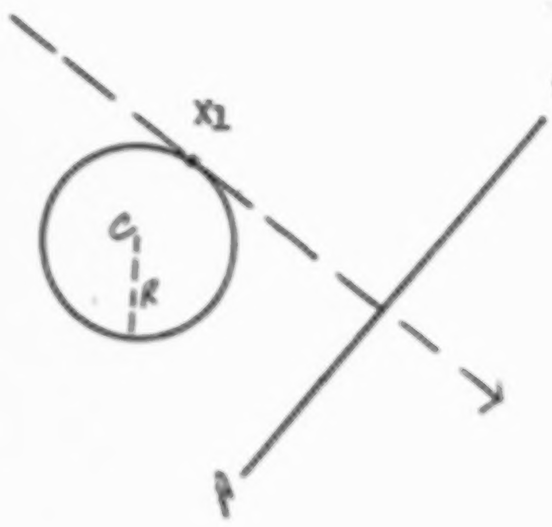


① Could be the points x and y.

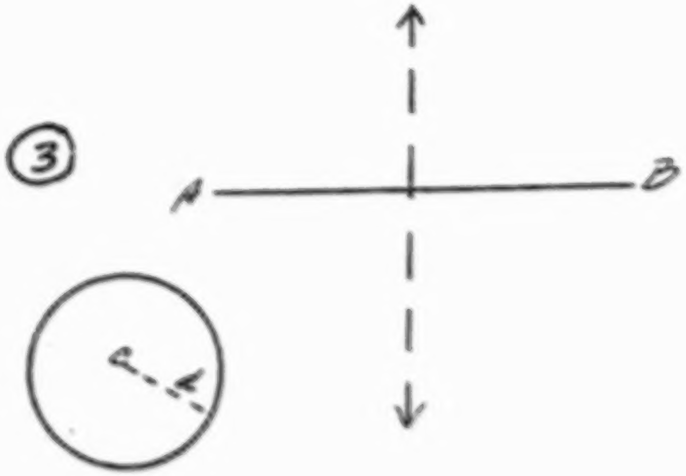
② Could be the point of tangency of line and circle (x_1)

③ Could be no points in common (empty set)

②



③



D. Other Loci
    1. Concurrent lines
    2. Circumcenter
    3. Orthocenter
    4. Incenter
    5. Centroid

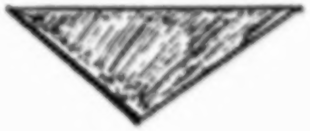
The very definitions of the terms listed here establish them as types of loci.

1. Lines passing through the same point (concurrent lines).
2. The location of the ⊥ bis. of the sides of a △ (circumcenter).
3. The point of intersection of the altitudes of a △ (orthocenter)
4. The point of intersection of the bisectors of the angles of a △ (Incenter).
5. The point of intersection of the medians of a △ (centroid)

If the student has to prove too many locus problems (step 4-A) it takes all the fun out of loci. Real enjoyment comes with compound loci when the student draws a figure and then starts saying to himself, "yes, but what if?"

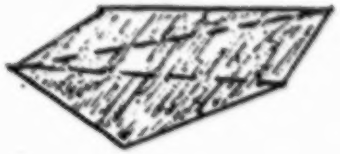
IX. Areas of Polygons and Circles
    A. Regions
        1. Triangular
        2. Polygonal

Region is a comparitively new term used in the teaching of area, and will need explanation and illustration for students who have not had the "new math." A triangular region is a figure that consists of a triangle plus its interior

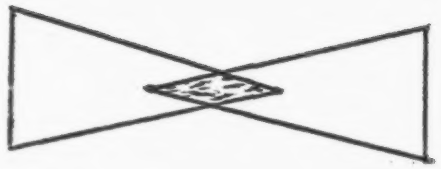


This is the union of a triangle and its interior.

A polygonal region is a figure which can be divided or "cut" into triangular regions.



A union of a finite number of triangular regions so that any intersection is either a segment or a point.

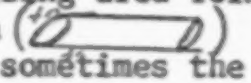


There cannot be any overlapping.
Neither a segment nor a point.

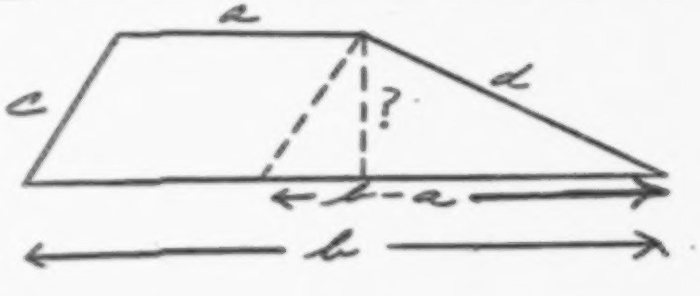
B. Areas of triangles
    1. Any given the base and altitude.
    2. Equilateral
    3. Scalene

Neatness and accuracy are two big problems when students begin this unit. If a figure is drawn neatly it is more easily read, and speed and accuracy of computation usually follow.

C. Areas of Quadrilaterals
    1. Rectangle
    2. Parallelograms
    3. Square
    4. Rhombus
    5. Trapezoid

Students also resist memorizing area formulas, and are careless in checking to be sure that all dimensions are expressed in  terms of just one unit before solving the problem. This unit is the best, and sometimes the only review of arithmetic the student has in high school.

Problems should be varied, and a sufficient number should be given so that the student becomes efficient. The text by Reichjate and Spiller has some excellent practice problems. Many geometry books word all the problems so that the students solve for the area only. Practice should be had in finding the base given the area and altitude of a △, and in a trapezoid, given the non-parallel sides and both bases to find the altitude.



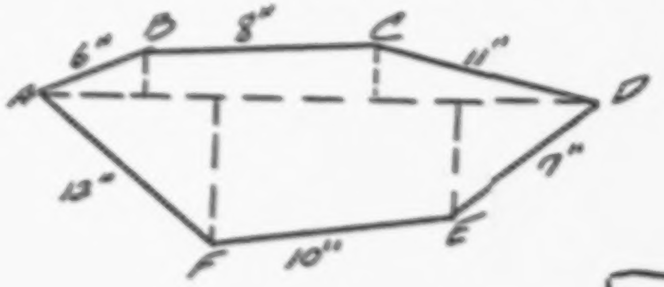
D. Areas of Regular Polygons
   1. Hexagon
   2. N-gon

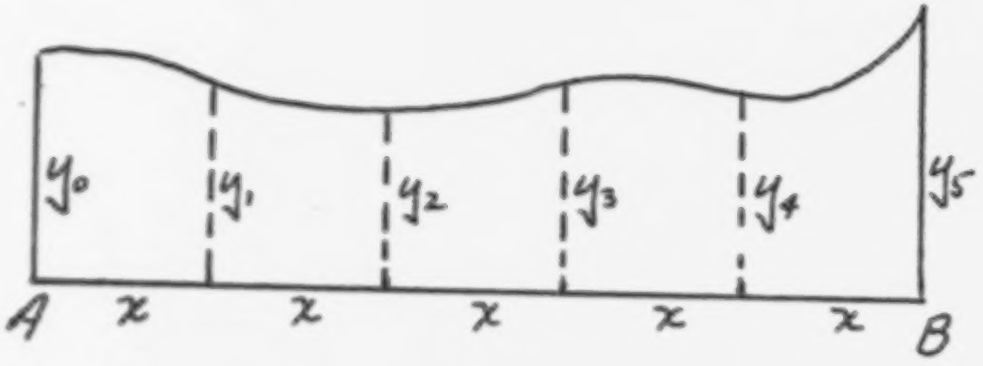
Have the students inscribe in a circle and circumscribe by a circle a regular polygon and explain and demonstrate the radius and apothem.

Students will almost always choose a hexagon or an octagon, but they should be required to use a pentagon and heptagon also.

E. Approximate Areas of Irregular Polygons

The student should have practice in separating a polygon into triangles and trapezoids, and in reaching an approximate area when the figure does not lend itself to known figures.



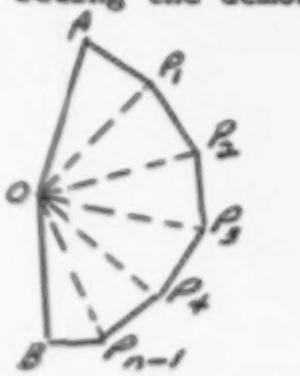


Divide AB into any number of equal segments. Draw $\perp^s$ to AB and extend the $\perp$ to meet the curve CD. Extend the trapezoid formula and use

$$A = x \left(\frac{y^0 + yn}{2} + y_1 + y_2 + y_3 + \text{---}\ y_n - 1\right)$$

F. Circles
   1. Area
   2. Length of an arc
   3. Area of Sector
   4. Area of Segment

Have students draw several figures after seeing the demonstration of the meaning of the length of an arc.

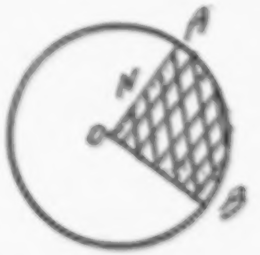


The length of the arc $\widehat{AB}$ is the limit of the sum of the lengths of the chords as n increases indefinitely.

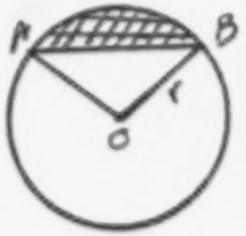
(Recall from geometric thread designs that a polygon approaches a circle as a limit)

Length of an arc = $\dfrac{\text{degree measure of the arc}}{360^\circ}(2\pi r)$

Students will understand sectors and segments better if they draw a circle, draw in two radii, and shade the area to show a sector and a segment.



area = $\dfrac{\text{degrees in central}\angle}{360^\circ}(\pi r^2)$

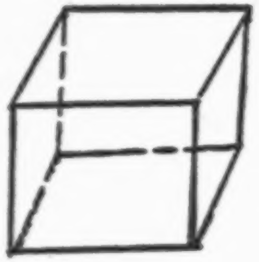


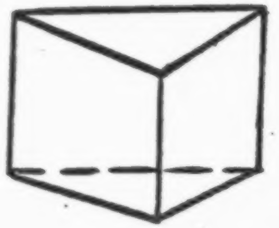
Area of segment =
area of sector - area of △

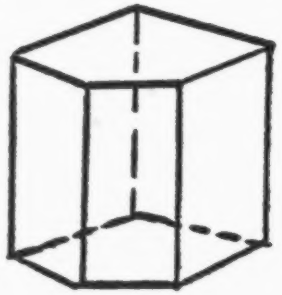
XII. Areas and Volumes of Solids

A. Vocabulary
  1. Prisms and Pyramids
    a. Lateral faces
    b. lateral edges
    c. right and oblique prisms
    d. altitude
    e. slant height
    f. regular pyramid
    g. lateral area
    h. total area
    i. volume
    j. frustum of a pyramid

Practice in drawing prisms and pyramids neatly will make the solution of problems much easier and will save time.

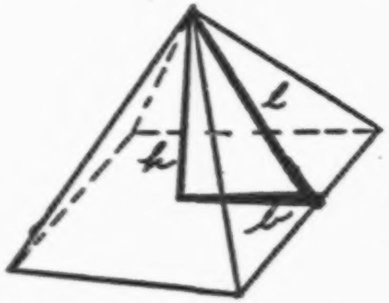






Students should have the vocabulary firmly fixes in their minds before any problem assignments are made.

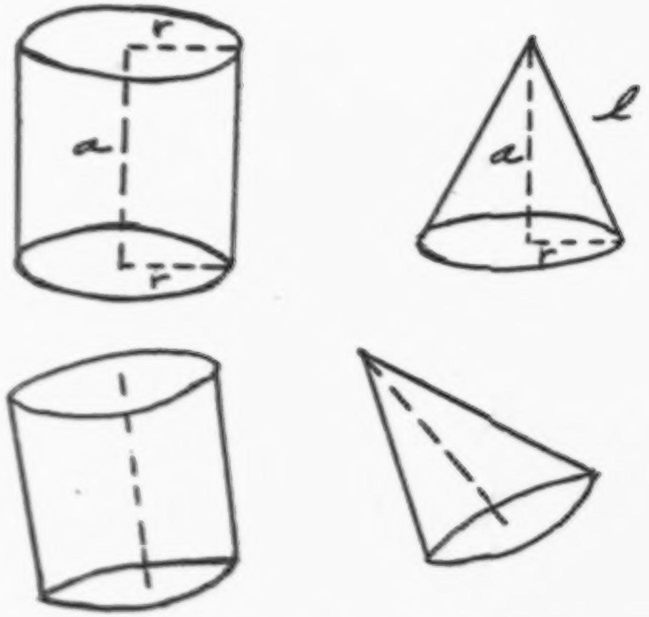
Have the students bring in illustrated articles from magazines showing the uses of geometric solids.



regular pyramid
ℓ = slant height
h = altitude
b = $\frac{1}{2}$ the length of a side of the base of a pyramid.

The students should understand the formulas for area and volume, and should be required to memorize them.

2. Cylinders and Cones
   a. circular cylinder, right and oblique
   b. circular cones, right and oblique
   c. axis of cone and cylinder
   d. slant height of a cone
   e. lateral surface
   f. volume
   h. frustum of a cone



r = radius of circular base
a = axis (height) of cylinder

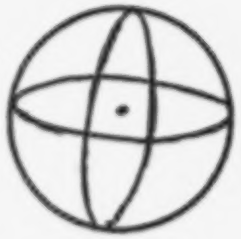
ℓ = slant height
a = axis (height)
r = radius of the base

Again the student should draw figures and label them until he has a "friendly" feeling toward them.

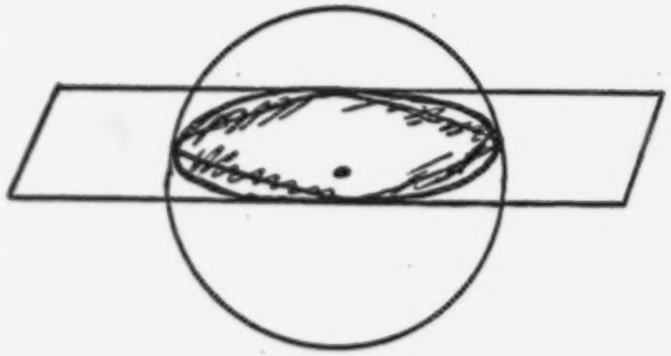
Students usually work with regular figures. They should understand that a right circular cylinder is generated by rotating a rectangle about one of its sides as an axis, and that a right circular cone is generated by rotating a right triangle about one of its legs as an axis. Have the students make models using acetate, balsa wood, etc.

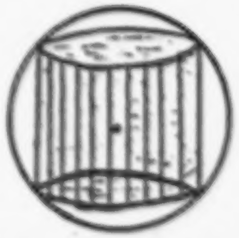
There are many elegant commercial models to use for class demonstrations, but students enjoy making models, and they "learn by doing."

3. Spheres
    a. intersection of a plane and a sphere
    b. circumscribed and inscribed sphere
    c. spherical sector
    d. spherical segment
    e. lateral area
    f. volume
        1. sphere
        2. segment
        3. sector
        4. cone

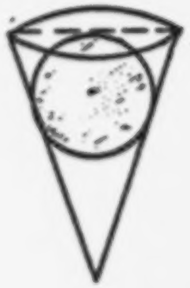


cut by a plane

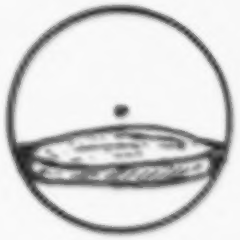




Circumscribed

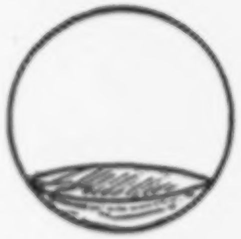


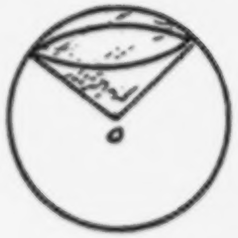
Inscribed



Segment of two bases

Segment of
one base





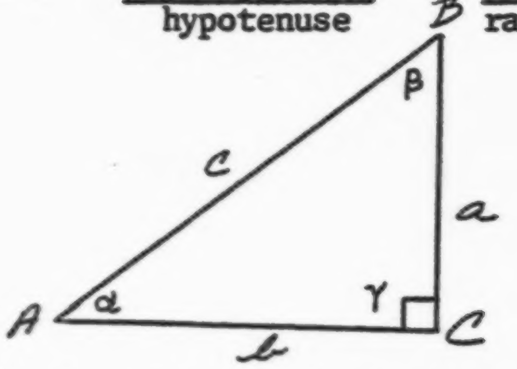
Spherical sector
and cone

Have the students practice drawing figures, and be sure they understand the formulas for the volume of a segment, sector, and cone before assigning problems to be solved.

XIII. Numerical Trigonometry
- A. The Trigonometric Ratios
    1. Sine ratio
    2. Cosine ratio
    3. Tangent ratio

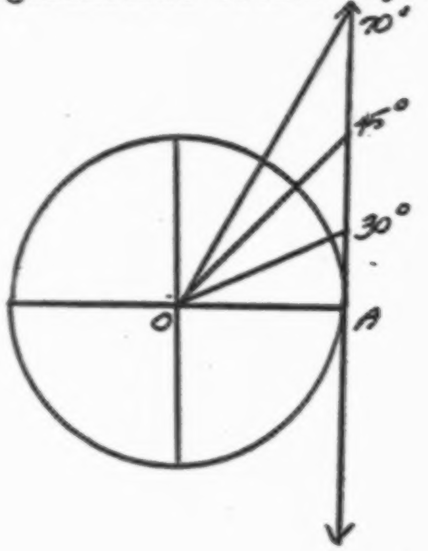
Draw, label, and explain the right triangle in relationship to the trigonometric ratios. The rectangular coordinate system should be used in illustrations.

$$\sin \alpha = \frac{\text{side opp. } A}{\text{hypotenuse}} = \frac{\text{ordinate}}{\text{radius vector}} = \frac{a}{c}$$

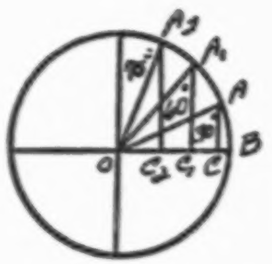


The student must understand that it is the ratio of the length of the side opposite A to the length of the hypotenuse.

Use a graph board, draw a circle, use a protractor to measure angles of 30 , 45 , and 70 . Explain how values of the trigonometric ratios may be estimated.



values of tangent ratio



Estimating values of the sine ratio.

Students have some difficulty in seeing that the rules which apply to proportions in arithmetic and algebra also apply to the trigonometric ratios.

$$\sin\alpha = a/c$$

$$c \sin\alpha = a$$

$$c = \frac{a}{\sin\alpha}$$

B. How to read the table of trigonometric ratios
  1. Natural functions

Mathematical tables from the "Handbook of Chemistry and Physics" are available in inexpensive paperbacks, and are a great help to students here and in later classes in mathematics and physics. Many students prefer to buy their own, but a set could be provided for each class.

Students should understand that the values in these tables are approximate numbers, usually carried to four decimal places.

Oral and written practice in finding the sine, cosine, and tangent of various angles expressed in degrees and minutes, and practice in finding the measure of the angle when the function is known, will detect any misunderstanding on the part of the student.

C. The reciprocal trigonometric ratios
   1. Cosecant ratio
   2. Secant ratio
   3. Cotangent ratio

Explain the reciprocal relations and give examples where they might simplify the solution of a problem (Students find it easier to multiply than to divide) Give simple problems such as $a = 6$, $b = 8$, $\alpha = ?$.

$$\alpha = 35^\circ 15', \; c = 15, \; a = ?, \; b = ?$$

until the student has confidence in his ability to use the tables, and then give examples such as "What is the slope of a river bank in degrees and minutes, if it rises 8 feet vertically for every 3 feet horizontally." The students should have many problems dealing with the right triangle.

Students enjoy working with logarithms and they learn to be careful with computations. This unit is more for fun than for practical use. It could be omitted.

D. Logarithms
   1. Common logarithms
      a. Computation with logarithms

Define a logarithm as an exponent. The equation $10^x = N$ can be written to show the relation between a number N and its logarithm X. $\log_{10} x = N$ expresses the same relationship and is read "the log of x to the base 10 is N."

Explain to the student that the laws of exponents learned in algebra will apply to logarithms.

$(x^2)(x^3) = x^5$ In multiplication, the exponents are added. If two numbers, A and B are to be multiplied, find the logarithm of each number and add. If the same two numbers (A, B) are to be divided, find the logarithm of each number and subtract.

If a number, x, is to be raised to the nth power, find the log of x, multiply by n, and find the antilog.

If the root of a number x is to be found, find the log of x, divide by n, and find the antilog.

2. The two parts of a logarithm
   a. characteristic
   b. mantissa

Students have the greatest difficulty with the characteristic of a logarithm of a number. One method to find the characteristic is to convert the number to scientific notation: $1.56 \rightarrow 1.56 \times 10^{0}$, $27.6 \rightarrow 2.76 \times 10^{1}$, 3250 3.25×10^{3}, .0268 2.68×10^{-2} .

The exponent of 10 is the characteristic of the log of the number.

The student should have lots of practice giving the characteristic of numbers before dealing with the mantissa.

This is a good oral exercise: Hand out a dittoed sheet and have the students call out the characteristic of the numbers.

The mantissa can be found in the math tables book, and is usually expressed as a five place decimal (Most textbooks have four place decimals).

log 1.56 = 0.19312 (five place)
log 1.56 = 0.1931 (four place)
log 27.6 = 1.44091
log 3250 = 3.51188

The students should have practice multiplying, dividing, raising to a power, and finding the root of a number until they become sure of themselves in the use of logarithms.

E. Logarithms of the Values of Trigonometric Functions

Values of trigonometric functions are real numbers and logarithms can be found for any such values that are positive.

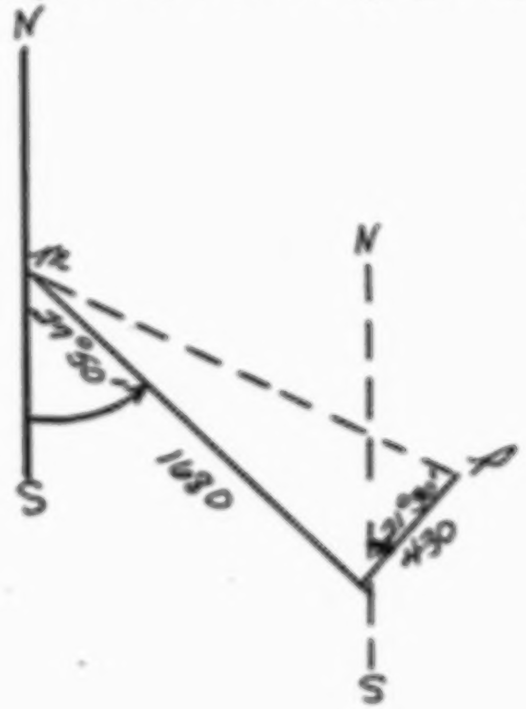
The math tables book gives five place logarithms for sin θ, cos θ, tan θ, cot θ in degrees and minutes.
log sin $35^{\circ}\ 18'$ = 9.76182

The book explains that you subtract 10 from the characteristic of each logarithm. (9.76182 - 10)

F. Solving Oblique Triangles
   1. Law of sines
   2. Law of cosines
   3. Law of tangents
   4. Tangent half-angle law

This can be an optional topic. Many of the new trigonometry textbooks spend little time on the solution of oblique triangles. The tangent half angle law is omitted completely from some books.

Probably time and the interest of the students should determine whether the topic is introduced. Some of the brighter students will thoroughly enjoy working surveying and navigations problems. Example: Find the distance and bearing of P from M if a surveyor goes from M to P by the indicated path. First 1680 yds. s 27°50' E; then 430 yds. N 21°30' E



The student could use
the law of tangents and
the law of sines and
check by measurement.

Answer: 1437 yds.
S.40° 57'E

XIV. Vectors and Air Navigation - Optional
   A. Introduction

This topic is not covered in geometry textbooks any more, but it is a topic of interest to a great many students, and physics teachers appreciate it if students have some knowledge of vectors, velocity, the parallelogram of vectors, etc. The problems in this unit are a very good exercise in careful drawing and measuring.

B. True Heading, True Course, Drift

True heading is the angular direction of the longitudinal axis (axis from tail to nose) of the plane with respect to true north.

True course is the direction over the surface of the earth, expressed as an angle with respect to true north, which the aircraft intends to fly (If is the direction as laid out on a map or chart)

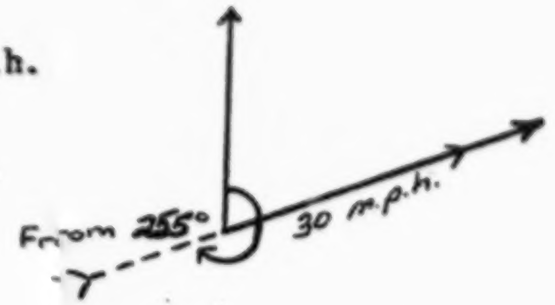
C. Vector

1. magnitude
2. direction

Vectors and Air Navigation

A vector may be a quantity which has both magnitude and direction. It is represented by a line segment to show direction (measured in degrees clockwise from North), and the length of the line segment drawn to some convenient scale shows the magnitude of the vector.

Scale: 1 in. = 20 m.p.h.



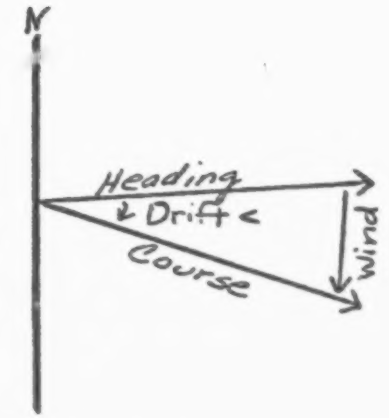
a vector representing
a 30 m.p.h. wind blowing
from direction 255°.

D. Velocity

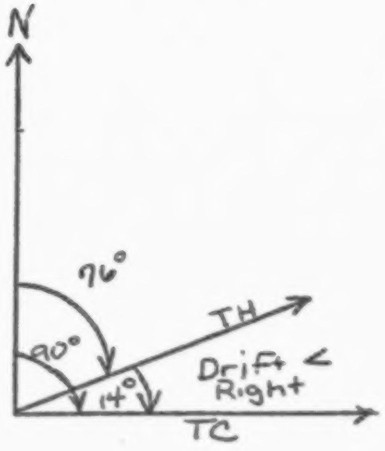
A velocity consists of both speed and direction. It is a rate of change of position in a given direction.

The figure directly above is a graphical representation of a wind velocity.

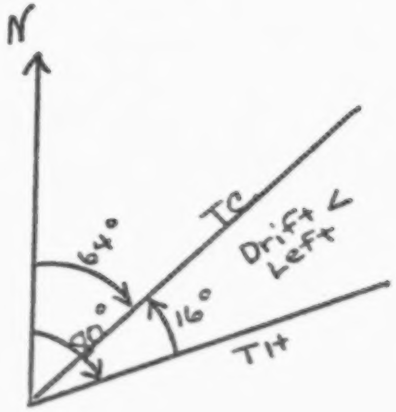
Drift, or drift angle, is the angle between the heading and the course. If the drift angle is clockwise, it is called right drift; if this angle is counterclockwise, it is called left drift.



TH→true heading

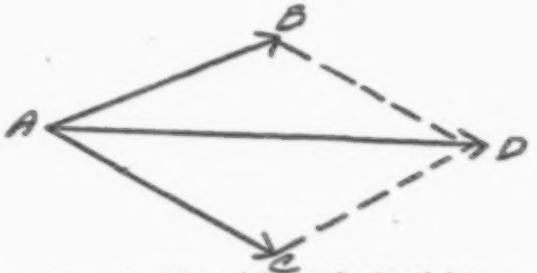


TC→true course



E. Addition of Vectors
   1. Vector sum
   2. Resultant

When an object is acted upon at the same time by two forces that are not parallel or skew, the object moves in a direction different from that of either force. This direction may be represented by a single force which has the same effect on the object as did the two forces working together. This single force is called the vector sum of the two forces; the single vector is called the resultant of the two vectors representing the original forces. The parallelogram of forces (a law of Physics) illustrates the above.

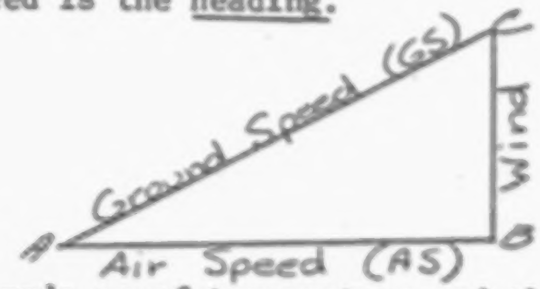


Give the two vectors AB and AC, construct $BD \parallel AC$ and $CD \parallel AB$. The diagonal AD is the resultant representing the vector sum of the vectors AB and AC.

In air navigation, it is not always necessary to construct the complete ▱ to find the resultant. Construct only one of the △s. The two △s of the parallelogram are ≅ (Proved in an earlier unit).

F. Triangle of velocity
   1. air speed
   2. ground speed
   3. six parts to the triangle of velocity

An aircraft is subject to two velocities: Wind velocity and velocity through the air due to the force of the propeller. The amount of magnitude of the velocity due to the propeller is called the air speed. The direction of the air speed is the heading.



AB → air speed vector
BC → winds vector
AC → resultant

The triangle of velocity is made up of two vectors and their sum or resultant. AB and BC are the components of AC.

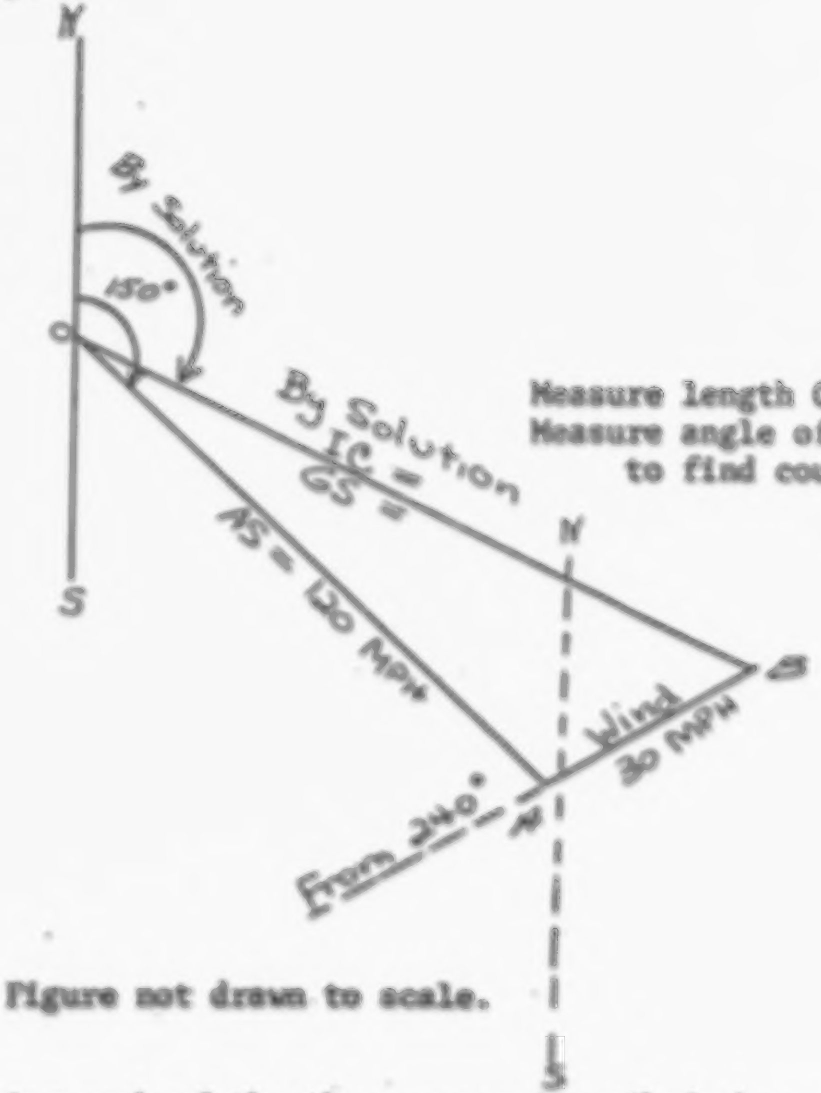
Ground speed is the actual speed of the plane over the surface of the earth.

Note carefully: Air speed is in the same direction as the heading. Ground speed is in the same direction as the course. There are six parts to any triangle of velocity. These parts consist of three lengths and three directions. Given four parts, we can find the other two parts. Three parts is sufficient, if one of the three is a length.

G. Three elementary cases in which the △ of velocity is used.

I. Given: Wind, 30 m.p.h. from 240, TH 150°, AS 120 m.p.h.
Find: GS and TC

Scale: 1" = 30 m.p.h.



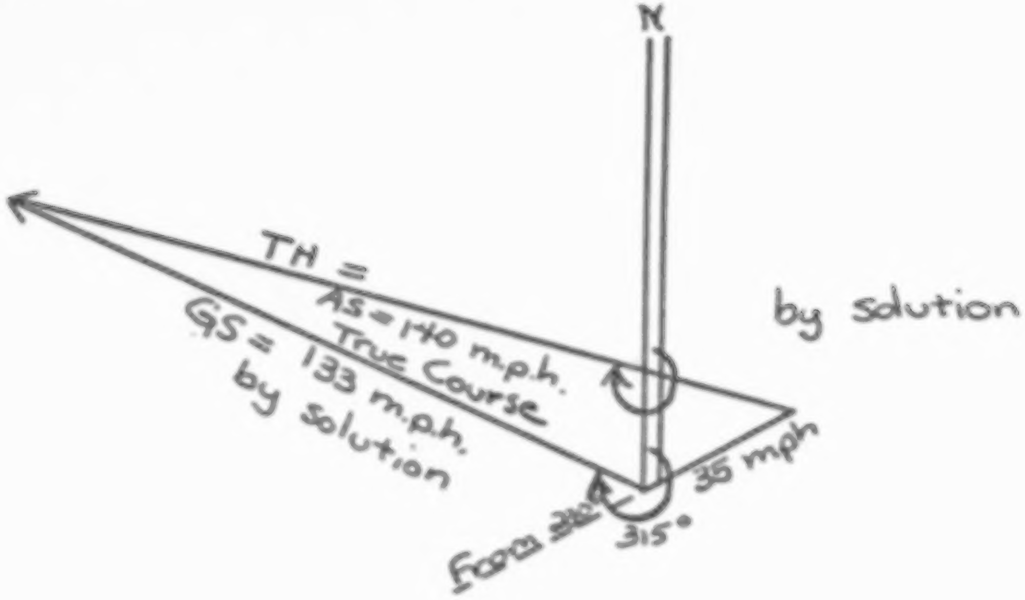
Measure length OB to find GS.
Measure angle of OB from North
to find course (TC).

Figure not drawn to scale.

Have students work several problems using each of the three cases described above.

II. **Given:** Wind, 35 mph from 230°.
Air speed 140 mph
Course 315
Find: GS, TH

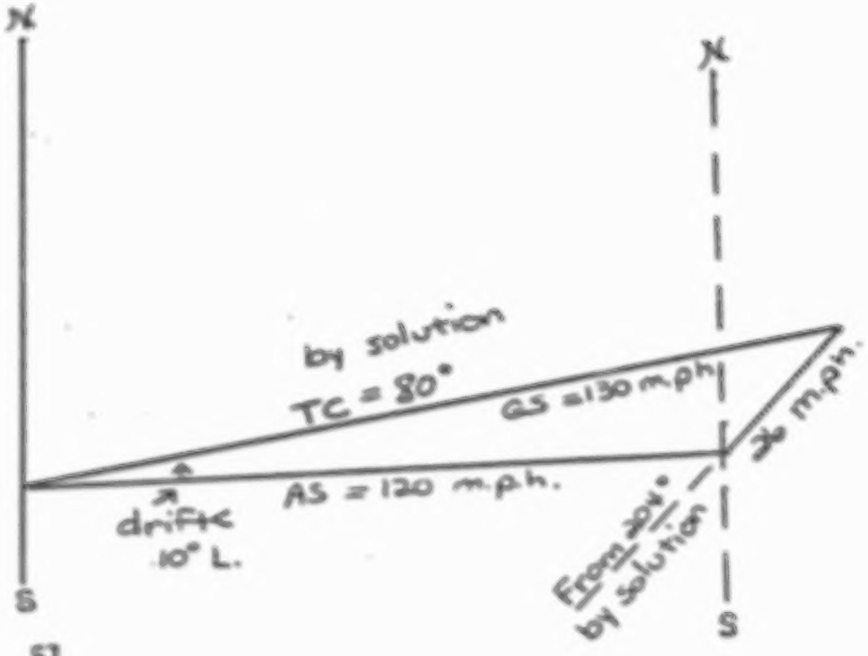
Scale: 1" = 35 m.p.h.



III. **Given:** True headings, 90
Air speed 120 m.p.h.
Ground speed 130 m.p.h.
Drift 10° left

Scale: 1" = 30 m.p.h.

Figure not drawn to scale.



Reference Books

1. S.M.S.G. Geometry. Yale University Press
2. Modern Geometry. Jurgensen-Donnelly-Dolciani, Houghton-Miffin Company
3. Geometry. Moise-Downs, Addision Wesley Company
4. Plane Geometry. Welchons-Krickenberger-Pearson, Ginn and Company
5. Plane Geometry. Fehr, D.C. Heath
6. Modern Geometry. Jurgenson-Weeks, Houghton Mifflin Company
8. High School Mathematics Course 2. Beberman-Vaugh, D.C. Heath

ED0 26270

O.C.S.E.I.P. SYLLABUS

Trigonometry and Introduction to College Mathematics

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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INTRODUCTION

The Orange County Science Education Improvement Program (O.C.S.E.I.P.) is sponsored by the National Science Foundation and hosted by U.C. Irvine. It is a cooperative venture undertaken by the University of California, Irvine, California State College at Fullerton, the Orange County Schools Office and local school districts throughout Orange County. This syllabus was written by O.C.S.E.I.P. to help teachers teach the best aspects of recent mathematics programs. It is not meant to be another textbook for a new program. Instead, it is meant to be a sharing and synthesis of effective teaching methods. The outline of topics is a minimum coverage which is common to all schools in Orange County. Topics adequately covered in the majority of texts in use are given a minimum treatment in the syllabus.

The first draft of this syllabus was written during an 8 week session at University of California, Irvine during the summer of 1966 by:

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PREFACE

ANALYTIC TRIGONOMETRY AND INTRODUCTORY COLLEGE MATHEMATICS

The purpose of this syllabus is to present topics to be covered rather than as an aid to teaching, although some suggestions and comments are made.

With reorganization within the mathematics department, there is confusion as to what should be included in the fourth year of high school mathematics.

It is the hope of this committee that the topics presented here will prepare the student to enter calculus and will also serve as a terminal course for other students.

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ANALYTIC TRIGONOMETRY

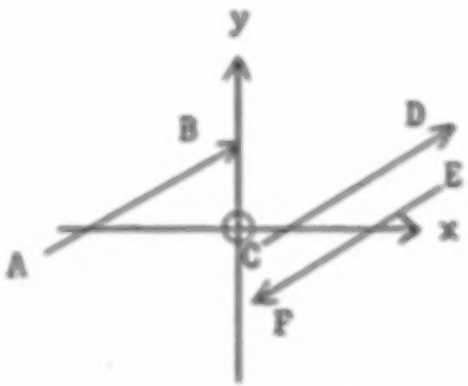
I. Vector Quantities

A. Definition and symbol for vectors

Explain that a quantity that involves both magnitude (the property of having size) and direction is said to be a vector quantity. Force, displacement, velocity, and acceleration are vector quantities.

B. Equivalent vectors

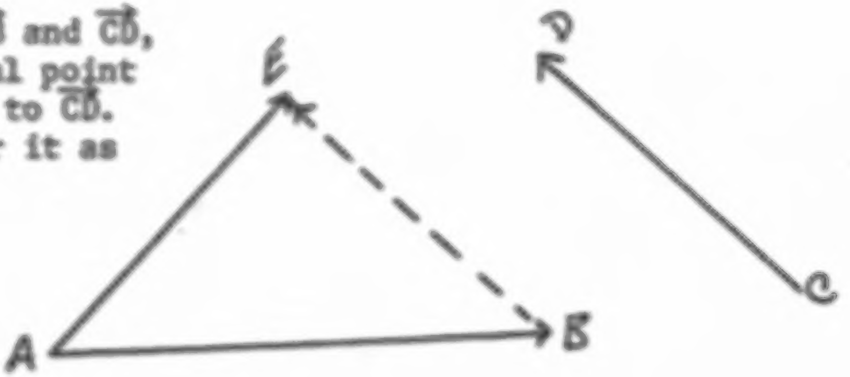
A directed line segment $\overrightarrow{AB}$ is often used to represent a vector. We may name a vector by first naming its initial point and then naming its terminal point. A two-dimensional vector is an ordered pair (x,y) of real numbers.



Demonstrate that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent vectors since they have = lengths and the same direction. $\overrightarrow{EF}$ is not equivalent to either $\overrightarrow{AB}$ or $\overrightarrow{CD}$ since, although it appears to have the same length, it is not heading in the same direction. We may say that $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{EF} \neq \overrightarrow{AB}$ and $\overrightarrow{EF} \neq \overrightarrow{CD}$. The sense of the vector $\overrightarrow{EF}$ is the opposite of the sense of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. To indicate that $\overrightarrow{AB}$ (or $\overrightarrow{CD}$) have the same length but opposite directions we may write $\overrightarrow{AB} = -\overrightarrow{EF}$.

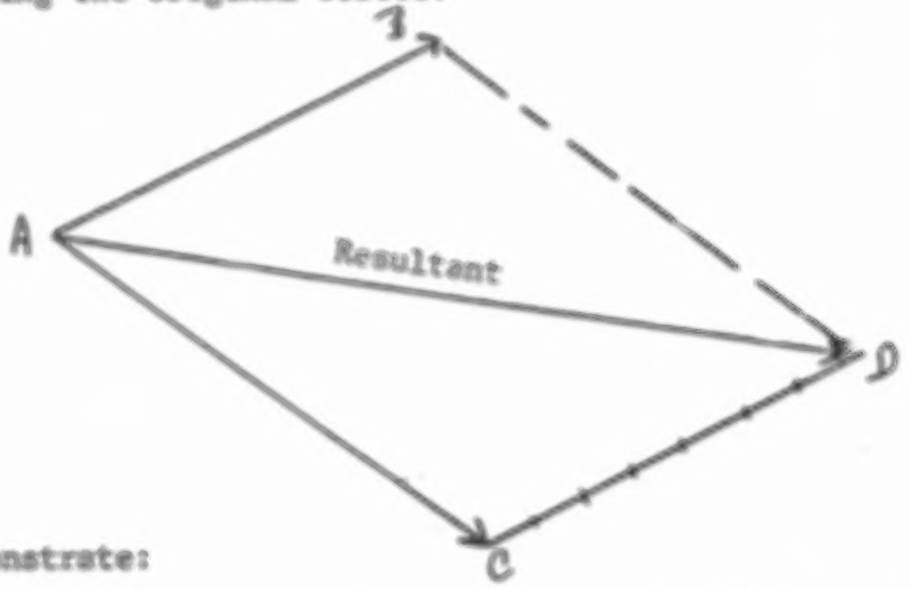
C. Adding Vectors

To add any two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, first through the terminal point of $\overrightarrow{AB}$ draw $\overrightarrow{BE}$ equivalent to $\overrightarrow{CD}$. Then draw $\overrightarrow{AE}$ and consider it as the sum $\overrightarrow{AB} + \overrightarrow{BE}$.



This is sometimes called the parallelogram law of vectors. Many Vector Quantities in Physics obey this law.

When an object is acted upon at the same time by two forces that are not parallel, the object moves in a direction different from either force. This direction may be represented by a single vector that shows a single force which has the same effect upon the object as did the two forces working together. This single force is called the resultant of the two forces; the single vector is called the vector sum of the two vectors representing the original forces.

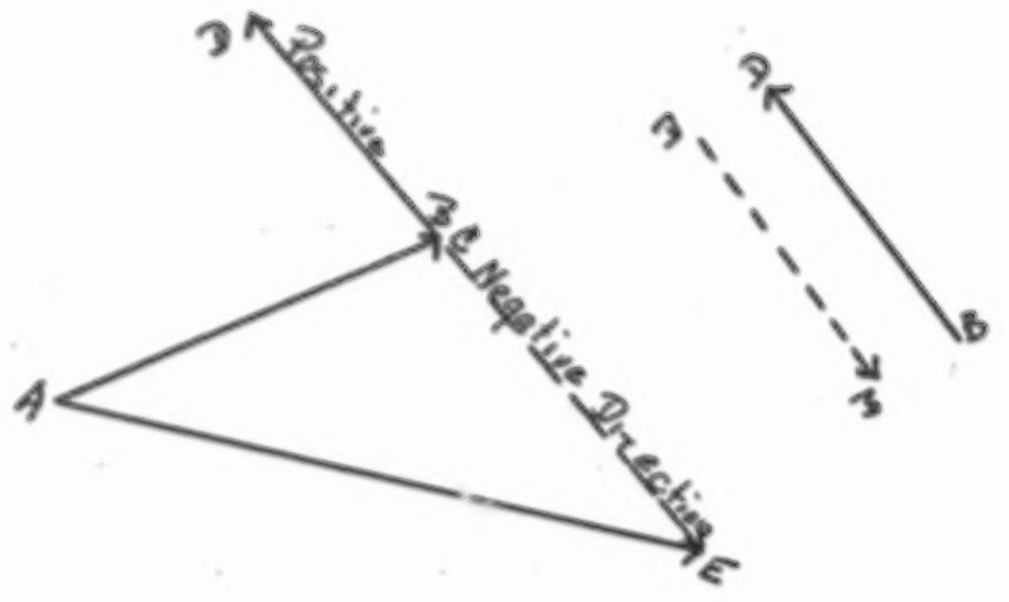


D. Subtracting

Demonstrate:

Subtraction is the inverse operation of addition. Thus if $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ then $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$ and $\overrightarrow{AC} - \overrightarrow{BC} = AB$

To illustrate: To find $\overrightarrow{AB} - \overrightarrow{BD}$. $-\overrightarrow{BD} = \overrightarrow{DB}$ $\overrightarrow{AC} - \overrightarrow{BD} = \overrightarrow{AC} + \overrightarrow{DB}$



$\overrightarrow{AC} - \overrightarrow{BD} = \overrightarrow{AE}$

E. Operations with vectors

Explain and give an example:

The multiplication of a real number and a vector is called scalar multiplication. The number by which you multiply is called a scalar. Multiplying (or dividing) a vector by scalar means multiplying (or dividing) the magnitude by the absolute value.

The scalar product (inner product and dot product)

$V_1 \cdot V_2 = X_1 X_2 + Y_1 Y_2$

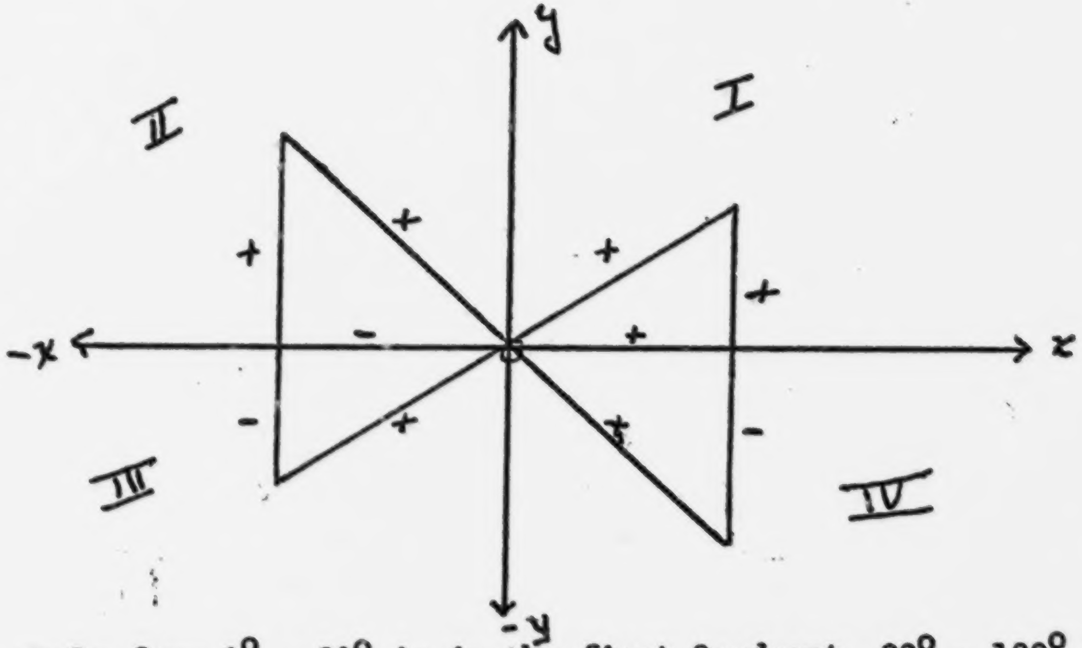
If $V_1 = 3i - 2j$ and $V_2 = 5i - 2j$

$V_1 \cdot V_2 = (3)(5) + (-2)(-2) = 15 + 4 = 19$

II. Trigonometric Functions of Angles

A. Standard position of angles and size of angles

To explain this unit, introduce the unit circle, place the angle in standard position (the vertex at the origin of a rectangular coordinate system and its initial side on the positive end of the x-axis.



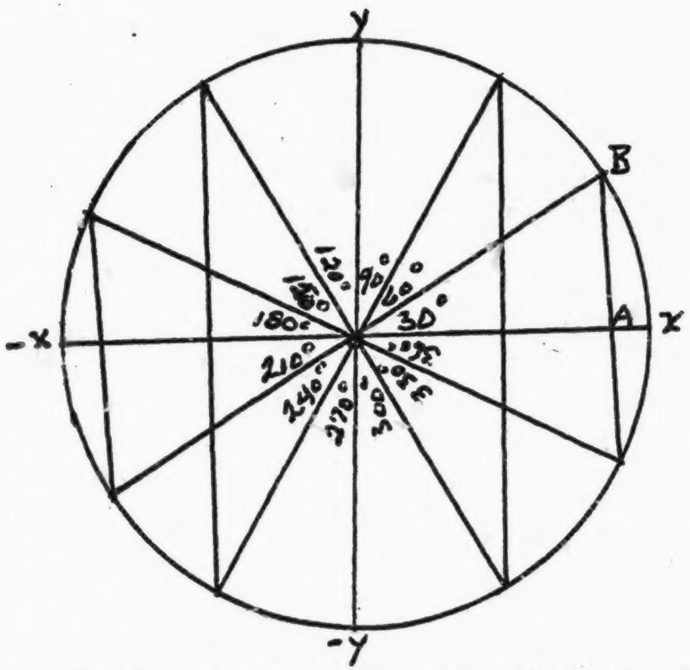
An angle from 0° - 90° is in the first Quadrant, 90° - 180° in the second, 180° - 270° in the third, and 270° - 360° in the fourth.

B. Relations, functions domains, ranges

Explain relations as a set of ordered pairs, the set of first members known as the domain of the relation, and the set of second members known as the range of the relation.

If for each member of the domain of a relation there is exactly one member of the range of the relation, the relation is a function. For each x value in an ordered pair, there is exactly one value of y.

C. Values of functions



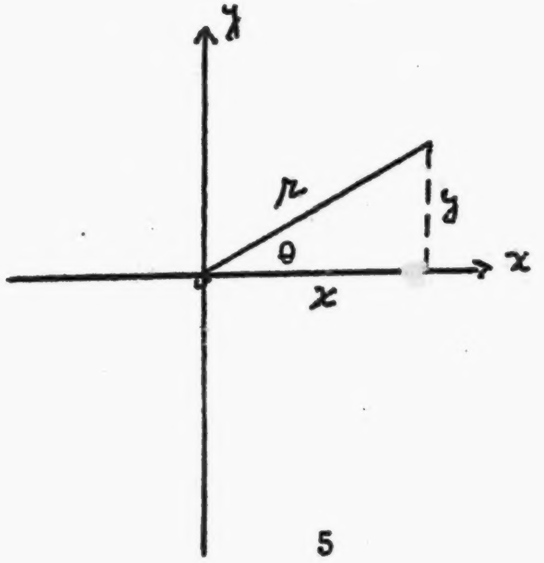
Discuss:

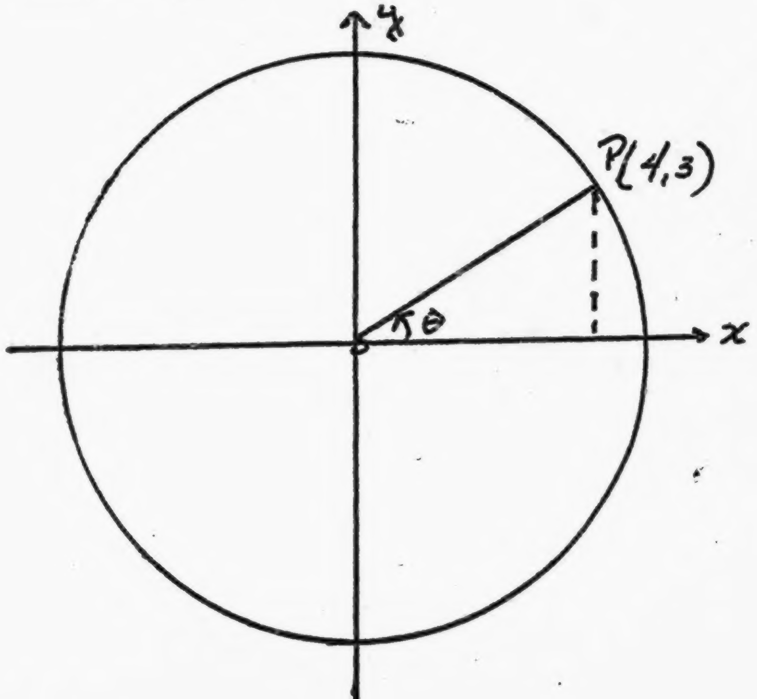
As a point P moves from $0^o - 360^o$, notice what happens to Sin θ, Cos θ, and tan θ

As Sin θ increases from 0 to 1, Cos θ decreases from 1 to 0.

($\overline{AB}$ = Sin θ, $\overline{OA}$ = Cos θ) etc.

Students should have nractice finding the values of the six trigonometric functions of θ using figures such as





$$\text{Sin } \theta = \frac{y}{r} = \frac{\text{Ordinate}}{\text{Radius Vector}}$$

$$\text{Cos } \theta = \frac{x}{r} = \frac{\text{Abscissa}}{\text{Radius Vector}}$$

$$\text{Tan } \theta = \frac{y}{x} = \frac{\text{Ordinate}}{\text{Abscissa}}$$

D. Functions of quadrantal angles

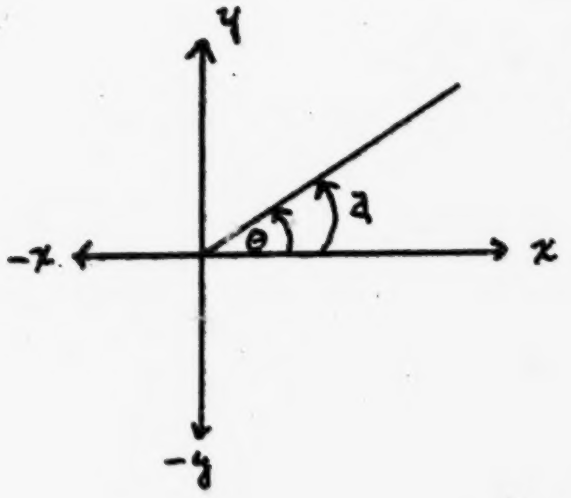
Students should become familiar with the following table.

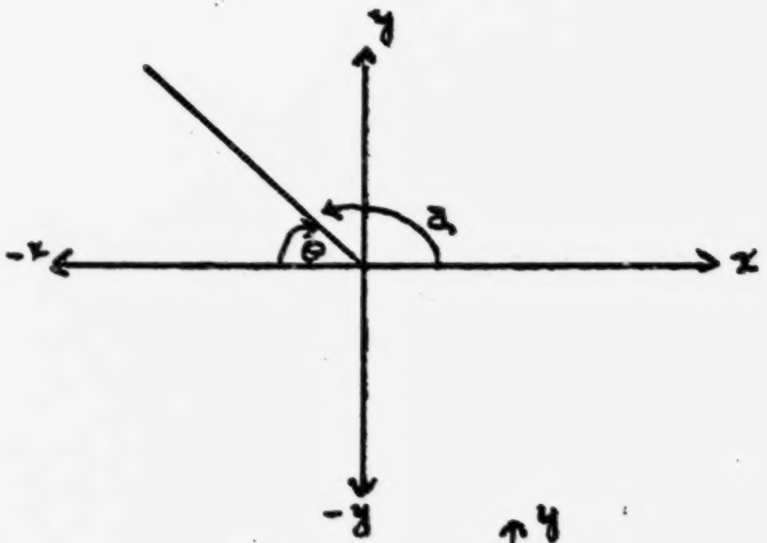
Angle	Sin	Cos	Tan	Cot	Sec	Csc
0°	0	1	0	$\frac{x}{0}$	1	$\frac{x}{0}$
90°	1	0	$\frac{x}{0}$	0	$\frac{x}{0}$	1
180°	0	-1	0	$\frac{x}{0}$	-1	$\frac{x}{0}$
270°	-1	0	$\frac{x}{0}$	0	$\frac{x}{0}$	-1

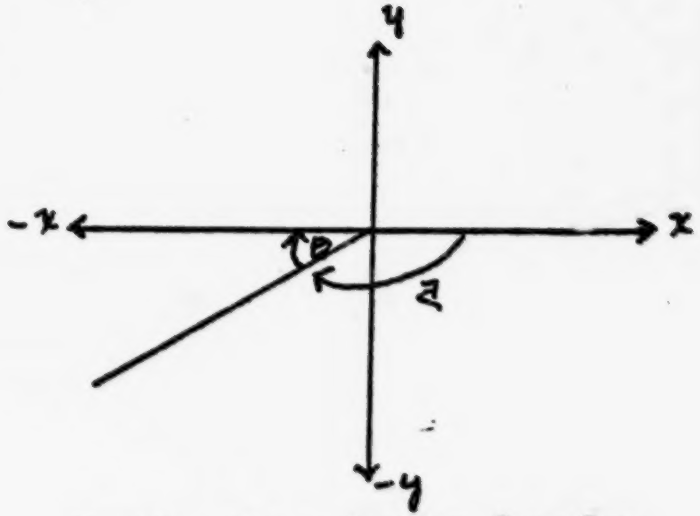
(undefined. $\frac{x}{0}$)

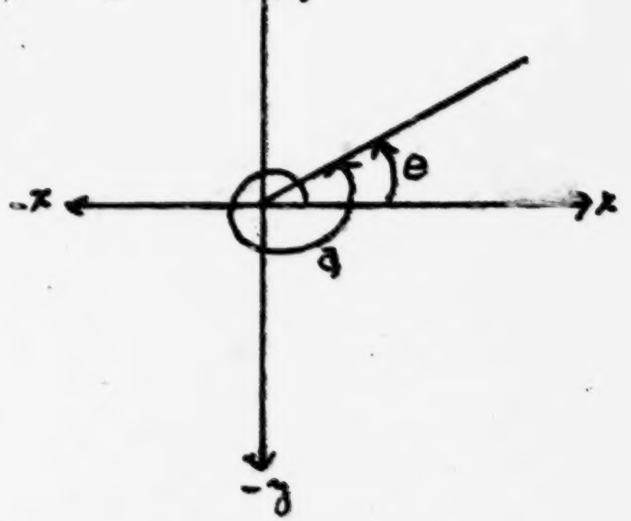
E. Reference angle

If students have math tables books, little time will have to be spent on the reference angle, but it should be explained that the reference angle is the acute angle θ between the terminal side of a and the x-axis.









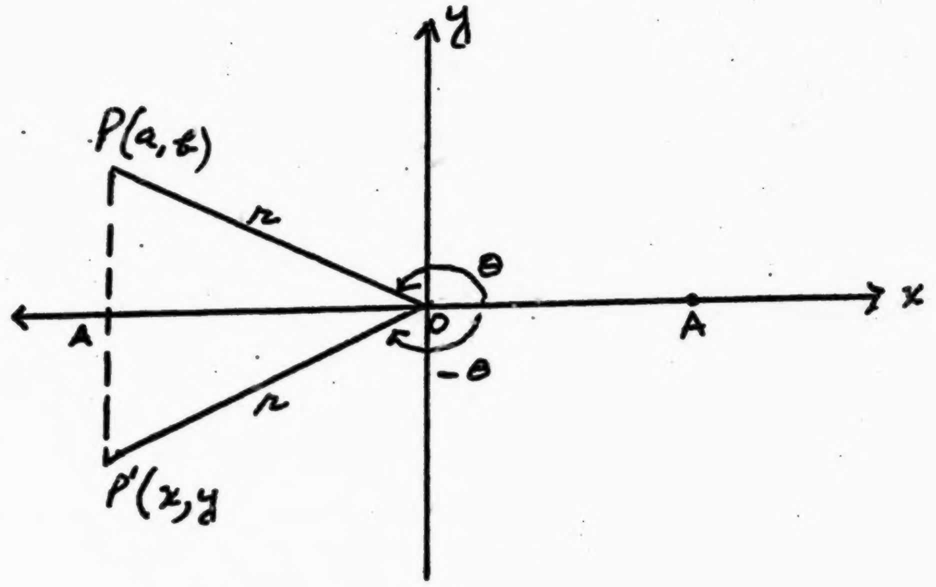
Mention the reduction formulas

$\left[\text{Any function of an (even multiple of } 90^{\circ}) \pm \theta\right] = \pm \text{(Same function of } \theta)$

$\left[\text{Any function of an (odd multiple of) } 90^{\circ} \pm \theta\right] = \pm \text{(Cofunction of } \theta)$

F. Functions of negative angles

Explain to the students that regardless of the size of θ ($\theta \neq \pi$), P and P', points on the terminal sides of θ and negative θ, always lie on opposite sides of the x-axis and on the same side of the y-axis.



$\angle AOP = \angle AOP'$, $\triangle AOP \cong \triangle AOP'$

$\angle PAO = \angle P'AO$, and $PP' \perp$ x-axis

$x = a$ and $y = -b$

$\text{Sin}\ (-\theta) = \frac{y}{r} = -\frac{b}{r} = -\text{Sin}\ \theta$

$\text{Cos}\ (-\theta)\ \frac{x}{r} = \frac{a}{r} = \text{Cos}\ \theta$

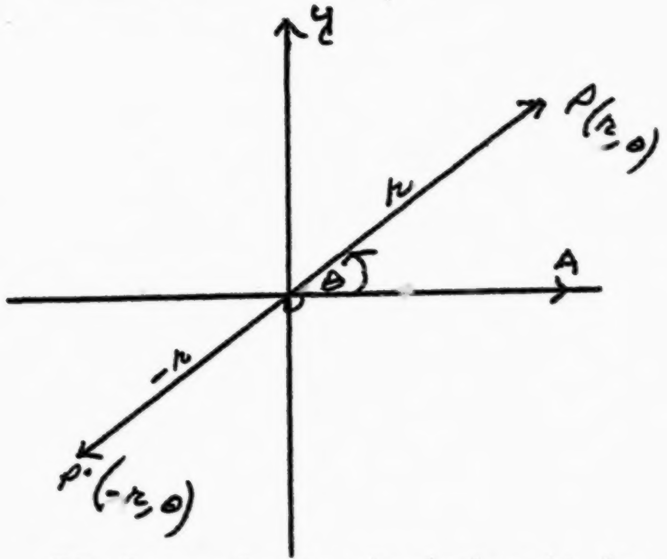
$\bar{\tan}\ (-\theta) = \frac{y}{x} = -\frac{b}{a} = -\tan\ \theta$

III. Polar Coordinates

A. Definitions

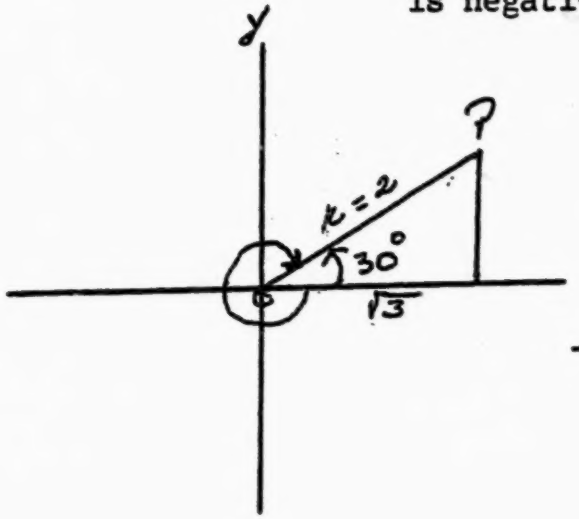
Explain and demonstrate:

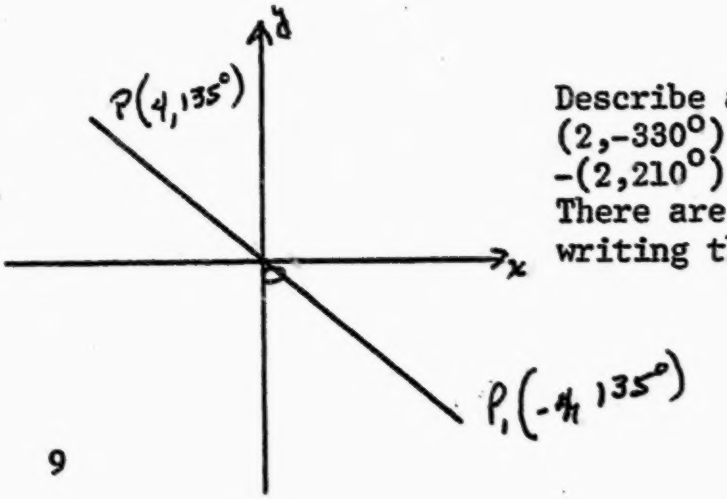
In the system of polar coordinates the position of a point P is the distance r from the origin or pole, and the angle θ that ray $\overrightarrow{OP}$ makes with a fixed ray $\overrightarrow{OA}$ (positive direction) called the polar axis θ is sometimes called the direction angle, and r the polar distance.



r is restricted to be non-negative by some authors

If the angle is clockwise it is negative. When r is measured along the extension of the terminal side through the origin, it is negative.





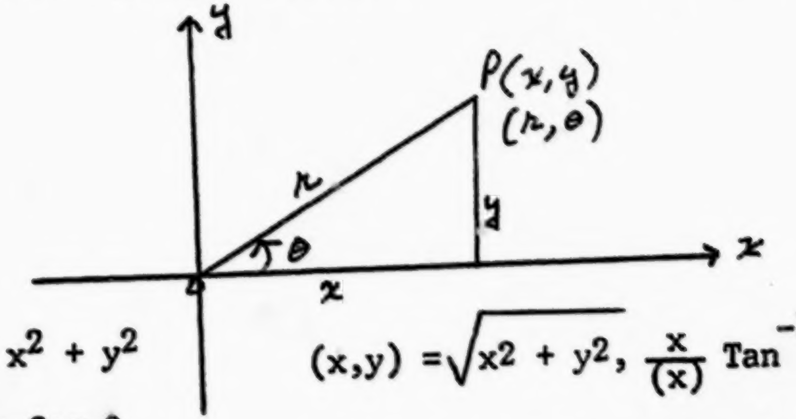
Describe as $(2,30^{\circ})$ and $(2,-330^{\circ})$ or describe $-(2,210^{\circ})$ $(-2,150^{\circ})$. There are many ways of writing the same point.

B. Changing systems

1. Rectangular to polar

2. Polar to rectangular

Students have had the trigonometric functions in Geometry and Algebra II so there should be no difficulty in the explanation of the change from rectangular to polar coordinates.



$r^2 = x^2 + y^2$

$x = r \text{ Cos } \theta$

$y = r \text{ Sin } \theta$

$(x,y) = \sqrt{x2 + y2}, \frac{x}{(x)} \text{ Tan}^{-1} \frac{y}{x}$

$(r, \theta) = (r \text{ Cos } \theta, r \text{ Sin } \theta)$

Students should have practice problems of the type: Given rectangular coordinates P(4,3), change to polar coordinates, and Given polar coordinates (2,60°) change to rectangular coordinates.

Example: P(4,3)

$r = \sqrt{(4)^2 + (3)^2} \qquad r = 5$

$\tan \theta = \frac{3}{4} = .7500$

$\theta = 36^{\circ}52'$ Polar coordinates $(5, 36^{\circ}52')$

IV. Complex Numbers

If students have complex numbers in Algebra II, a brief review here should be sufficient.

A. Definition

Example: A complex number is a number of the form a + bi, where a and b are real numbers and i represents $\sqrt{-1}$

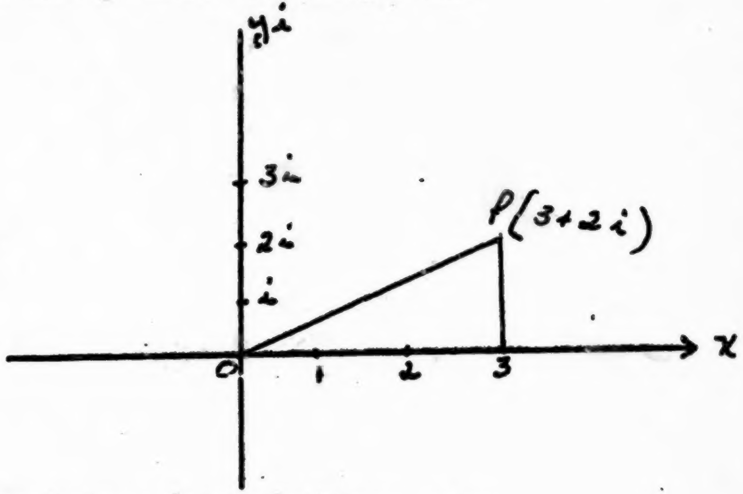
B. Graphing complex numbers

Explain that all numbers represented by a + 0i may be graphed on the x-axis in the coordinate system and that the x-axis is called the Axis of reals. All numbers represented by 0 + bi may be graphed as points on the y-axis, and the y-axis is called the Axis of imaginaries.

Complex plane

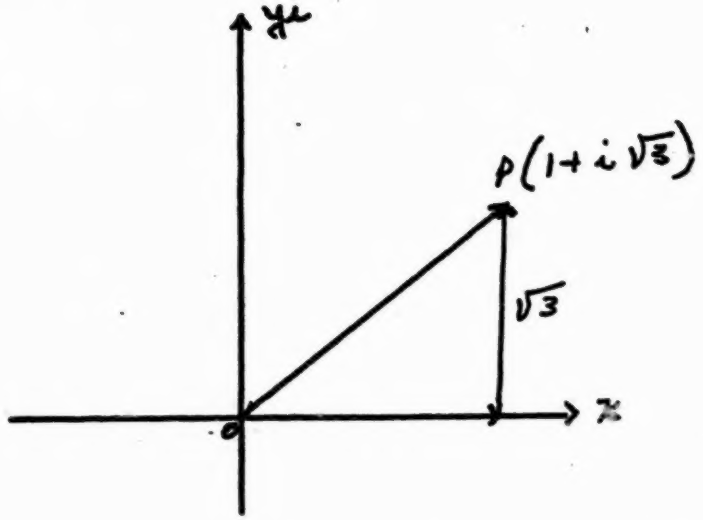
Sometimes called Argand diagram

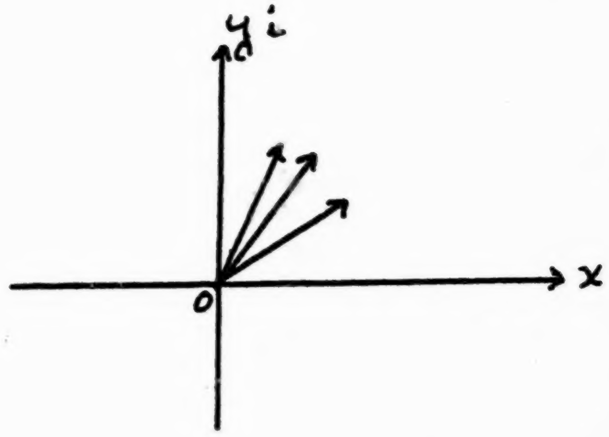
(J.R. Argand, French mathematician)



C. Complex numbers and vectors

Explain that there is a one-to-one correspondence between complex numbers and points on a plane, and a one-to-one correspondence between points on the plane and vectors. There is the same correspondence between complex numbers and vectors. This is especially apparent if the vectors are represented by directed line segments with tails at the origin.

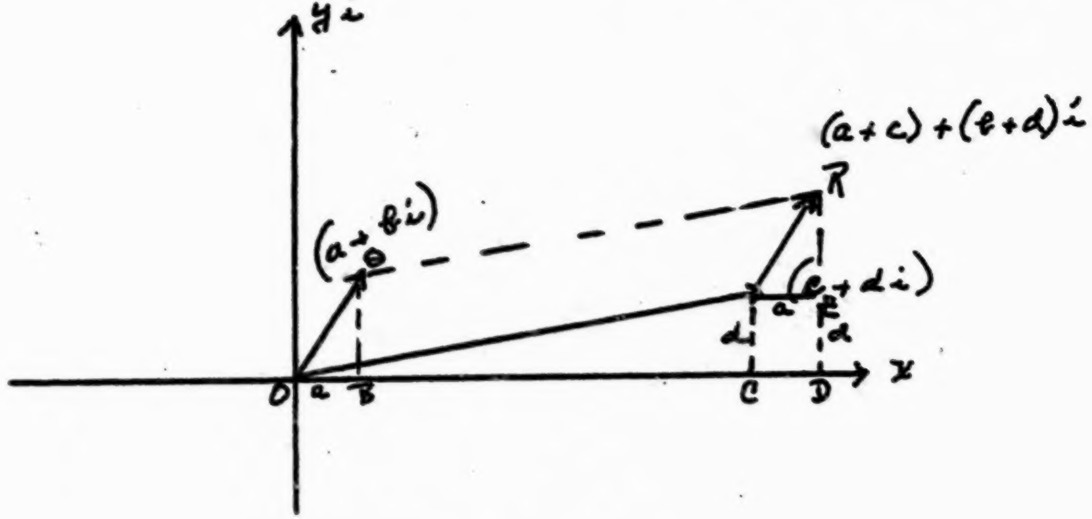




Point P, whose rectangular coordinates are $(1, \sqrt{3})$ [ordered pair], is the terminal point of vector $\overrightarrow{OP}$. P can also represent the complex number $1 + i\sqrt{3}$, so $\overrightarrow{OP}$ corresponds to the complex number $1 + i\sqrt{3}$. The 1 of the complex number expresses the x-component of $\overrightarrow{OP}$ and $\sqrt{3}$ expresses the y-component. It should be pointed out to the student that real numbers are useful in expressing size, but complex numbers are useful for expressing both size and the direction of the vector quantity.

D. Adding and subtracting complex numbers with vectors

In unit I, addition of vectors was explained. The student should see that to add two vectors he has only to add the complex numbers to which they correspond.



$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are two vectors to be added. $\overrightarrow{OQ}$ corresponds to the complex number (a + bi) and $\overrightarrow{OP}$ to (c + di)

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ}$$

To subtract one vector from another, subtract the x-component of one from the x-component of the other, and the y-component of one from the y-component of the other.

E. Polar form of complex number

The student should know how to change x + yi to its polar form.

Example: $x = r \text{ Cos } \theta$

$y = r \text{ Sin } \theta$

Substituting these values in x + yi

$$r \text{ Cos } \theta + ri \text{ Sin } \theta = r\ (\text{Cos } \theta + i \text{ Sin } \theta)$$

r is called the modulus and θ the argument (sometimes called amplitude

$r\ (\text{Cos } \theta + i \text{ Sin } \theta) = r \text{ Cis } \theta$. The word "Cis" means Cosine, i sine.

Example: Express −5 + 2i in polar form

$x = -5,\ y = 2$

$r = \sqrt{x^2 + y^2} \quad = \sqrt{29} \quad = 5.38$

$\tan \theta = \frac{2}{-5} = .4 = 158^\circ\ 10'$

$r \text{ cis } \theta = 5.38 \text{ cis } 150^\circ\ 10'$

Students should have a sufficient number of problems to become efficient.

F. Product of two complex numbers in polar form

Explain the theorem "The product of two complex numbers is a complex number whose modulus is the product of the moduli and whose amplitude or argument is the sum of the argument or amplitudes of the given complex number."

$x_1 = r_1 (\text{Cos } \theta_1 + i \text{ Sin } 0_1)$

$x_2 = r_2 (\text{Cos } \theta_2 + i \text{ Sin } \theta_2)$

$X_1 X_2 = r_1 r_2 \Big[(\text{Cos } \theta_1 \text{ Cos } \theta_2 - \text{Sin } \theta_1 \text{ Sin } \theta_2) +$

$i (\text{Sin } \theta_1 \text{ Cos } \theta_2 + \text{Cos } \theta_1 \text{ Sin } \theta_2)\Big] =$

$r_1 r_2 \Big[\text{Cos } (\theta_1 + \theta_2) + i \text{ Sin } (\theta_1 + 0_2)\Big]$ or

$x_1 x_2 = r_1 r_2 \text{ Cis } (\theta + 0_2)\Big]$

Example: Find the product of $2\sqrt{3} + 2i$ and $-3 + 3i$

Change $2\sqrt{3} + 2i$ to polar form

$r_1 = \sqrt{12 + 4} = 4$ and $\theta_1 = \text{arc tan} \frac{1}{\sqrt{3}} = 30^\circ$

Then $2\sqrt{3} + 2i = 4 \text{ cis } 30^\circ$

Change $-3 + 3i$ to polar form

$r_2 = \sqrt{9 + 9} = 3\sqrt{2}$ and θ_2 arc tan $(-1) = 135^\circ$ or

$3\sqrt{2} \text{ cis } 135^\circ$

$(2\sqrt{3} + 2i)(-3 + 3i) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2) =$

$12\sqrt{2} \text{ cis } 165^\circ$

V. Fundamental Relations

A. Reciprocal quotients pythagorean

The importance of memorizing these relations should be stressed. (In some textbooks these are called fundamental identities.) Practice in simplification of trigonometric expressions should be extensive.

Examples: Simplify Cos θ tan θ   Restrictions (θ not an odd multiple of 90°)

1. $\text{Cos } \theta \tan \theta \longrightarrow \text{Cos } \theta \cdot \dfrac{\text{Sin } \theta}{\text{Cos } \theta} \longrightarrow \text{Sin } \theta$

2. $\dfrac{\text{Sin } \theta}{\text{Tan } \theta} - \dfrac{\text{Tan } \theta}{\text{Sec } \theta} \longrightarrow \dfrac{\text{Sin } \theta}{\dfrac{\text{Sin } \theta}{\text{Cos } \theta}} - \dfrac{\dfrac{\text{Sin } \theta}{\text{Cos } \theta}}{\dfrac{1}{\text{Cos } \theta}}$

$$\text{Sin } \theta \cdot \frac{\text{Cos } \theta}{\text{Sin } \theta} - \frac{\text{Sin } \theta}{\text{Cos } \theta} \cdot \frac{\text{Cos } \theta}{1}$$

$$\text{Cos } \theta - \text{Sin } \theta$$

B. Proving identities

Suggest that the more complicated member be transformed so that it reduces to the simpler member. If no other attack is indicated, express all functions in terms of sine and cosine. Students become very involved if they introduce radicals in these problems. Stay away from indeterminate forms.

Example 1: $\dfrac{\text{Cos } \theta + \text{Sin } \theta}{1 + \text{Tan } \theta} = \text{Cos } \theta \quad \dfrac{\text{Cos } \theta + \text{Sin } \theta}{1 + \dfrac{\text{Sin } \theta}{\text{Cos } \theta}} \longrightarrow \dfrac{\text{Cos } \theta + \text{Sin } \theta}{\dfrac{\text{Cos } \theta + \text{Sin } \theta}{\text{Cos } \theta}}$

$$\text{Cos } \theta + \text{Sin } \theta \cdot \frac{\text{Cos } \theta}{\text{Cos } \theta + \text{Sin } \theta} = \text{Cos } \theta$$

The complete solution should appear without explanation.

Example 2: $(\tan\phi + \text{Cot}\,\phi)(\text{Sec}\,\phi - \text{Cos}\,\phi) = \text{Sec}\,\phi\,\text{Tan}\,\phi$

$(\tan\phi + \text{Cot}\,\phi)(\text{Sec}\,\phi - \text{Cos}\,\phi)$ Step 1

$\left(\frac{\text{Sin}\,\phi}{\text{Cos}\,\phi} + \frac{\text{Cos}\,\phi}{\text{Sin}\,\phi}\right)\left(\frac{1}{\text{Cos}\,\phi} - \text{Cos}\,\phi\right)$ Step 2

$\left(\frac{\text{Sin}^2\phi + \text{Cos}^2\phi}{\text{Cos}\,\phi\;\text{Sin}\,\phi}\right)\left(\frac{1 - \text{Cos}^2\phi}{\text{Cos}\,\phi}\right)$ Step 3

$\left(\frac{1}{\text{Cos}\,\phi\;\text{Sin}\,\phi}\right)\left(\frac{\text{Sin}^2\phi}{\text{Cos}\phi}\right)$ Step 4

$\frac{\text{Sin}^2\phi}{\text{Cos}^2\phi\;\text{Sin}\,\phi}$ Step 5

$\frac{\text{Sin}\,\phi}{\text{Cos}^2\phi} \longrightarrow \frac{\text{Sin}\,\phi}{\text{Cos}\,\phi} \cdot \frac{1}{\text{Cos}\,\phi} \longrightarrow \tan\phi\;\text{Sec}\,\phi$ Step 6

C. Solving equations

Students should have extensive practice in solving trigonometric equations such as:

$\tan^2\theta + 3 = 3\sec\theta$

for $0 \leq \theta < 360^\circ$

Solution

$\tan^2\theta + 3 = 3\sec\theta$

$\text{Sec}^2\theta - 1 + 3 = 3\sec\theta$

$\text{Sec}^2\theta - 3\,\text{Sec}\,\theta + 2 = 0$

$(\text{Sec}\,\theta - 2)(\text{Sec}\,\theta - 1) = 0$

$\text{Sec } \theta = 2$ $\qquad$ $\text{Sec } \theta = 1$

$\theta = 60^\circ, 300^\circ$ $\qquad$ $\theta = 0^\circ$

Example 2: $2 \text{ Cos}^2 \theta = 4 \text{ Cos } \theta - 1$

$$2 \text{ Cos}^2 \theta - 4 \text{ Cos } \theta - 1 = 0$$

$$\text{Cos } \theta = \frac{4 \pm \sqrt{16 - 8}}{4} \longrightarrow \frac{2 \pm \sqrt{2}}{2} \longrightarrow 1.7071$$

or .2929 (The value of Cos θ cannot exceed 1, therefore no angle corresponds to Cos θ = 1.7071.)

$\text{Cos } \theta = .2929$

$\theta = 72^\circ\ 58', 287^\circ\ 2'$

VI. Functions of Two Angles

The student should write all of these formulas (A-F) on a card (3" x 5") and memorize them as soon as possible.

A. The Cosine of the sum and difference of two Angles

B. The Sine of the sum and difference of two Angles

A sufficient number of problems should be given so that the student becomes efficient in the use of these formulas

C. The tangent of the sum and difference of two Angles

Any trigonometry book explains and illustrates sufficiently well.

D. Functions of double Angles

These topics are listed here because they should be included in the course.

E. Functions of Half Angles

In some schools a trigonometry textbook is not used and some of these topics are omitted.

F. Sum and difference of two Sines or two Cosines

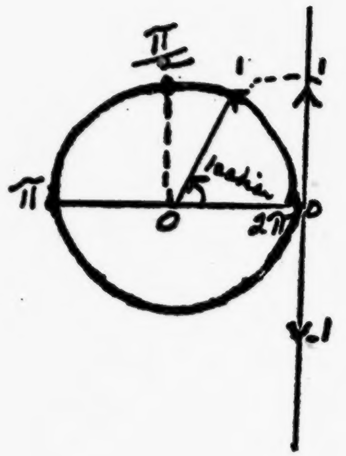
Perhaps here would be a good time to have students develop the reduction formulas.

VII. Circular Functions - Definition

Students should be adept at changing degrees to radians and radians to degrees.

$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ \; 17' \; 45''$ $\qquad \pi \text{ radians} = 180^\circ$

$1 \text{ degree} = \frac{\pi}{180^\circ} \text{ radian} = .01745 \text{ radian}$ $\qquad 2\pi \text{ radians} = 360^\circ$



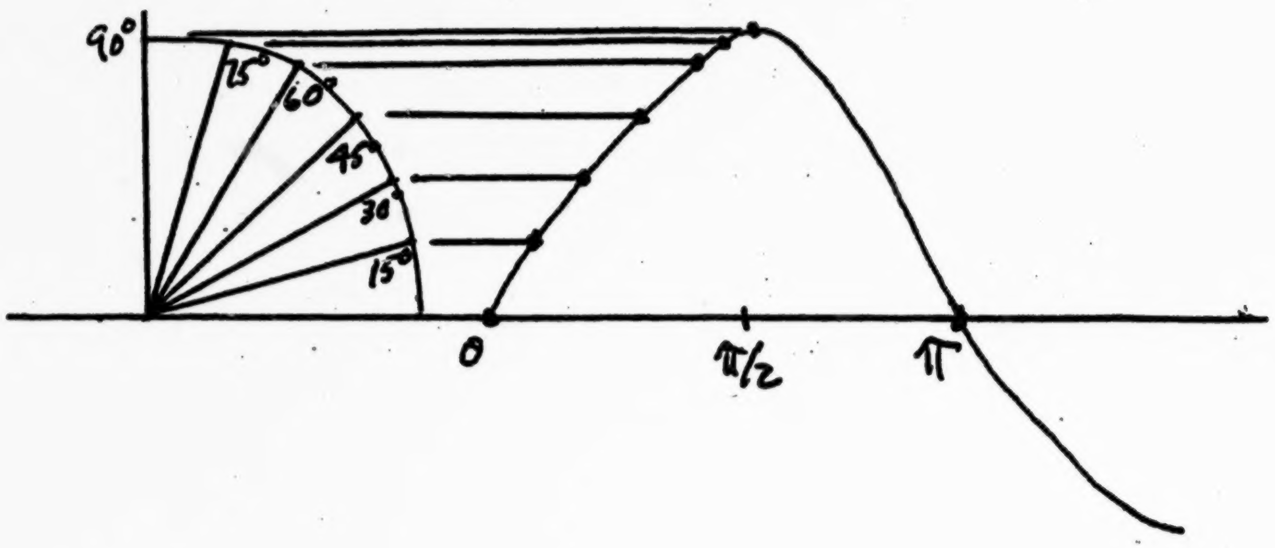
Mention that the positive half of the number line is wrapped counterclockwise around the circle.

B. Comparison of functions of real numbers and functions of Angles

The student may consider the trigonometric functions as sets of ordered pairs (n, Cos n) and (n, Sin n) etc. Where the domains are sets of angles, the ordered pairs are (θ, Cos θ) and (θ, Sin θ) etc. (In some texts, the basic idea of a function is the rule, not the ordered pair.)

C. Graphs of Sine and Cosine functions

Students should have sufficient practice in graphing so that they become very familiar with relating the curve to the unit circle.



It is suggested that Sin θ and Cos θ be graphed on a single axis to point out that as Sin θ increases, Cos θ decreases, and that if the circle were rotated through 90°, the Cos θ would result. The graph could also be used as a reference to show the relationship between the values of Sin θ and Cos θ. (The Sin 30° = Cos 60°)

D. Graphs of other functions

Students should graph the reciprocal relations on one axis to show the relationship between Cos θ and Sec θ; tan θ and Cot θ, etc.

A single graph showing $y = \text{Sin } x$, $y = \text{Sin } 2x$, $y = \text{Sin } \frac{1}{2}x$ will illustrate periods of the function, and a single graph showing $y = \text{Cos } x$, $y = 3 \text{ Cos } x$ etc. Explain amplitude carefully.

A relationship between $y = \text{Sin } x$ and $y = 2 \text{ Sin } (3x + \pi)$ on the same graph will show displacement, amplitude, and period.

It should be pointed out to students that the circular functions are applied in problems dealing with waves, vibrations, oscillations, orbits, cycles, and harmonics.

VIII. Inverse Functions

A. Definition and example

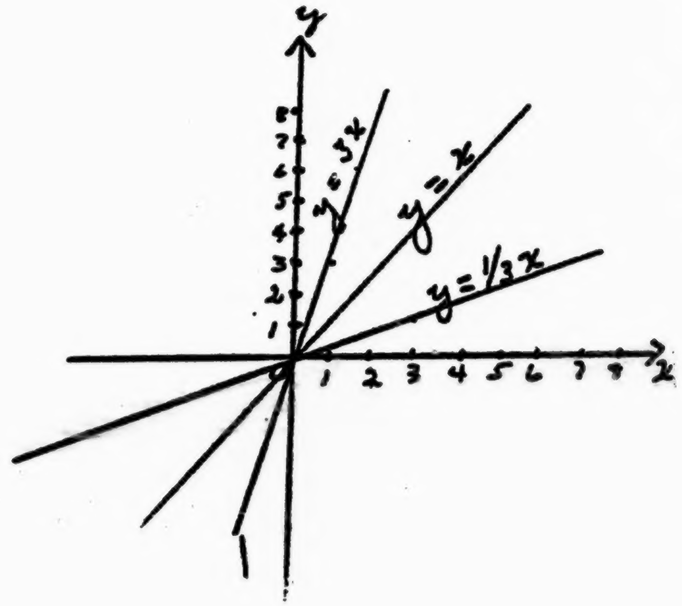
Since functions are relations, the students may define the inverse of a function as a relation which is obtained by interchanging the elements of each ordered pair in the set that makes up the function.

Relations may or may not be functions, but the line test will determine if it is a function. (A relation R is a function if, and only if, no vertical line meets the graph of R in more than one point)

Example: Give the relation R defined as $y = 3x$. Find the inverse relation R'

Interchanging coordinates, $x = 3y$ solve for y, $y = 1/3\ x$, $y = \frac{x}{3}$.

The inverse of a function is also defined: f^{-1} is the inverse of f if $ff^{-1} = I$ (the identity function)

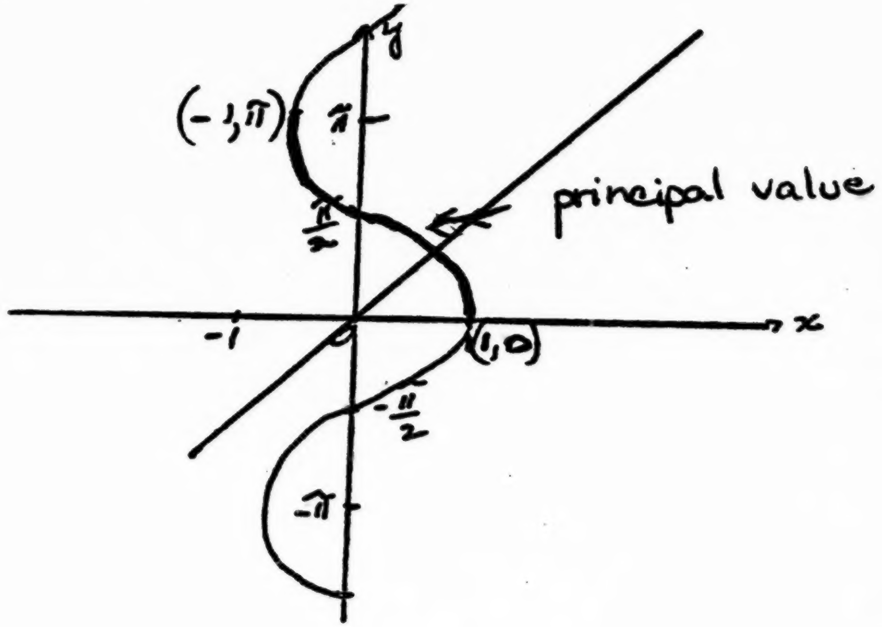


B. Graphs of inverse functions

Explain to the students that a restriction is usually placed on the range, so that a relation can be defined as the principal value function. The range is usually

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ for y = arc sin x. For y = arc cos x and range is usually $0 \leq y \leq \pi$.

The student may note that the graph of either function is a reflection of the other in the graph y = x



y = arc Cos x
y = x

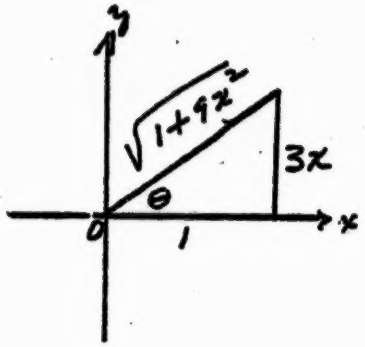
C. Identities and Equations Involving Inverse functions

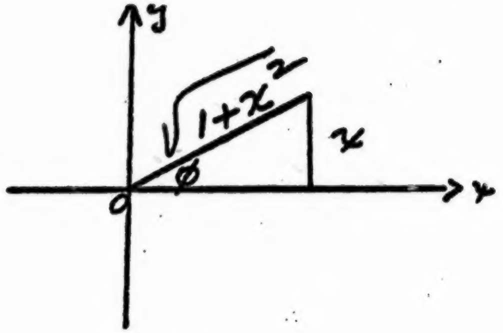
Remind the student that in any equation the roots must always be checked by substituting in the original equation.

The sine and Arc Sine functions are inverses. Their composition in either order is the identity function. Sin (Arc Sin x) = x Arc Sin (Sin x) = x

Example: Solve Arc Tan 3x + Arc Tan x = $\frac{\pi}{2}$, $x > 0$

Let θ = Arc Tan 3x; ϕ = Arc Tan x and $\theta + \phi = \frac{\pi}{2}$
Sin $(\theta + \phi)$ = Sin $\frac{\pi}{2}$
Sin θ Cos ϕ + Cos θ Sin ϕ = 1





Equation becomes by substitution $(3x^2 - 1)(3x^2 - 1) = 0$
$x = \pm \frac{\sqrt{3}}{3}$

Substituting in the original equation
Arc Tan $\sqrt{3}$ + Arc Tan $\sqrt{3}/3 = \frac{\pi}{2}$

$$\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{\pi}{2}$$

$x = \frac{\sqrt{3}}{3}$ is a solution
$x = -\frac{\sqrt{3}}{3}$ is not a solution

IX. Exponential and Logarithmic Equations

A. Definition and Examples

Explain that an exponential equation is one in which the variables appear in an exponent, and a logarithmic equation is one in which there is a logarithm of some expression involving the variable.

Exponential Equation

Example: (1) Solve for x in $2^x = 16$ or
(2) Solve for x in $4^{-x} = 64$

Logarithmetic Equation

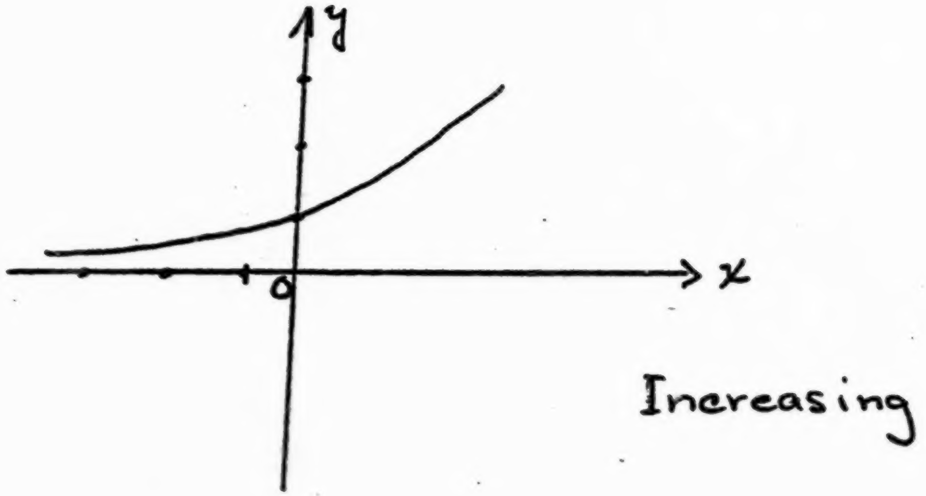
Example: (1) Solve $\log (x + 2) - \log (x - 2) = \log 5$;
(2) Solve $\log x^2 - \log \frac{2x}{5} = 7.58$

B. Solving Equations

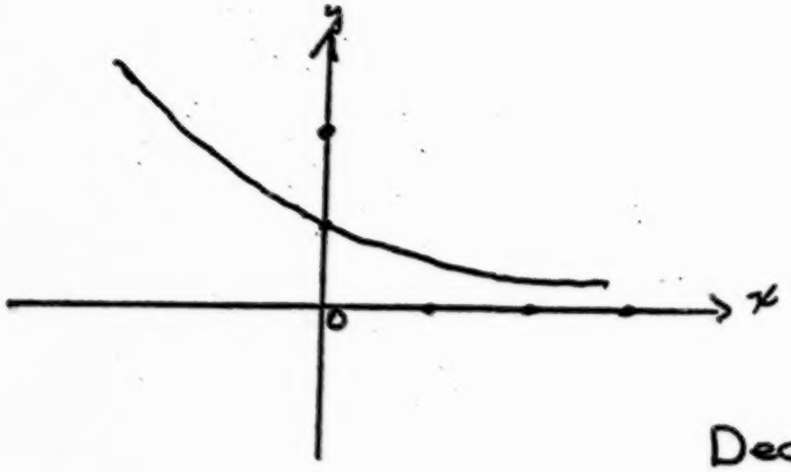
The student should become efficient in solving both types of equations listed above.

C. Graphing Exponential Functions

Example: (1) $y = a^x$; $a > 1$
$y = a^x$; $a > 1$



(2) $y = a^x$; $a < 1$

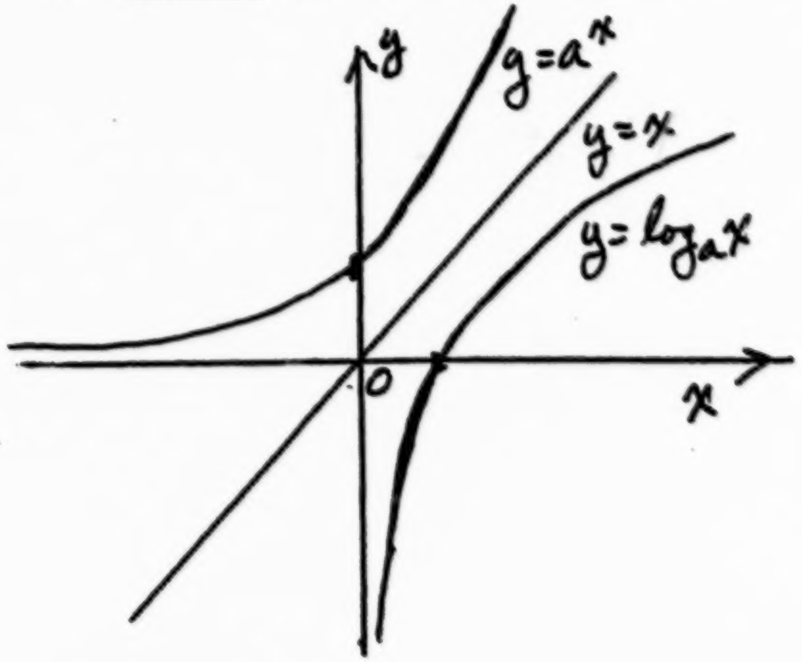


Decreasing

Point out that any base, not zero, raised to the zero power would be 1.

D. Graphing Logarithmic Functions

The inverse function of $y = a^x$ is called the logarithmic function ($y = \log a^x$)



The reflection about the line $y = x$

a^x is the inverse of $\log {}_a x$.

The student should have practice in converting from the exponential form to the logarithmic form and vice versa. Many graphs should be done.

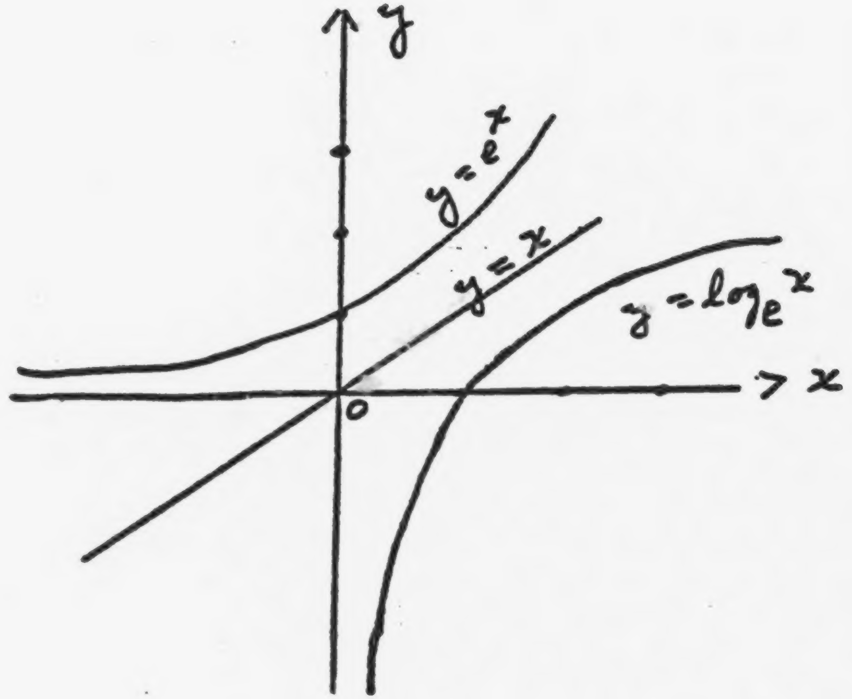
Example: $\log_3 1/3 = -1$

$3^{-1} = 1/3$

E. Natural Logarithms and Base e

The irrational number $e \approx (2.71828)$ is important in mathematics. Logarithms to the base e are the natural logarithms. Logarithms to the base 10 are called common logarithms.

The approximation to e can be expressed: $e \approx 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} \dots + \frac{x^n}{n!}$



X. Space Geometry

A. Review of Planes and lines in space

The students should spend some time drawing horizontal and vertical planes, intersecting planes, objects drawn in perspective, prisms, etc. There are many good commercial models to use during explanations, and students enjoy making models using acetate or bolsa wood or cardboard.

Be sure that the student understands that any line in a plane separates the plane into two regions called half-planes. Refer to Birkhoff's axioms in Geometry (S.M.S.G.)

B. Discuss vocabulary of space geometry

Definitions of dihedral, trihedral and polyhedral angles, tetrahedrons, polyhedrons, parallelepiped, etc. are parts of the vocabulary of solid geometry and the students should be familiar with them. For instance, these terms are used in aeronautical construction, engineering drawing, and in architecture.

C. Types of practice exercises

To help the students visualize space figures, have them sketch a figure and draw conclusions from exercises of the following type.

1. Plane m cuts plane n in AB and plane r in CD. n is parallel to r.
2. Line $AB \perp$ plane m. AB is in plane n.
3. In trihedral $\angle 0 - XYZ$, $\angle XOY = 40^\circ$, and $\angle YOZ = 60^\circ$

A discussion of locus of points in space is another good exercise.

Example (1): What is the locus of points in space equidistant from two intersecting planes.

(2): Find points equidistant from parallel planes MN and RS and also equidistant from given points A and B.

(3) Can the projection on a plane of a curved line or of a broken line be a straight line? Discuss.

E. Spherical Geometry

This topic should be thoroughly covered because of its importance in navigation, geography, and astronomy. The students are familiar with the location of points on the surface of the earth by means of longitude and latitude with radar equipment and with "great circle" flights in space travel as seen on television, but only the "old" Solid Geometry book covers the topics to be included here.

(1) Definitions

The students should understand spherical distance, great circle and small circle, polar distance, the spherical angle, spherical degree, spherical excess, zone, lune, spherical triangles, and polar triangle.

(2) Areas

Students should have some work with areas of zones, lunes, spherical triangles, spherical polygons, wedge, spherical pyramid, and spherical cone, spherical sector and spherical segments.

F. Solution of right spherical triangles

The students should draw a figure and prove the following formulas:

I. $\text{Sin } A = \dfrac{\text{Sin } a}{\text{Sin } c}$

II. $\text{Cos } A = \text{Sin } a \ \text{Cos } a$

III. $\text{Sin } B = \dfrac{\text{Sin } b}{\text{Sin } c}$

IV. $\text{Cos } B = \text{Sin } a \quad \text{Cos } b$

V. $\text{Cos } C = \text{Cos } a \quad \text{Cos } b$

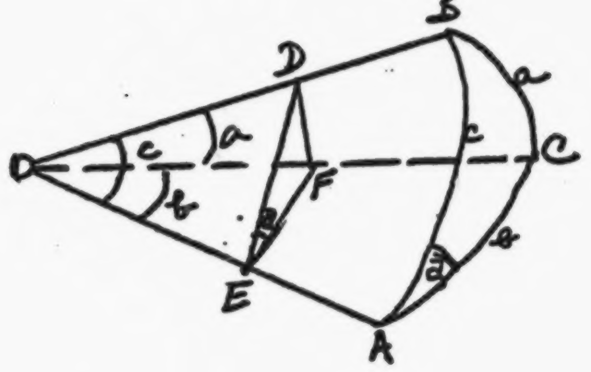
VI. $\text{Cos } A = \dfrac{\text{Tan } b}{\text{Tan } c}$

VII. $\text{Tan } A = \text{Sin } b \ \ \text{Tan } a$

IX. $\text{Tan } B = \text{Sin } a \ \ \text{Tan } b$

VIII. $\text{Cos } B = \frac{\text{Tan } a}{\text{Tan } c}$

X. $\text{Cos } C = \text{Cot } a \ \ \text{Cot } b$



Have students use logarithms to solve problems of the type:

If in Right Triangle ABC, $A = 32^{\circ}24'$ and $C = 49^{\circ}15'$, find b, a, B (Use formula V to find b, I to find A, IV to find B.

G. Solution of Oblique Spherical Triangles

Students should have some experience with the following formulas:

$$\text{Cos } a = \text{Cos } b \ \ \text{Cos } c + \text{Sin } b \ \ \text{Sin } c \ \ \text{Cos } A$$
$$\text{Cos } b = \text{Cos } a \ \ \text{Cos } c + \text{Sin } a \ \ \text{Sin } c \ \ \text{Cos } B$$
$$\text{Cos } c = \text{Cos } a \ \ \text{Cos } b + \text{Sin } a \ \ \text{Sin } b \ \ \text{Cos } C$$
$$\text{Cos } A = -\text{Cos } B \ \ \text{Cos } C + \text{Sin } B \ \ \text{Sin } C \ \ \text{Cos } a$$
$$\text{Cos } B = -\text{Cos } A \ \ \text{Cos } C + \text{Sin } A \ \ \text{Sin } C \ \ \text{Cos } b$$
$$\text{Cos } C = -\text{Cos } A \ \ \text{Cos } B + \text{Sin } A \ \ \text{Sin } B \ \ \text{Cos } c$$

Applications of spherical triangles can be of the type:

1. Find the great circle distance in nautical miles between two cities (Los Angeles and Dublin).
2. To find the time of day (solar time) of an observer whose latitude is known, when the sun's declination and altitude are given.
3. In a given latitude, to find the altitude and azimuth of a celestial object whose declination and hour angle are known.

XI. Coordinate Geometry

These topics have been introduced in Algebra I, Plane Geometry, and Algebra II, so a very brief review should be sufficient.

A. The straight line

1. Definition

(a) Slope, inclination, rise-run

Some textbooks explain slope using the terms rise-run. Rise is defined as $y_2 - y_1$ and run is defined as $x_2 - x_1$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

(b) Slope-intercept
(c) Two-point
(d) Distance
(e) Mid-point
(f) Parallel and Perpendicular lines

B. Absolute Value and Inequalities

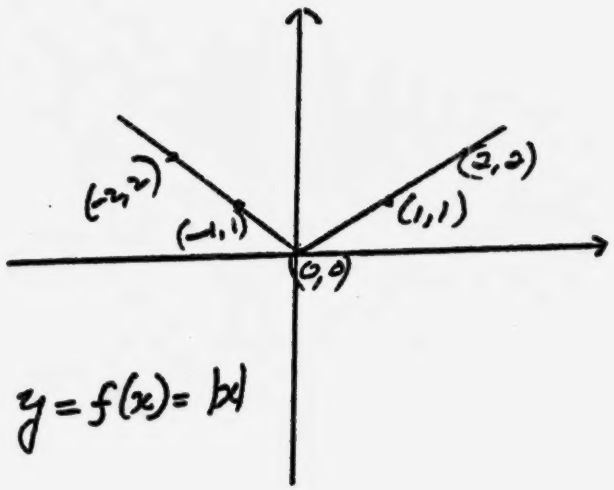
Some students have difficulty with the absolute value of a function in equations of the type: $|x - 7| = 3$. They should remember that this equation expresses the fact that $x - 7$ must be 3 or -3.

Graphically speaking, the absolute value of x is the distance between o and x on the number scale, regardless of whether x lies to the left or to the right ot 0.

1. Graph of absolute value

The student should have some experience with the graph of an equation of the type $f(x) = |x|$

x	2	1	0	-1	-2
f(x)	2	1	0	1	2



Graph some equations: $y = |x| + 1$, $y = |x + 1|$, $y = -|x|$

2. Graphs of Linear Inequalities

Explain that every line divides the plane into two regions, one on each side of the lines. All the points in one of these regions satisfy the same inequality and all the points in the other region satisfy the opposite inequality.

Give problems of the type:

(1) $2x - y - 4 > 0$ and $3x + y - 1 > 0$

(2) $x + 6y - 11 > 0$, $3x - 4y + 11 \quad 0$, $-2x - y + 11 > 0$

Use slope-intercept form to test linear inequalities

Have students graph these inequalities and describe the region of points (if any) which satisfy the system of linear inequalities.

C. The Conic Sections (Commercial models are excellent to use as illustrations of conics)

A Conic should be defined as the locus of points whose undirected distances from a fixed point are in a constant ratio to their undirected distances from a fixed line. The fixed point is called a focus, the fixed line the directrix, and the constant ratio (e.) is called the eccentricity. A chord of the conic that is perpendicular to the axis at a focus is called a latus rectum.

Have students derive the equation of the circle from the distance formula.

Give problems of the type:

1. The Circle

   (a) Find the center and radius of a circle whose equation is $(x - 4)^2 + (y + 2)^2 = 15$
   (b) Sketch the graph of the circle whose equation is $x^2 + 6x + y^2 - 4y = 12$
   (c) Find the equation of the circle that passes through (2,9), (6,1), (-3,4)

2. Parabola

   Define the parabola as the locus of points which moves so that it remains equidistant from the given point F and given line ℓ. The fixed point F is the focus and the line ℓ the directrix.

Have the student derive the equation for a parabola.

Give problems of the type:

(1) Find the equation of the locus of a point that is equidistant from (4, -1) and $y = -3$.
(2) Find the equation of the parabola whose focus is (3, 2) and whose directrix is $x = 1$.
(3) Sketch the graph of $y = 2x^2 - 12x + 16$

As a practical application - the cable supporting a suspension bridge hangs in the shape of a parabola if the bridge is uniformly loaded.

The path of a projectile takes the form of a parabola if there is no wind resistance. Reflecting telescopes, and the head lights of cars take the shape of a parabola.

3. Ellipse

Have the students construct an ellipse mechanically, and also have them derive an equation for the ellipse.

Practice problems of the type:

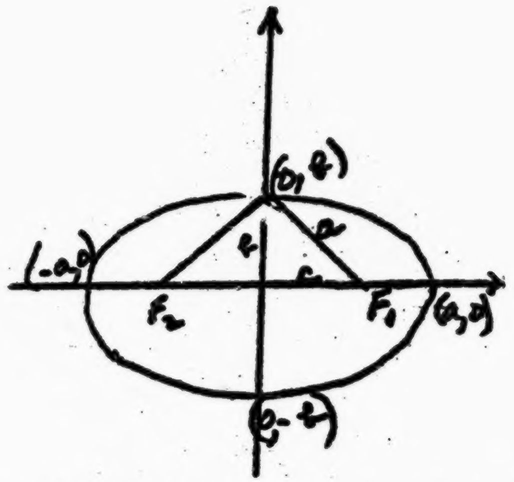
(a) foci
(b) major and minor axis
(c) eccentricity

Practice problems of the type:

(1) Find the equation of the ellipse whose foci F_1 (-4,0) F_2 (4, 0) and whose constant term is 10.

(2) Find the lengths of the major and minor axes, the coordinates of the foci and vertices, and the eccentricity of $16x^2 + 25y^2 = 400$

(The ellipse intersects the x-axis in two points (a, 0), (-a, 0) called the vertices.) The distances between these points (2a) is the length of the major axis. The ellipse intersects the y-axis at (0, b) and (0, -b). The distance between them (2b) is called the length of the minor axis.



The eccentricity (e) is defined as e = c/semi-major axis.

Practical applications of the ellipse should be discussed. The orbits in which the planets, including the earth, revolve around the sun are ellipses. The orbits reached by the astronauts are ellipses and the crescent moon is a semi-ellipse. Define apogee - the point in the orbit of a satellite at the greatest distance from the center of the earth. Perigee - least distance from the center of earth.

4. Hyperbola

(a) Transverse Axis
(b) Conjugate Axis
(c) Asymptotes (Latus rectum)

Have the students derive the formula for the hyperbola.

Explain that the line passing through the foci F_1 and F_2 is called the Conjugate Axis.

The hyperbola approaches but never touches the asymptotes.

Work problems of the type:

(1) Find the equation of the hyperbola whose foci are (-5,0)(5,0) and whose and whose constant differences is 6.

(2) Sketch the graph of the hyperbola $x^2 - y^2 = 9$ by finding the x and y intercepts (if any) and drawing the asymptotes $x^2 - y^2 = 0$ (or $x + y = 0$, $x - y = 0$)

(3) Sketch the graph of the hyperbola $xy = 12$, whose asymptotes are the x and y axis.

(4) Determine the vertices, foci, eccentricity, length of the latus rectum (a chord through either focus ⊥ to the transverse axis) and the equations of the asymptotes given $16x^2 - 9y^2 = 144$.

Practical applications to discuss are Boyle's Law and the general law of gravitation and construction of certain telescopic lenses. In navigation, a pilot guides his aircraft by maintaining a constant difference between his distances from two fixed points, representing radio sending stations. The curve along which the pilot flies is called a branch of a hyperbola.

The students should know that if the eccentricity (constant ratio) of a conic is given, the type of conic is determined by

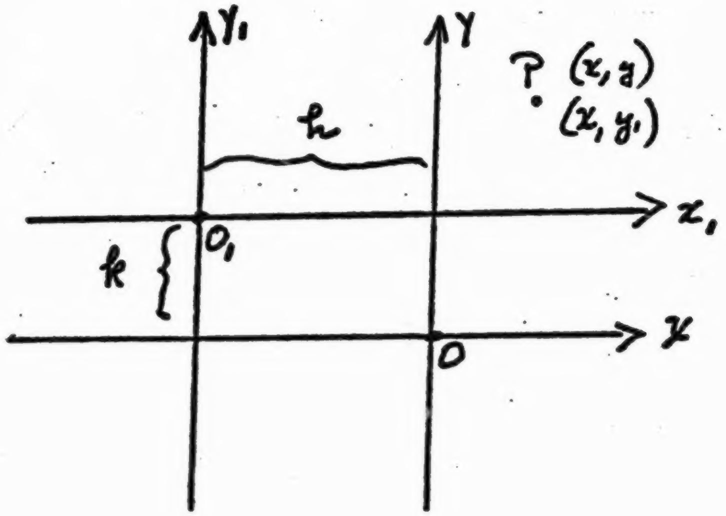
$e < 1$ = ellipse
$e = 1$ = parabola
$e > 1$ = hyperbola

D. Translation of Axes (See last page of this section for derivation of the equation of a circle for simple illustration.)

Explain to students that some curves (parabolas, hyperbolas, etc.) are not situated so conveniently to the xy axes as in the examples preceeding. (Center at the origin.)

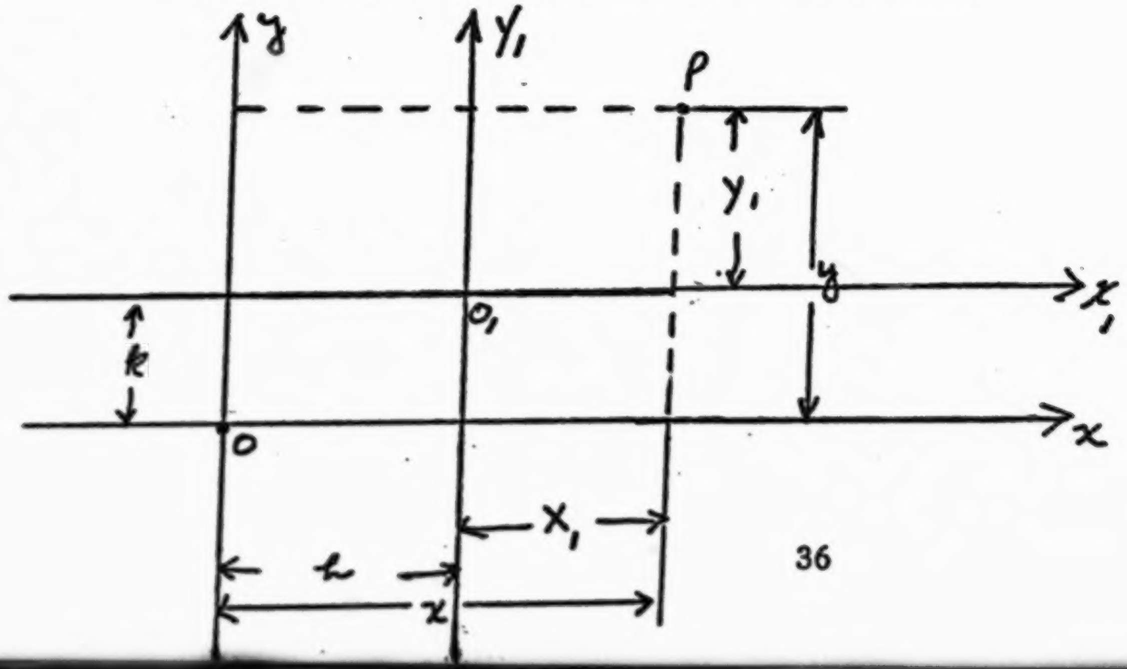
The Coordinate system can be changed in order to have the curve at a convenient and familiar location. The process of making this change is called Transformation of Coordinates.

An additional Coordinate system is now introduced with axis x_1 parallel to x and k units away, and axis y_1 parallel to y and h units away. The origin of the new coordinate system (O_1) will now have the coordinates (h, k) in the original system.



If P is any point in the plane its coordinates would be (x, y) on the original axis and (x_1 y_1) on the translated axis.

When P has coordinates (x, y) in the original system and (x_1, y_1) in the new system, they are related as shown here.



x_1 = x-h; y_1 = y-k
or
x = x_1 + h; y = y_1 + k

Example: Given the equation $9x^2 + 25y^2 + 18x - 100y - 116 = 0$ by using a translation of axes determine whether the equation is a parabola, ellipse, or hyperbola.

Complete the square in x and y.

$9(x^2 + 2x) + 25\ (y^2 - 4y) = 116$

$9(x^2 + 2x + 1) + 25(y^2 - 4y + 4) = 225$

1 was added then $9(1) = 9$, 4 was added than $25(4) = 100$

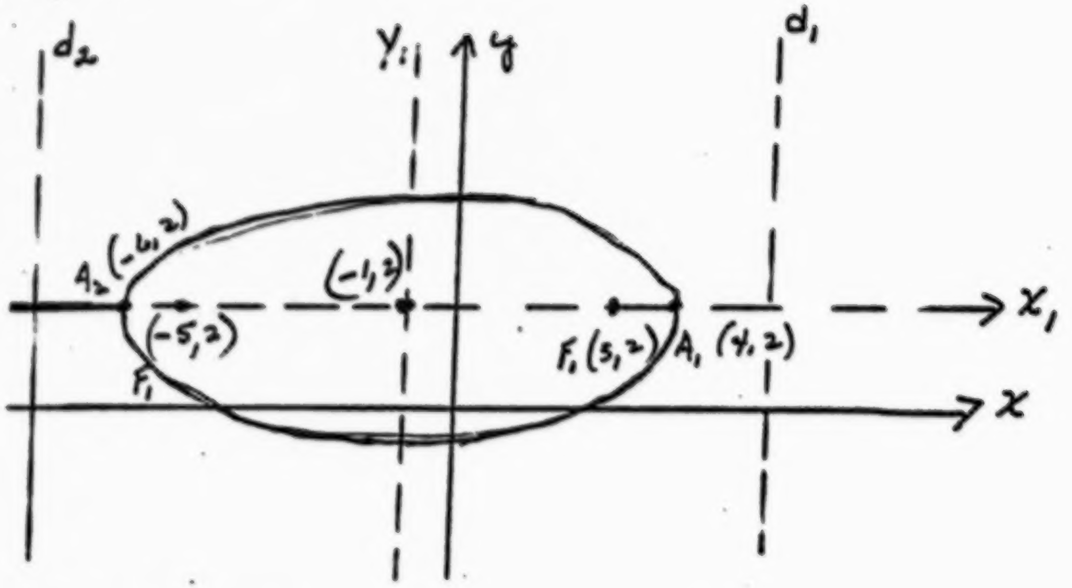
$116 + 9 + 100 = 225$

$9(x + 1)^2 + 25(y - 2)^2 = 25.$

∴ in the translation of axes

$x_1 = x + 1$ and $y_1 = y - 2$

$h = -1,\ k = 2$

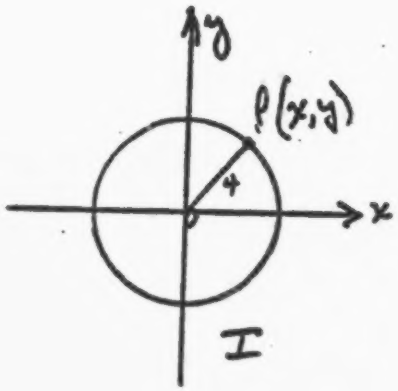


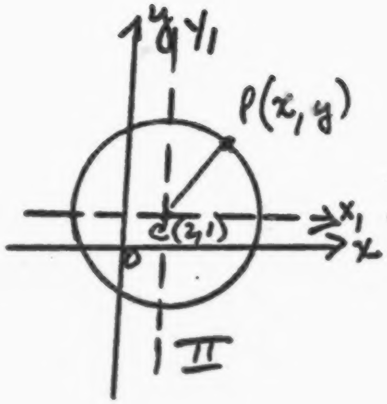
The equation becomes $9{x_1}^2 + 25{y_1}^2 = 225$ with the translation. Divide by 225

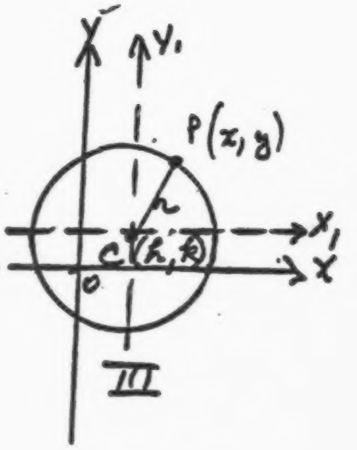
$$\frac{{x_1}^2}{25} + \frac{{y_1}^2}{9} = 1 \text{ (an ellipse)}$$

E. Deriving the equation of the circle from the distance formula

Simple Illustration of Translation of Axes







In I, P(x, y) is any point 4 units from 0

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = 4$$

$$x^2 + y^2 = 16$$

In II, P(x, y) is any point 4 units from center (2, 1)

$$OP = \sqrt{(x-2)^2 + (y-1)^2} = 4$$

$$(x - 2)^2 + (y - 1)^2 = 16$$

In III, P(x, y) is any point r units from center (h, k)

$$OP = \sqrt{(x-h)^2 + (y-k)^2} = r$$

$$(x-h)^2 + (y-k)^2 = r^2$$

F. Algebraic Curves

The student has seen that, if the equation of a curve is of first degree in x and y,

$$Ax + By + C = 0$$

the curve is a line, and $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, its equation is of the second degree.

the locus of the equation (if it exists) is a conic section.

If the given equation cannot be reduced to either of the above forms, its locus is said to be a higher plane curve.

1. Symmetric Curve

A curve is symmetric with respect to a given line or to a given point if, when $P(x, y)$ is any point on the curve, then its symmetric point with respect to the given line or the given point also lies on the curve.

Have the students test for symmetry as follows: (Useful in graphing)

(a) For symmetry with the y-axis. The graph of a function $f(x, y) = 0$ is symmetric to the y-axis if substituting $(-x)$ for x does not destroy the equality. The point symmetric to $P(x, y)$ with respect to the y-axis is $(-x, y)$, with respect to the x-axis is $(x, -y)$, with respect to the origin is $(-x, -y)$.

Exercise: Let the student verify that $y^2 - 4x + 4 = 0$ is symmetrical to the x axis; that $xy = 1$ is symmetrical to the origin; that $x^2 - y^2 = 16$ is symmetrical to both axes and to the origin.

2. Extent of Values

The student should understand that there is no set or formal rules for determining extent of values, but that the usual procedure is to express each variable in terms of the other and then by inspection find which values of one give real values of the other. Values of x which make y imaginary are excluded values of x when graphing.

3. Intercepts

To find the y-intercepts of $f(x, y) = 0$, substitute zero for x and solve for y.

To find the x-intercepts of $f(x, y) = 0$, substitute zero for y and solve for x.

Example: Find the intercepts of $y - \frac{x^2 - 1}{x^2 - 4} = 0, x \neq 2 \text{ or } -2$

Let $x = 0$; $y = \frac{0-1}{0-4} = \frac{1}{4}$. The y-intercept is $(0, \frac{1}{4})$

Let $y = 0$; $0 = \frac{x^2 - 1}{x^2 - 4}$ $x^2 - 1 = 0, x = \pm 1$

The x-intercepts are (1, 0) and (-1, 0).

4. The students should understand the rule for vertical asymptotes: To find the vertical asymptotes of the locus of $f(x, y) = 0$, solve for y in terms of x; if the result is a fraction, set its denominator equal to zero and solve for x; equating x to each of the real solutions gives the equations of the vertical asymptotes.

They should also understand the rule for horizontal asymptotes: To find the horizontal asymptotes of the locus of $f(x, y) = 0$, solve for x in terms of y; if the result is a fraction, set its denominator equal to zero and solve for y; equating y to each of the real solutions gives the equation of the horizontal asymptotes.

5. A rational function of x

The expression $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are any two polynomials in x in called a rational function of x. The graph of the equation $y = \frac{P(x)}{Q(x)}$ is called the graph of the rational function.

I. Test the following curves for symmetry:

1. $x^2y = 4$
2. $x^2 + y^2 + x + y = 0$
3. $x^2y^2 - x^2 - y^2 = 0$

II. Discuss and sketch the following curves:

1. $y = x^2 - 3x^2 + 2x$
2. $y = x^4 - 3x^3$
3. $y = x^3 + 4$

III. Discuss and sketch the following curves:

1. $y = \frac{x}{x^2 - 4}$
2. $y^2 = \frac{x}{x - 2}$
3. $y = \frac{2x - x^2}{x^2 + x - 2}$

The students sketch many problems of the above types using the preceeding sections on intercepts, asymptotes, symmetry, etc.

XII. Polynomials of Higher Degree

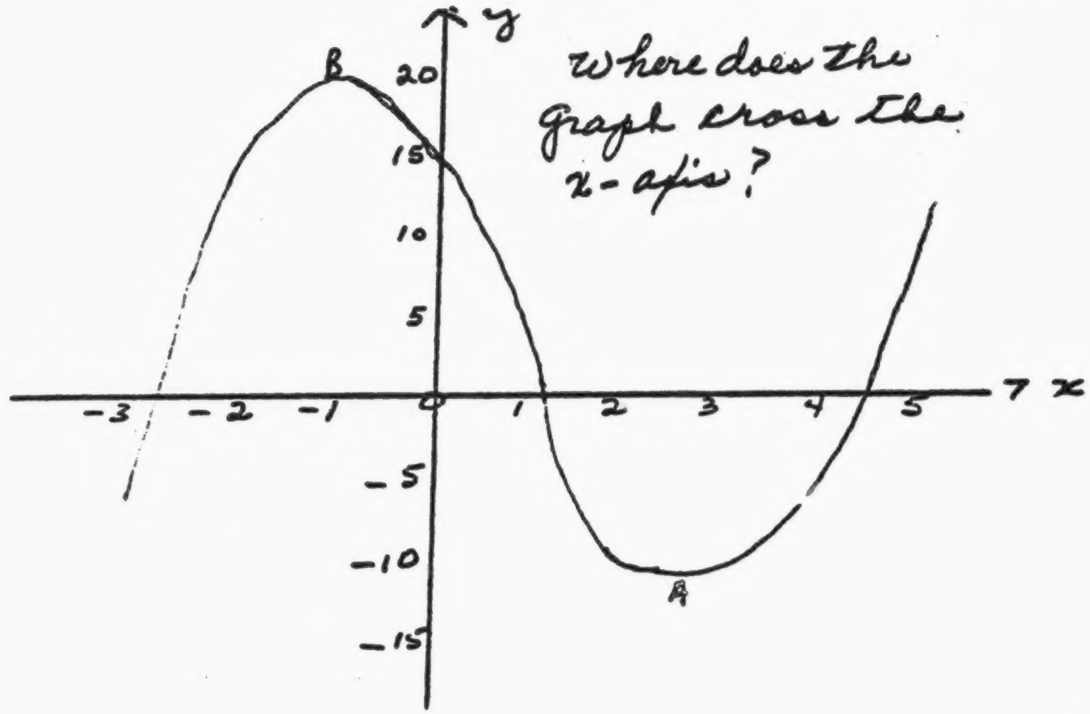
A. Graphing the cubic equation

Students should spend some time graphing cubic equations and finding the approximate roots of the equation from the graphs.

Example: $x^3 - 3x^2 - 9x + 15 = 0$

$y = f(x) = x^3 - 3x^2 - 9x + 15$

x	-3	-2	-1	0	1	2	3	4	5
y	-12	13	20	15	4	-7	-12	-5	20


Where does the graph cross the x-axis?

Roots are approximately 1.3, 4.3, -2.6

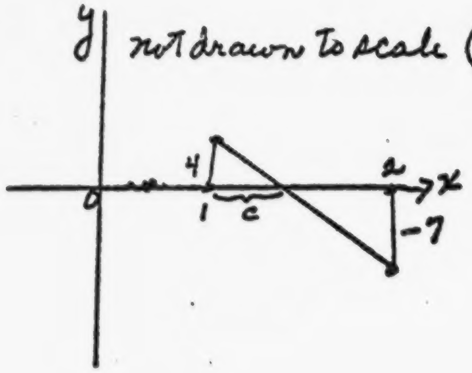
The illustrative example can be reduced to a problem of computation rather than one of careful drawing.

In the following solution, using the functional notation, watch for a pair of values of x for which f(x), or y, changes from + to - or from - to +. Such a change in f(x) means that the graph must cross the x-axis somewhere between two values and therefore indicates the presence of a root. This root will then be approximated by replacing the portion of the curve in question by its chord, and using interpolation.

$$f(x) = x^3 - 3x^2 - 9x + 15$$

Let c = the correction

$f(0) = 15$
$f(1) = 4$
$f(2) = -7$



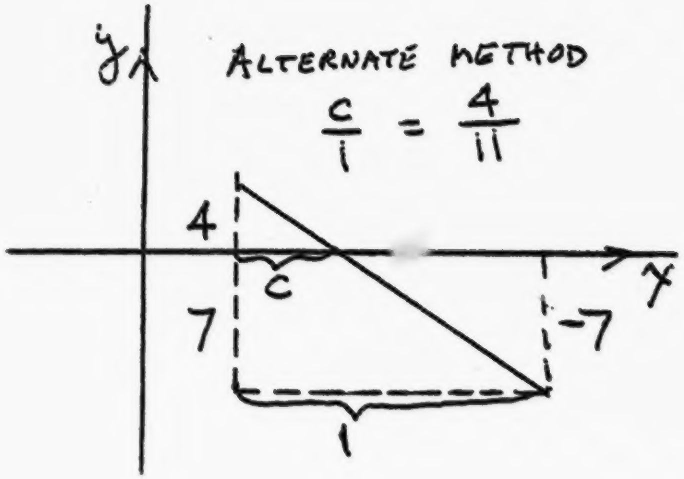
not drawn to scale (Parts of graph on previous page.)

$$\frac{c}{1-c} = \frac{4}{7}$$

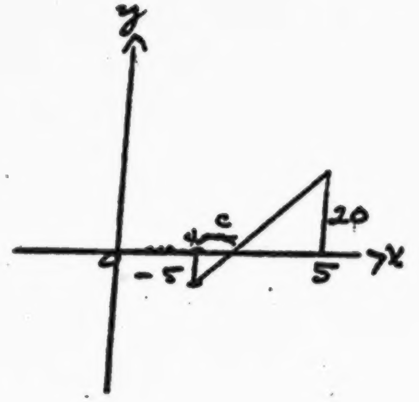
$c = .4$
$r_1 = 1.4$

ALTERNATE METHOD

$$\frac{c}{1} = \frac{4}{11}$$



f(3) = -12
f(4) = -5
f(5) = +20



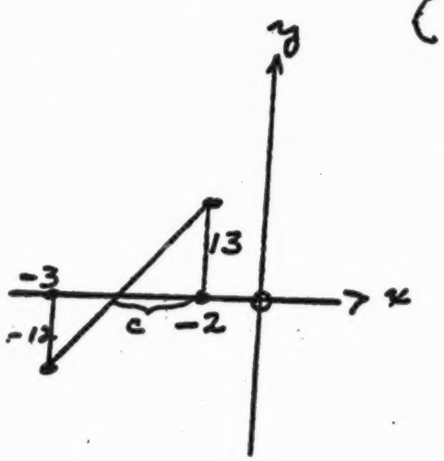
$$\frac{c}{1-c} = \frac{5}{20}$$

$c = .2$

$r_2 = 4.2$

(SEE ALTERNATE METHOD)

f(-1) = 20
f(-2) = 13
f(-3) = -12



$$\frac{c}{1-c} = \frac{13}{12}$$

$c = .5$

$r_3 = -2.5$

The students should work several examples to become efficient in both the graphing and computational method of finding the approximate roots of cubic equations.

XIII. Theory of Equations

A. Remainder Theorem

Explain that the remainder theorem refers to the properties of the remainder secured when a polynomial is divided by $x - a$. (Introduced in Algebra II - Reviewed here) The value of the polynomial function $P(x)$ for $x = a$ is the same as the remainder when $P(x)$ is divided by $x - a$.

Problems of the type:

(1) Find the remainder when $x^5 - 7$ is divided by $x - 2$.

Solution: $P(x) = x^5 - 7$
$P(2) = 2^5 - 7 = 25$

The remainder is 25.

B. Factor Theorem

If c is a root of the equation $P(x) = 0$, then $x - c$ is a factor of $P(x)$.

Problems of the type:

(1) Find the remainder when $x^4 + 3x^3 - 5x^2 + 7x - 2$ is divided by $x + 2$.

Solution: $P(x) = x^4 + 3x^3 - 5x^2 + 7x - 2$

The divisor $x - c$, in this case $x + 2$, $c = -2$
$R = P(-2) = (-2)^4 + 3(-2)^3 - 5(-2)^2 + 7(-2) - 2$
$R = -44$

C. Synthetic division

Explain that with the aid of the factor theorem you can determine one of the binomial factors of the polynomial function, but to complete the solution of the equation, it would be necessary to divide the polynomial by the binomial of the form $x - c$.

An abridged way for doing this long division is known as synthetic division.

By eliminating the x's and retaining only the numerical coefficients in $2x^3 - 7x^2 + 16x - 12$ divided by $x - 2$ this would become:

$$\xrightarrow{\quad} \underline{1} - 2 \,\overline{\big)\, 2 - 7 + 16 - 12}$$

Since $\underline{1}$ is always 1, it is eliminated.

The synthetic division now becomes:

$$\begin{array}{r|l}
 & \underline{2 - 3 + 10} \\
\underline{-2} & 2 - 7 + 16 - 12 \\
 & \underline{- 4} \\
 & -3 \\
 & \underline{+ 6} \\
 & 10 \\
 & \underline{- 20} \\
 & 8
\end{array}$$

and the equation has become depressed from a Cubic equation to a quadratic equation and the quotient is $2x^2 - 3x + 10 + \frac{8}{x - 2}$.

A better form for synthetic division is:

$$\begin{array}{r|rrrr}
\underline{-2} & 2 & -\ 7 & +\ 16 & -\ 12 \\
 & & -\ 4 & +\ 6 & -\ 20 \\
\hline
 & 2 & -\ 3 & +\ 10 & +\ 8
\end{array}$$

$$2x^2 - 3x + 10 + \frac{8}{x-2}$$

D. Polynomials in factored form

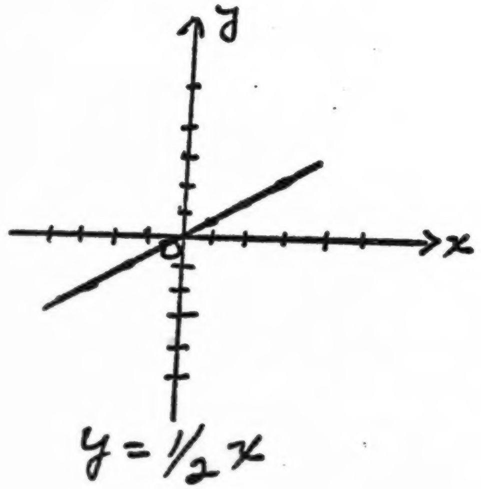
When a polynomial can be written in factored form, sketching the graph is greatly simplified.

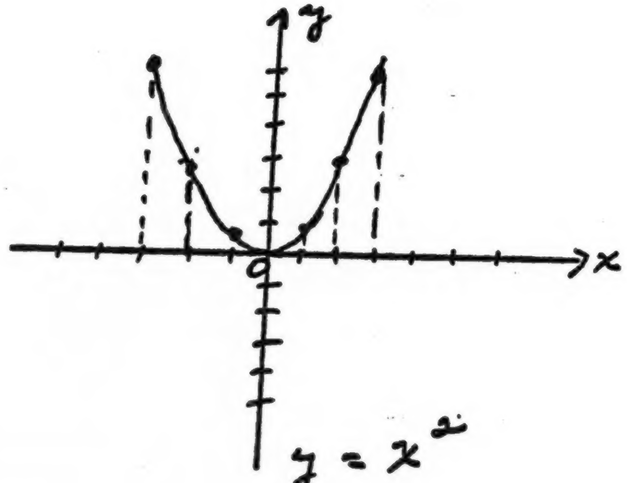
Example: Sketch the graph of $y = (x - 3)^2$

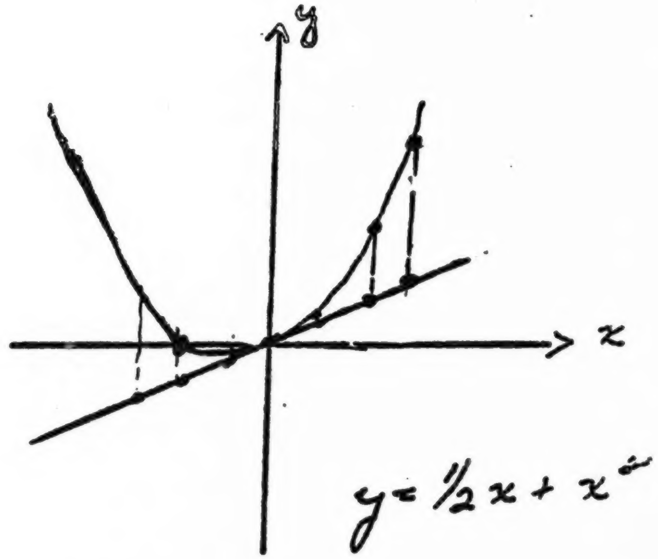
Solution: The y-intercept is $(-3)(-3) = 9$. The square of $x - 3$ corresponds to the tangency at (3, 0). (A multiple root implies tangency)

E. Addition of Ordinates

Explain that a curve of the type $y = \frac{1}{2}x + x^2$ may be considered to be a composite of two curves: $y = \frac{1}{2}x$ and $y = x^2$. Sketch the graph of each and then add the Ordinates of $y = x^2$ to those of $y = \frac{1}{2}x$ for corresponding values chosen for x as shown in the following sketch.







This can also be obtained by completing the square and translating the axis.

F. Rational Roots

Explain that the set of possible rational roots of $P(x) = 3x^3 + 4x^2 - x - 2 = 0$ is reduced to fractions (a/b). The numerator is a factor of 2 and the denominator is a factor of 3. $\{1, -1, 2, -2, 1/3, -1/3, 2/3, -2/3\}$

A general statement would be "In a polynomial equation with integral coefficients, a is a factor of the constant term c, and b is a factor of the coefficient of the highest-degree term."

G. Descartes' Rule of Signs

By Descartes' Rule, there can be no more positive roots in the polynomial equation $f(x) = 0$ than the number of variations of signs in $f(x)$ or less than that number by some positive even integer.

Example: $x^3 \overset{\curvearrowright}{+ x} - 3 = f(x)$ One variation of sign.

Conclusion: There can be no more than 1 positive root.

There can be no more negative roots in $f(x) = 0$ than there are variations of sign in $f(-x)$, or less than that number by some positive even integer.

Example: $f(x) = x^3 + x - 3$
$f(-x) = -x^3 - x - 3$ No variation of sign

There can be no negative roots. $\therefore$ there must be 1 positive root and 2 complex roots, since the equation is a cubic equation and must have 3 roots. There will be a zero root if the constant term is zero.

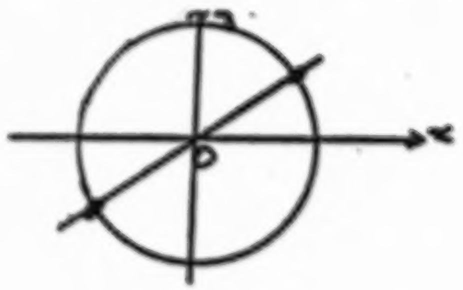
Example: $f(x) = x^3 - 12x^2 + 32x - 15$ Three variations of sign.
There are three positive roots.

H. Systems of Equations

Introduce graphical interpretation of the roots of systems of equations.

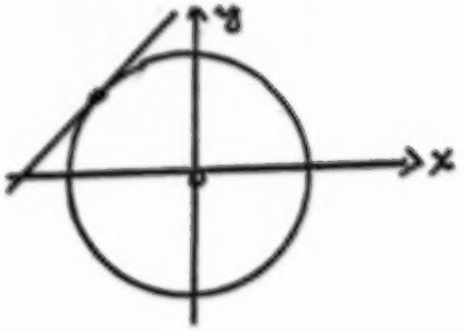
Problems of the type:

(1) Find the solution set of $y = 3/4\ x$ and $x^2 + y^2 = 25$



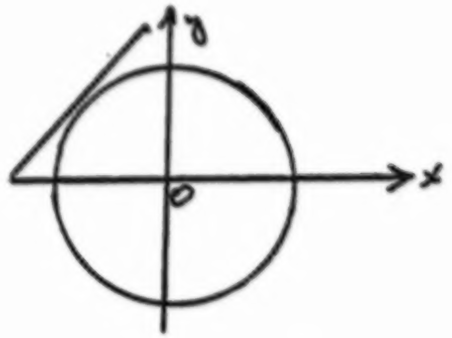
2 ordered pairs of real numbers
in common

(2) $y = 3/4\ x = 6\frac{1}{4}$ and $x^2 + y^2 = 25$



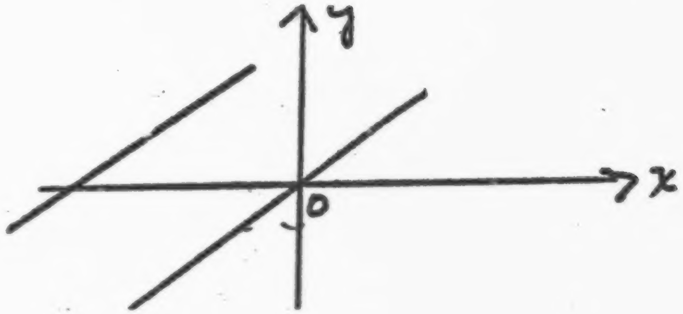
One ordered pair of real numbers
in common.

(3) $y = 3/4\ x + 8$ and $x^2 + y^2 = 25$



No ordered pair of real numbers
in common - $\emptyset$

(4) $y = 3/4\ x$ and $y = 3/4\ x + 8$



No ordered pair in common -
either real or imaginary - $\emptyset$

I. Matrices and Determinants

1. Matrices Definition

One reason that matrix notation is useful is because it saves time by omitting the x's, y's and z's in solving a system of linear equations.

A matrix is a rectangular array of numbers written in rows and columns.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{or} \quad \begin{Vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{Vmatrix}$$

A matrix does not have quantitative value. It is not a symbolic representation of some polynomial.

2. Determinant Definition

A determinant is a function with its domain a set of square matrices of the same order, and its range all numbers.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

3. Addition and Scalar Multiplication

Problems of the type:

(1) Addition

$$\begin{bmatrix} 3 & 1 & 5 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} (3+0) & (1+2) & (5+3) \\ (1-1) & (-1+1) & (0+2) \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$$

(2) Scalar Multiplication

(Any real number is called a scalar when working with matrices)

$$x \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} x & 2x \\ -x & 3x \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 5 & 10 \\ -5 & 15 \end{bmatrix}$$

4. **Matrix Multiplication**

Problems of the type: Matrices

(1) $\begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (1 \times 2) + (3 \times 1) + (-1 \times 3) \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$

(2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

(3)

$$\begin{array}{cc} & \begin{array}{ccc} C^1_1 & C^1_2 & C^1_3 \end{array} \\ \begin{array}{c} R^1_1 \\ R^1_2 \\ R^1_2 \end{array} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} \begin{array}{ccc} C^2_1 & C^2_2 & C^2_3 \end{array} & \\ \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} R^2_1 \\ R^2_2 \\ R^2_3 \end{array} \end{array}$$

$$\begin{bmatrix} (1)(-1) + (0)(3) + (-1)(1) \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix}$$

$$\begin{array}{cc} \begin{array}{c} R^1_1 \\ R^1_2 \\ R^1_3 \end{array} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} \begin{array}{ccc} C^2_1 & C^2_3 & C^2_3 \end{array} & \\ \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} R^2_1 \\ R^2_2 \\ R^2_3 \end{array} \end{array}$$

$$\begin{bmatrix} (1)(0) + (0)(1) + (-1)(0) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{array}{cc}
 & \\
R^1_1 & \\
R^1_2 & \\
R^1_3 &
\end{array}
\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}
\quad
\begin{array}{ccc} C^2_1 & C^2_2 & C^2_3 \end{array}
\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \end{bmatrix}
\begin{array}{c} R^2_1 \\ R^2_2 \\ R^2_3 \end{array}$$

$$\begin{bmatrix}(1)(2) + (0)(-4) + (-1)(0)\end{bmatrix} = \begin{bmatrix}2\end{bmatrix}$$

$$\begin{array}{c} R_1^{\ 1} \\ R^1_2 \\ R^1_2 \end{array}
\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}
\quad
\begin{array}{ccc} C^2_1 & C^2_2 & C^2_3 \end{array}
\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \end{bmatrix}
\begin{array}{c} R^2_1 \\ R^2_2 \\ R^2_3 \end{array}$$

$$\begin{bmatrix}(2)(-1) + (1)(3) + (3)(1)\end{bmatrix} = \begin{bmatrix}4\end{bmatrix}$$

$$\begin{array}{c} R^1_1 \\ R^1_2 \\ R^1_3 \end{array}
\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}
\quad
\begin{array}{ccc} C^2_1 & C^2_2 & C^3_3 \end{array}
\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix}(2)(0) + (1)(1) + 3(0)\end{bmatrix} = \begin{bmatrix}1\end{bmatrix}$$

$$\begin{array}{c} R^1_1 \\ R^1_2 \\ R^1_3 \end{array} \begin{bmatrix} 1 & 0 & -1 \\ \boxed{2} & \boxed{1} & \boxed{3} \\ 3 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \boxed{2} \\ 3 & 1 & \boxed{-4} \\ 1 & 0 & \boxed{0} \end{bmatrix}$$

$$\left[(2)(2) + (1)(-4) + (3)(0)\right] = \left[\,0\,\right]$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ \boxed{3} & \boxed{-1} & \boxed{2} \end{bmatrix} \cdot \begin{bmatrix} \boxed{1} & 0 & 2 \\ \boxed{3} & 1 & -4 \\ \boxed{1} & 0 & 0 \end{bmatrix}$$

$$\left[(3)(1) + (-1)(3) + (2)(1)\right] = \left[\,2\,\right]$$

Answer: $\begin{bmatrix} -2 & 0 & 2 \\ 4 & 1 & 0 \\ 2 & -1 & 10 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ \boxed{3} & \boxed{-1} & \boxed{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & \boxed{0} & 2 \\ 3 & \boxed{1} & -4 \\ 1 & \boxed{0} & 0 \end{bmatrix}$$

$$\left[(3)(0) + (-1)(1) + 2(0)\right] = \left[-1\right]$$

$$\left[\begin{array}{ccc} 1 & 0 & -1 \\ 2 & 1 & 3 \\ \hline \multicolumn{1}{|c}{3} & -1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\right] \qquad \left[\begin{array}{cc|c|} \cline{3-3} 1 & 0 & 2 \\ 3 & 1 & -4 \\ 1 & 0 & 0 \\ \cline{3-3} \end{array}\right]$$

$$\Big[(3)(2) + (-1)(-4) + (2)(0)\Big] \quad = \quad \Big[10\Big]$$

Some work with determinants has been done in Algebra 1, and it is suggested that expansion by minors be reviewed here.

5. Minors

The minor of any given element of a determinant is the determinant that remains when the column and the row containing the given element are deleted.

Example: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The minor of a_1: $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ The minor of a_2: $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$

The minor of c_3: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Any row or column may be selected as the one in terms of whose minors the determinant is to be expanded.

The first column was chosen for the example. Since a_1 was in the first row and the first row and first column $(1 + 1 = 2$, an even number), the sign of the product is +

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Since a_2 was in the second row and first column $(2 + 1 = 3$, an odd number), the sign of the product is -

$$-a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

Since a_3 was in the third row and first column $(3 + 1 = 4$, an even number) the sign of the product is +

$$a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

The value of the determinant is: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$ (continued on next page)

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$a_1 (b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) =$$

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

The students should work some third and fourth order determinants by expansion by minors until they become efficient in the solutions.

XIV. Solid Analytic Geometry

Note: This unit is covered very briefly in a few textbooks, and not at all in others. This material will be given in more detail than in previous units in this syllabus so that it may be used as a student text.

A. Rectangular Coordinates

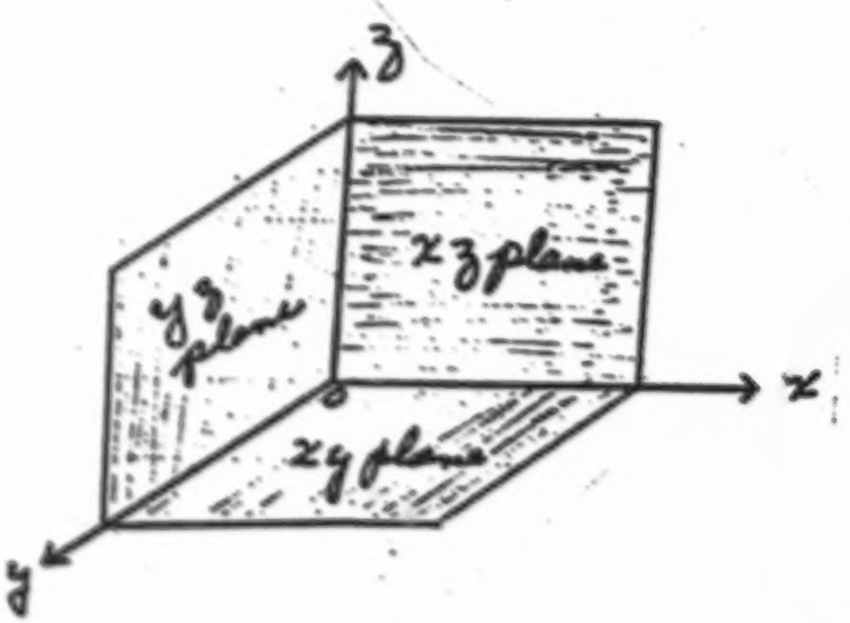
The foundation of a rectangular system of coordinates is three mutually perpendicular planes called coordinate planes, which intersect in pairs in three mutually perpendicular lines called coordinate axes. The point common to all of these planes and lines is called the origin (0, 0, 0).

The coordinate axes are usually designated as x, y, z axes and the coordinate planes as xy, yz, xz planes corresponding to the axes that they contain. Let a positive direction be assigned to each coordinate axis. Each axis is then a directed line and determines the positive side of the coordinate plane perpendicular to it.

The Coordinates (x, y, z) of any point are defined to be its directed distance from the coordinate planes: x is the directed distance from the yz plane, y is the directed distance from the xz plane, and z is the directed distance from the xy plane.

The Coordinates (x, y, z) are always to be written in that order and so constitute an ordered triad of real numbers.

This is a left-hand coordinate system.



Octants

	x	y	z
I	+	+	+
II	-	+	+
III	-	-	+
IV	+	-	+
V	+	+	-
VI	-	+	-
VII	-	-	-
VIII	+	-	-

In order to represent point, lines, and other figures in space by means of drawings on paper, certain conventions must be adopted.

One axis is drawn horizontal and directed to the right, one is drawn vertical and directed upward, and one axis is drawn at an angle of 135° with each of the other two and directed toward the observer.

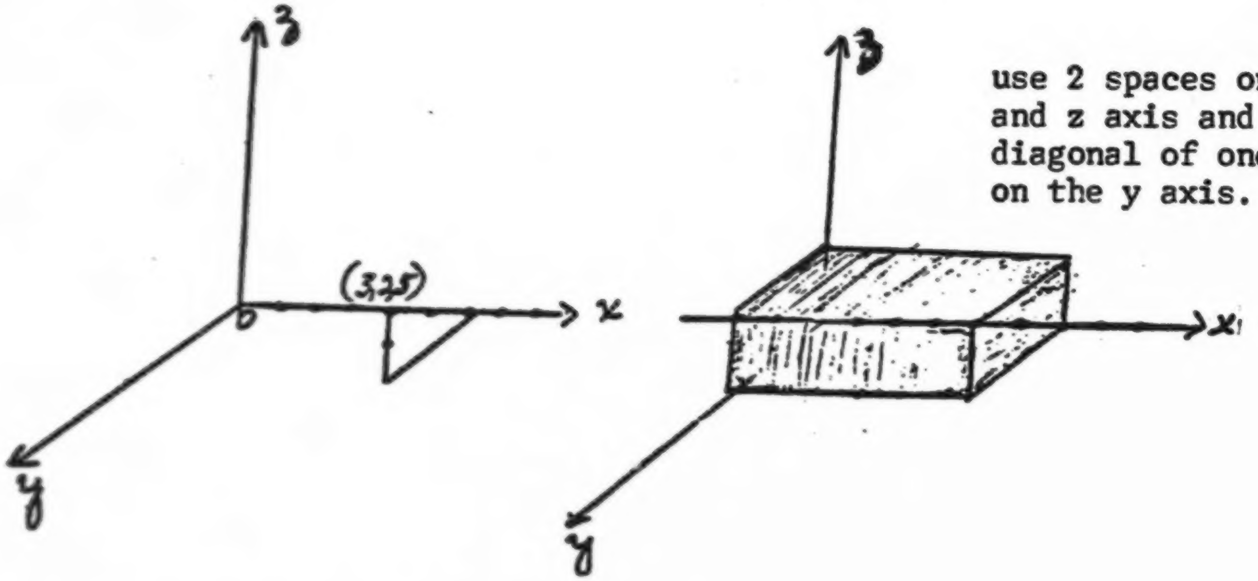
The x-coordinate is + when a point P is to the right of the yz plane, negative when to the left, and 0 when in the plane.

The z-coordinate is + when P is above the xy plane, negative when below, and 0 when in the plane. This choice of a coordinate system is for conformity.

For convenience, use 2 spaces on the x and z axes, and the diagonal of one square for the unit on the y-axis. (Graph paper).

Problems: Plot the following points in two ways (broken line and parallelpiped).

(3, 2, 5)



use 2 spaces on the x and z axis and the diagonal of one square on the y axis.

At least ten problems of this type should be assigned so that students become adept at graphing.

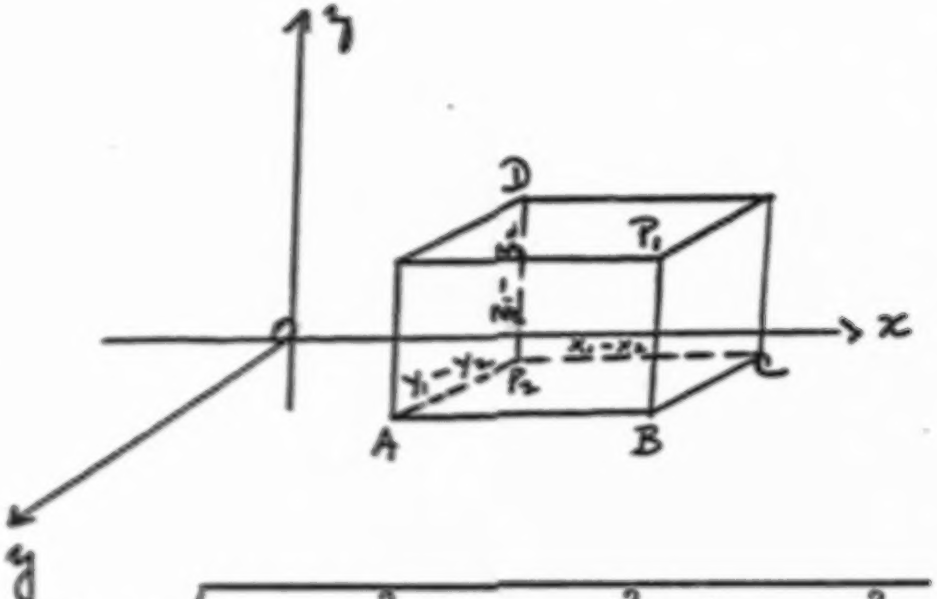
1) (1, -1, 1)
2) (-2, 2, -2)
3) (4, -2, -1)
4) (2, 0, 4)
5) (-2, 2, 3)
6) (-2, -3, 4)
7) (3, 5, -1)
8) (-3, -2, -2)
9) (5, 0, 0)
10) (3, -2, -4)

B. Distance between two points

Let P_1 $(x_1y_1z_1)$ and P_2 $(x_2y_2z_2)$ be any points in space. Then P_1 and P_2 are opposite vertices of a rectangular parallelepiped (which may collapse to a rectangle, a line segment, or a point).

The edges of this parallelepiped are parallel to the coordinate axes and equal in length to the projections of the line segment P_1P_2 on the Axes.

P_1P_2 is a diagonal of the parallelepiped, and the length of the diagonal can be found.



$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Problems to be assigned.

Graph and find the distance P_1P_2

1) P_1 (4, -1, 9) P_2(-2, 1, 6)
2) P_1 (0, -2, 2) P_2(4, -9, 6)
3) P_1 (2, 5, 8) P_2(1, 2, 6)
4) P_1(2, -3, 1) P_2(5, -6, -2)
5) P_1(2, -3, 8) P_2(7, 4, -1)

C. Projections

A point P in space may be projected on a plane $\underline{A}$ or on line $\underline{\ell}$. The line through P $\perp$ to $\underline{A}$ intersects $\underline{A}$ in a point called the orthogonal projection of P on $\underline{A}$.

The plane through P $\perp$ to $\underline{\ell}$ intersects $\underline{\ell}$ in a point called the orthogonal projection of P on $\underline{\ell}$.

The projection of any set of points is defined to be the collection of all projections on the individual points of the set. The projection of a point is a point, and the projection of a line segment may be a line segment or a point.

$$\text{Proj}_x\, P_1\, P_2 = x_2 - x_1 \qquad \text{Proj}_y\, P_1\, P_2 = Y_2 - Y_1$$

$$\text{Proj}_z\, P_1\, P_2 = z_2 - z_1$$

Find the projections of P_1P_2 on the Coordinate Axes (3,2,1) (-2,2,3)

Problems:

Find the projection P_1P_2 on the coordinate axes.

(1) P_1 $(-2,-3,4)$ $P_2(0,-3,0)$

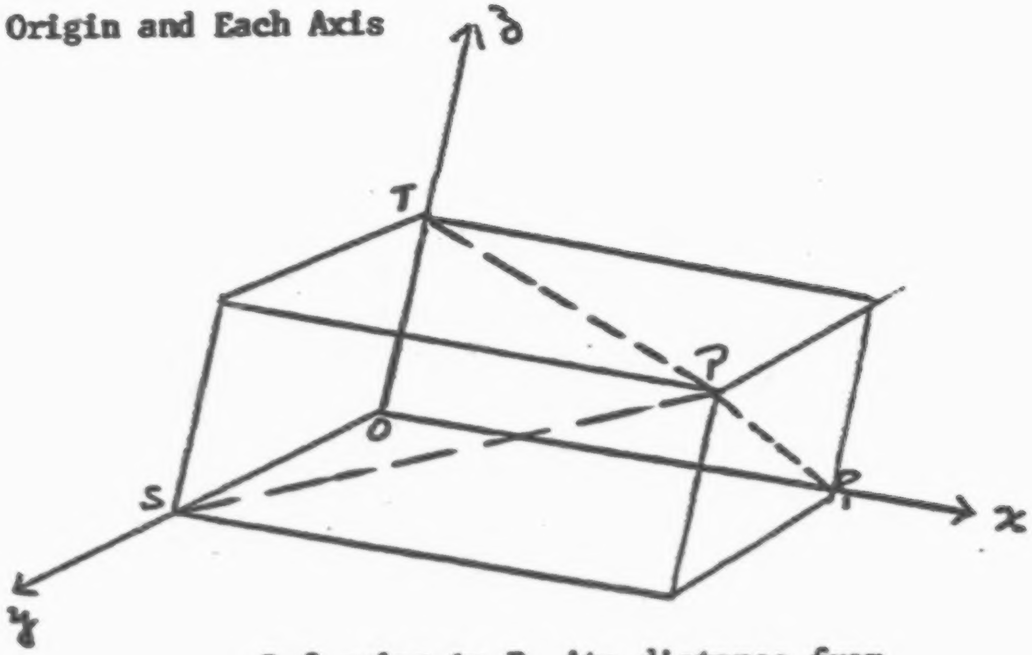
(2) $P_1(3,-2,-4)$ $P_2(2,0,4)$

(3) $P_1(1,-1, 1)$ $P_2(3,-2,-4)$

(4) $P_1(-2,2,3)$ $P_2(3, -2, -4)$

(5) P_1 $(-3, -2, -2)$ $P_2(4, 3, -2)$

D. Distance from Origin and Each Axis



Referring to P, its distance from

OX is $RP = y^2 + z^2$,

OY is $SP = x^2 + z^2$,

OZ is $TP = x^2 + y^2$ and from the origin is $OP = x^2 + y^2 + z^2$

Problems:

Have the students use the figures they drew for the distance between two points and the same points to find the distance from the origin and each axis.

Example: (4, -1, 9)

$RP = \sqrt{1 + 81}$ $\qquad SP = \sqrt{16 + 81}$

$RP = \sqrt{82}$ $\qquad SP = \sqrt{97}$

$TP = \sqrt{16 + 1}$ $\qquad OP = \sqrt{16 + 1 + 81}$

$TP = \sqrt{17}$ $\qquad OP = 7\sqrt{2}$

E. Direction Cosines

The angles α, β, γ that a directed line (or line segment) makes with the positive x, y, and z axes, respectively, are known as the direction angles of the line (or line segment), and the Cosines of these angles are the direction Cosines of the line (or line segment).

The notation to be used here is: $\quad \cos\alpha \;, \; \cos\beta \;, \; \cos\gamma$.

If $P_1 P_2$ is any directed line segment in space with length d and direction angles

$$(1) \quad \cos\alpha = \frac{x_2 - x_1}{d} \qquad \cos\beta = \frac{y_2 - y_1}{d} \qquad \text{and } \cos\gamma = \frac{z_2 - z_1}{d}$$

$$\text{where } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Squaring these expressions for the direction cosines, adding, and using the formula for the distance d between two points, we have the fundamental relation between direction cosines of a directed line segment.

$$\text{Cos}^2 \alpha + \text{Cos}^2 \beta + \text{Cos}^2 \gamma = 1$$

If the directed line segment is chosen from the origin 0 to the point P(x, y, z) and we let p be the distance between 0 and P, then the relations given in (1) $\text{Cos}\ \alpha = \dfrac{x_2 - x_1}{p}$, etc.

Give formulas for the direction cosines of the radius vector of P.

$$\text{Cos}\ \alpha = \frac{x}{p}, \quad \text{Cos}\ \beta = \frac{y}{p}, \quad \text{Cos}\ \gamma = \frac{z}{p}$$

In particular, if P is a point on the unit sphere, then p = 1 and the coordinates of P are identical with the direction cosines of its radius vector.

Give problems of the type:

Find the direction cosines of each line:

(1) $P_1(5,4,-2)\ P_2(1,2,2)$

(2) $P_1(7,3,6)\ P_2(6,5,4)$

(3) $P_1(1,1,4)P_2(-2,0,7)$

(4) $P_1(2,-1,1)\ P_2(4,-7,-2)$

(5) $P_1(1,-3,-2)P_2(-1,11,3)$

Example: $P_1\ (2, 5, 8)$ $P_2\ (1, 2, 6)$

$1 - 2 = -1$ $d = \sqrt{1 + 9 + 4}$

$2 - 5 = -3$ $d = \sqrt{14}$

$6 - 8 = -2$

$$\text{Cos } \alpha = \frac{-1}{\sqrt{14}} = -.27027 \qquad \alpha = 105^\circ 41'$$

$$\text{Cos } \beta = \frac{-3}{\sqrt{14}} = -.81081 \qquad \beta = 144^\circ 11'$$

$$\text{Cos } \gamma = \frac{-2}{\sqrt{14}} = -.54054 \qquad \gamma = 122^\circ 43'$$

F. Direction Numbers of a Line

Any three real numbera a, b, c (not all zero) are called a set of direction numbers of a line if they are proportional to the direction cosines of the directed line.

If a, b, c are a set of direction numbers of a line whose direction Cosines are Cos α, Cos β, and Cos γ there must exist a constant $k \neq 0$ such that $a = k \cos \alpha$, $b = k \cos \beta$, $c = k \cos \gamma$.

To find the direction cosines of a line when a set of its direction numbers a, b, c are given, determine the factor of proportionality k.

By squaring the members of the equations $a = k \cos \alpha$ etc., adding and simplifying, we obtain:

$$a^2 + b^2 + c^2 = k^2\ (\text{Cos}^2 \alpha + \text{Cos}^2 \beta + \text{Cos}^2 \gamma) = k^2$$

$$k = \pm \sqrt{a^2 + b^2 + c^2}$$

Substitute this expression for k in $a = k \cos \alpha$ etc., and solve.

$$\cos \alpha = \frac{a}{\pm\sqrt{a^2 + b^2 + c^2}} \qquad \cos \beta = \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \gamma = \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}$$

Problems: Find the direction cosines of the lines having the given direction numbers.

1) (2, 1, 3)
2) (-1, -1, -1)
3) (-2, -3, 5)
4) (4, -2, -1)
5) (0, 0, -2)

Example: (1, -2, -5)

$$\cos \alpha = \frac{1}{\pm\sqrt{1 + 4 + 25}} = \frac{1}{\sqrt{30}} = .18182$$

$$\cos \beta = \frac{-2}{\pm\sqrt{30}} = \mp .36364$$

$$\cos \gamma = \frac{-5}{\pm\sqrt{30}} = \mp .90909$$

G. Equation of a Plane

The graph of any linear equation in three variables is a plane. Let L be any directed line having direction numbers ℓ, m, n, and let point $P_o(x_o\ y_o\ z_o)$ be any point on L. To write an equation of a plane passing through P_o and $\perp$ to L, we notice that such a plane is the locus of points P(x, y, z) satisfying the condition that P_oP is $\perp$ to L.

The projection of P_oP on the axis are $x - x_o$, $y - y_o$, $z - z_o$, and these projections furnish a set of direction numbers for the line segment P_oP.

The direction numbers of P_oP and line L must satisfy the relation $\ell(x - x_o) + m(y - y_o) + n(z - z_o) = 0$.

Expand and simplify $\ell x + my + nz = \ell x_o + my_o + nz_o$.

The right member is the constant and the equation may now be written $Ax + By + Cz = D$.

Problems:

Find the equation of the plane through the following points.

1. (2, 3, 5) (-1, -1, -1) (2, 6, 5)
2. (1, -3, 2) (3, 1, 4) (-1, -1, -2)
3. (1, 1, 6) (2, -1, -1) (5, -2, 3)
4. (2, 1, 3) (6, 3, 5) (-2, 2, 8)
5. (1, 2, 4) (5, 1, 1) (10, -5, 2)

Example: (1, 12, 1) (2, 7, -1) (-4, 5, 3)

$$Ax + By + Cz = D$$

$$\begin{array}{ll} A + 12B + C = D & \quad 2A + 24B + 2C = 2D \\ 2A + 7B - C = D & \quad \underline{-2A - 7B + C = -D} \\ -4A + 5B + 3C = D & \quad 17B + 3C = D \end{array}$$

$$\begin{array}{r} 4A + 48B + 4C = 4D \\ \underline{-4A + 5B + 3C = D} \\ 53B + 7C = 5D \end{array}$$

$$\begin{array}{r} A + 12B + C = D \\ \underline{2A + 7B - C = D} \\ 3A + 19B = 2D \end{array}$$

$$\begin{array}{r} 6A + 21B - 3C = 3D \\ \underline{-4A + 5B + 3C = D} \\ 2A + 26B = 4D \end{array}$$

$$\begin{array}{r} 6A + 38B = 4D \\ \underline{-6A - 78B = -12D} \\ -40B = -8D \\ B = \frac{D}{5} \end{array}$$

$$\begin{array}{r} 901B + 159C = 53D \\ \underline{-901B - 119C = 85D} \\ 40C = -32D \\ C = \frac{-32D}{40} \\ C = -\frac{4D}{5} \end{array}$$

$$3A = 2D - \frac{19D}{5}$$

$$3A = \frac{10D - 19D}{5} \rightarrow 3A = -\frac{9D}{5} \rightarrow A = -\frac{3D}{5}$$

The equation is $3x - y + 4z = -5$. These problems could also be solved by determinants.

Note: If there are any students who have not had "New Math," it is suggested that the "Worktext in Modern Mathematics" published by Harper and Row could be used as a unit in this course.

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O.C.S.E.I.P. SYLLABUS

Introductory Calculus and Linear Algebra

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PREFACE

This syllabus provides a comprehensive guide to the first two years of university mathematics and is written for the teachers' use, not for the students'. The committee interpreted university mathematics to mean that attention was focused on the calculus sequence rather than the entire mathematics curriculum for grades thirteen and fourteen.

The belief that the present mathematics curriculum has lagged behind the growth and uses of the subject was one of the ideas that motivated the committee in their writing. The committee thought that the development of mathematics and the broadening of its applications have outgrown the curriculum. We believe that the core of the subject will remain unchanged but that the syllabus will present to the instructor a reorganization of the material, some methods of approach, intuitive examples, applications, suggestions for additions and deletions, amount of mathematical rigor, proofs to be included as well as those to be omitted, and selected reference books. These changes will be relatively minor in overall content, but tremendously important in teaching emphasis.

The syllabus as the committee outlined it was not designed especially for use with the superior student but for use with the student who will use college mathematics and who comes to college with a substantial high school preparation for it. We kept in mind in the writing of the syllabus the OCSEIP recommendations for high school mathematics; that is we assume that the student entering our program has had analytical geometry including vectors in two dimensional space and three dimensional space. This course does not have to be the complete traditional analytical geometry course but could have such topics as conic sections omitted.

Though there is a tendency for some high schools to move calculus to the twelfth grade, it is the recommendation of this committee that the teaching of calculus should be restricted to the colleges. There are many ideas and topics that should and can be covered in high school that would give a student a much better background for the study of calculus. One such topic would be an introduction to vectors in two-dimensional space and three-dimensional space. Other topics which could be extremely useful from the point of view of engineering are elementary probability and statistics.

In offering the following outlines we have omitted many definitions and proofs. Since many of these proofs and definitions would be in the textbook, it was thought that this omission would allow the teacher flexibility in the use of the textbook. Included in the outlines are our comments on some of the problem areas, with suggested rearrangement of topics and emphasis on methods of approach.

This committee does not feel that there is only one possible sequence of topics. However, we have tried to formulate a basic list of topics. The syllabus is not intended to be a continuous sequence but is taken up topic by topic so that the order may be adjusted to the textbook being used. The amount of time spent on a given topic will also vary with the textbook.

The bibliography should be regarded as an attempt to include as many useful references as the committee had time to peruse during this writing. More than one reference is sometimes given in hopes that the instructor might have at least one of these books readily available. Since completion of the original bibliography other books have been added, in a supplementary bibliography. Books in this are listed with primes, as [4'].

It is the committee's opinion that the course should develop in the student the idea of what constitutes a proof, and the necessity for a proof, so that, by the second year's material, he will be prepared for a rigorous development. During the first year, time should be spent on a more intuitive justification, although care should be taken to point out any difficulties that might arise. Thus, it is the committee's intention that the course not be developed rigorously from the start, but that the student's understanding of rigor should be developed throughout the course.

An appendix listing some possible final exam questions for the first semester course is included, to give an idea of the committee's feelings as to course content. We have used four courses of four-semester hours each as the building blocks for our outlines. A variation of this form that was very appealing to the committee was a sequence consisting of two four-semester hour courses and then three three-semester hour courses. In this latter sequence the first two courses would be the same as our outlines. The other three courses would consist of linear algebra, differential equations and infinite series, and vector calculus. With some modifications the committee feels that the outline could very easily conform to this sequence. A source of possible course arrangements was the report to the Mathematical Association of America, "A General Curriculum in Mathematics for Colleges". This report was used for much of our basic philosophy and influenced the design of the courses. For a motivation of many of the topic selections and especially the placing of a formal course in linear algebra in the first semester of the second year, see the aforementioned report.

To select a textbook to fit each course outlined is a difficult task. The committee feels that there are textbooks available which give a fair approximation to the outlines, but none which we have reviewed that we would call a basic text. The majority of the committee recommends the careful examination of [7] and [23] (Calculus Bibliography).

For linear algebra the topic coverage in [10] (Linear Algebra Bibliography) is one of the most complete as far as this outline is concerned but we are not in complete agreement with the approach. In the vector calculus and differential equations course the books [8] and [12] (Calculus Bibliography) start out with good topic coverage but do not go as far as our outline. The book with the most complete topic coverage for this course is [4] (Calculus Bibliography).

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CALCULUS - 1st YEAR

(8 semester hours)

I. LIMITS AND CONTINUITY 3 hours

A. Limits of sums, products and quotients

1. Numerical and graphical illustrations to motivate an intuitive understanding of limits

B. Continuous functions

1. Suggested illustrations of discontinuous functions, contrasted with continuous, and continuous "smooth" functions

C. Extreme and intermediate value theorems (no proofs)

1. Very brief statement of the theorems (since they will be in the textbook in use)

II. DERIVATIVE 4 hours

A. Definition of derivative; tangents (general discussion)

1. Numerical example from plane geometry of tangent to a circle

B. Average and instantaneous rates of change used to define the derivative

C. Derivatives of polynomials by the Δ-process definition of the derivative used to define the tangent line to a curve

III. TECHNIQUES OF DIFFERENTIATION 10 hours

A. Formulas for derivatives of sums, products, quotients, rational functions and x^n (n rational)

B. Chain rule

C. Derivatives of implicit functions

D. Definitions of higher order derivatives

E. Definition of differentials and applications

IV. APPLICATIONS OF DERIVATIVES 10 hours

A. Graphical approach to extrema of continuous functions in an open interval

        1. Mean Value Theorem and Rolle's Theorem

    B. Extrema at end points of a closed interval or where derivatives fail to exist

    C. Interpretations and applications of first and second derivatives for extrema for continuous functions and to curve sketching; horizontal and vertical tangents

    D. Velocity and acceleration applied to rate problems

    E. Newton's Method

    F. Maxima and minima problems

V. ANTIDIFFERENTIATION 3 hours

    A. Antidifferential formulas

    B. Applications to acceleration, velocity and distance

VI. THE DEFINITE INTEGRAL 3 hours

    A. Intuitive definition of the definite integral

    B. Fundamental theorem of integral calculus

    C. Differentiation under the integral

VII. APPLICATIONS OF THE DEFINITE INTEGRAL 9 hours

    A. Area

        1. Area between curves and coordinate axis

        2. Area between two curves

    B. Volumes of solids of revolution, by disk, washer, and shell methods

    C. First moments and centers of gravity

        1. Moments of areas

        2. Moments of volumes

    D. Arc length of plane curve

    E. Area of a surface of revolution

VIII. TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS AND THEIR INVERSES 10 hours

A. Derivatives of the trigonometric functions

1. Limit theorems

2. Derivatives of other trigonometric functions

B. Derivatives of inverse trigonometric functions

1. Integrals

C. ${}_{1}^{x}\frac{dt}{t}$; derivative of $\log_e x$

D. Derivatives and integrals of a^x and e^x

IX. HYPERBOLIC FUNCTIONS 3 hours

A. Definition and derivatives

B. Hyperbolic radian

C. Inverse hyperbolic function

D. Integration

X. TECHNIQUES OF INTEGRATION 9 hours

A. Methods of substitution

1. Algebraic substitution

2. Partial fractions

3. Trigonometric substitution

B. Integration by parts

1. Inverse trigonometric functions

2. Tabular device

C. Approximations

1. Simpson's Rule and Trapezoidal Rule

2. Other numerical methods

XI. VECTORS 18 hours

- A. Introduction
- B. Vectors in the plane
- C. Vectors in three dimensions
  1. Definition
  2. Equations of lines and planes
  3. Scalar and cross products
  4. Motion on space curves
  5. Derivatives
  6. Curvature, tangent, and normal

XII. PARTIAL DIFFERENTIATION 10-11 hours

- A. Introduction
- B. Functions of several independent variables
  1. Scalar fields
  2. Neighborhoods and open sets
- C. Derivative of a function with respect to a vector
  1. The directional derivative
  2. The partial derivative
  3. Continuity and limits of a scalar field
  4. Mean-value theorem for scalar fields
  5. Linearity of the derivative
- D. The gradient of a scalar field: ∇f
  1. Definition
  2. Geometric interpretation of ∇f
  3. Tangent plane and normal line
  4. Maxima, minima, and saddle points

E. The increment of a scalar field: ∇f

  1. Chain rule for derivatives of scalar fields

  2. The total differential: df

F. Mixed partial derivatives

G. Implicit functions

H. Line integrals

  1. Line integrals in the plane

  2. Geometric interpretation

  3. Path independent line integrals

  4. Work

  5. Evaluation

XIII. MULTIPLE INTEGRATION 8 hours

A. Double integrals

  1. Motivation

  2. Definition and properties

  3. Evaluation

  4. A counterexample

  5. Applications

B. Change of variables

C. The triple integrals

  1. Motivation

  2. Definition and properties

  3. Evaluation

  4. Applications

CALCULUS - 1st Year 8 Semester Hours

I. LIMITS AND CONTINUITY

Limits and continuity of real valued functions of real variables are considered in elementary calculus.

A. Limits of sums, products and quotients

These are first considered from an intuitive approach, building on previous experiences, since the concept of limit is often elusive. Following the examples, which should include both algebraic and graphical examples, students may be able to verbalize the limit theorems for sums, products and quotients and compositions (or at least believe in their validity). The illustration from plane geometry of the relationship between the circumference of a circle and the perimeters of inscribed and circumscribed polygons can be recalled. Emphasize uniqueness of limits with a proof of this fact.

1. An example of limits from arithmetic is the fact that

$$\frac{1}{3} = 0.33 = 0.3 + 0.03 + \ldots$$

Many modern arithmetic texts at the college level will give other examples, and some of the students will have had geometric progressions and be able to use the formula

$$\lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

(in this case: $\frac{0.3}{1 - 0.1} = \frac{0.3}{0.9} = \frac{1}{3}$). For the rest write

$$\begin{array}{rl} 10\ S = & 3 + 0.3 + 0.03 + 0.003 + \ldots \\ S = & \phantom{3 +{}} 0.3 + 0.03 + 0.003 + \ldots \\ \hline 9\ S = & 3 \end{array}$$

and $S = \frac{3}{9} = \frac{1}{3}$ (Students are usually very willing to accept the one-to-one matching of the above infinite series.) Finally, the ϵ-δ definition should be discussed, and the limit of a constant given as the first application.

The theorem concerning the limit of a sum should be proved, and the student may be asked to reproduce it.

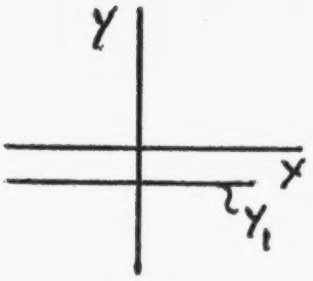
For an example of limits of sums we could start with a straight line such as y = x - 1, considering the values y assumes as x takes on values close to 2. Consider the table:

x = 1.99	1.999	1.99999
y = 0.99	0.999	0.99999

x = 2.01	2.001	2.00001
6 = 1.01	1.001	1.00001

together with the graphs of (1) $y_1 = -1$ (2) $y_2 = x$ (3) $y_3 = x - 1$

(1)



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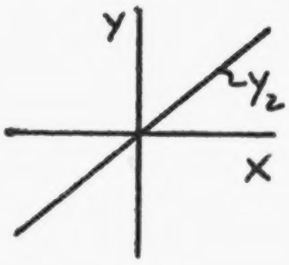

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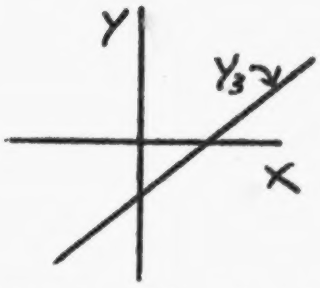


$\lim_{x \to 2} -1 = -1$ $\lim_{x \to 2} x = 2$ $\lim_{x \to 2} x - 1 = \lim_{x \to 2} x + \lim_{x \to 2} -1 = 1$

2. For motivation of an understanding of a product it is suggested that something on the order of $(x - 1)(x + 2) = x^2 + x - 2$ be considered by graphing both factors and then their product, separately as in the example on sums to show that

$$\lim_{x \to n} (x^2 + x - 2) = \lim_{x \to n} (x - 1) \cdot \lim_{x \to n} (x - 2),$$

(where n should include very large and very small numbers). An agreement as to how the handy but much abused symbol ∞ will be used in the course can be reached here (for example some texts use $-\infty < x < +\infty$ to mean all real numbers).

3. For quotients, two types are suggested with a graphical approach similar to that for products. Quotients with a constant in the denominator are somewhat trivial and should be treated as constant times a sum or product.

First a function such as $f(x) = \frac{x^2 - 1}{x - 1}$ might be considered drawing the graphs of $x^2 - 1$, $x - 1$ and $x + 1$ and considering limiting values of the quotient as compared to the limiting values of $x + 1$.

a. A second type is exemplified by $\frac{x^2 - 4}{x^2 - 5x + 6}$. Limiting values of this function as $x \to 0$, $x \to 2$, $x \to -2$, $x \to 3$ and for very large and very small values of x should be considered. Many other examples may be found in [32].

Separate graphs of $x^2 - 4$, $x^2 - 5x + 6$, and $\frac{x + 2}{x - 3}$ should be drawn and the intuitive development of the limit theorem for quotients continued. Examples of this sort should convince the students that the quotient of $\frac{0}{0}$ is not unique, and $\frac{a}{0}$ $(a \neq 0)$ is undefined. A formal statement of the limit theorems is omitted since these will be in any textbook. As noted above, the theorem concerning the limit of a sum should be proven. In a good class, the other theorems may be proven also.

b. Continuous functions

The same examples used in "a" will motivate a discussion on continuous functions, by examining the discontinuities present. The classical examples of $y = x^2$ and $y = x$ are in most textbooks. The definition of continuity in the class discussion of "approaches," "close to," "in the neighborhood of" and "nearly" can also be generated from the graphs in "a." It is strongly recommended that an agreement be reached that numbers "close to" or "in the neighborhood of" a number N be in the interval $n - \epsilon < n < n + \epsilon$ for small epsilon, in order to lay the foundation for future delta-epsilon use.

1) Some examples such as $f(x) = \begin{cases} -1 & ; x < 0 \\ 0 & ; x = 0 \\ x^2 & ; x > 0 \end{cases}$, f(x)

the signum function and step functions

$$f(x) = \begin{cases} 1 & ; x \text{ rational} \\ -1 & ; x \text{ irrational} \end{cases}$$

and other similar functions should be introduced here. An intuitive recognition that all polynomials of the form

$y = a_0x^n + a_1x^{n-1} + \dots + a_n$, ($a_i$; i = 0.1, ...

constants) are continuous can be generalized here, as well as conclusions as to the continuity of the sum, product, quotient of two continuous functions and composition of functions (use the examples from "a").

c. Extreme and intermediate value theorems (no proofs)

These theorems should be presented from an intuitive pictorial approach.

1) Extreme value theorem. If f is continuous over an interval $a \leqq x \leqq b$ it has a minimum value m and a maximum value M, that is, there are numbers x_1 and x_2 in the interval such that $m = f(x_1)$ and $M = f(x_2)$ and such that for all x in the interval the condition $m \leqq f(x) \leqq M$.

2) Intermediate value theorem. If f is continuous over an interval $a \leqq x \leqq b$ and if K is a constant for which $f(a) < K < f(b)$ or $f(a) > K > f(b)$ then there exists at least one number c for which $a < c < b$ and $f(c) = K$.

These theorems need only be stated at this time since further use of them will be made. All discussion of uniform continuity should be omitted.

II. DERIVATIVE

A. Definitions of derivative, tangents

Formal symbolic statements of the definitions of the derivative of a function and of the tangent to a curve will be in the textbook being used and are therefore omitted here. At some time during this discussion, the definitions in a mathematics dictionary should be examined, particularly to clarify the student's plane geometry concept of a tangent line to a curve. Notation for the derivative should be introduced to conform to the text. Use of dy/dx can be deferred until differentials are discussed, to avoid confusion.

1. The following numerical example uses the student's previous knowledge of a tangent to a circle (including the fact that the radius of the circle and the tangent to the circle are mutually perpendicular at the point of tangency):

   To show that the limiting value of the slope of the secant line $\overline{P_1P_2}$ is equal to the slope of the tangent to the circle at P_2.

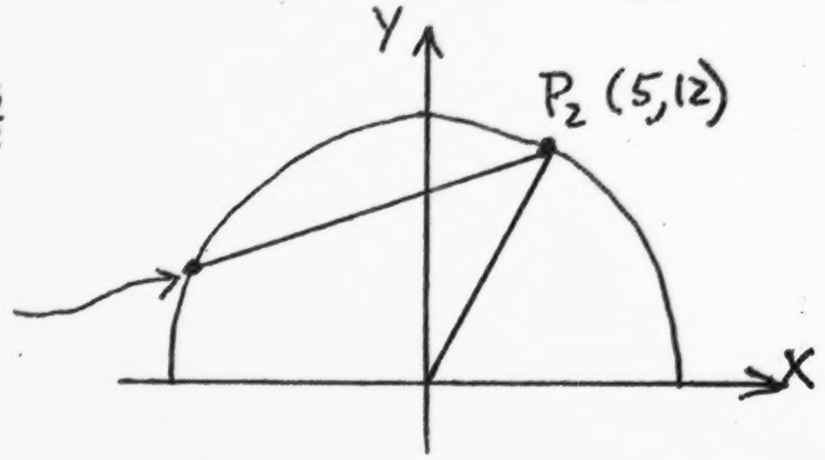
   Consider the semi-circle $y = 169 - x^2$ and choose a general point $P_1(x, \sqrt{169 - x^2})$ and a point $P_2(5, 12)$, and write the slope of the line segment P_1P_2 as $m_{\overline{P_1P_2}} = \dfrac{12 - \sqrt{169 - x^2}}{5 - x} =$

   $-\dfrac{x + 5}{12 + \sqrt{169 - x^2}}$ and take limits as $x \to 5$ of $m_{\overline{P_1P_2}}$.

   $$\lim_{x \to 5} m_{\overline{P_1P_2}} = \frac{-5}{12}$$

   and $m_{\overline{OP_2}} = \dfrac{12}{5}$

   $P_1(x, \sqrt{169 - x})$



   The pattern of $m_{\overline{P_1P_2}} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x + x) - f(x)}{x}$ where the coordinates of P_2 are taken more generally as (x_2, y_2), where $\Delta x = x_2 - x_1$, can be noted, together with the fact that the $\lim_{x_2 - x_1} m_{\overline{P_1P_2}}$ was equal to the slope of the tangent to the circle.

   The next examples generalize the pattern to other continuous functions.

B. Average and instantaneous rates of change

Student's familiarity with uniform motion problems and everyday experience is utilized to state that:

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

In general, however, we know that the velocity is not equal to the average velocity during the time elapsed; at various times it may be equal to, greater than and less than the average velocity.

Consider an object moving in a straight line, such that at any time t seconds, its distance s in feet is given by the formula $s = t^2 - 1$. Consider the following table of distance and time starting when t = 2 seconds:

time	t	2	1.1	1.01	1.001	1	$1 + h$
	$t^2 - 1$	3	.21	.0201	.002001	0	$2h + h^2$

Then consider the average velocity for the following time periods:

Time Period	Average Velocity
from 1 to 2 seconds	$\left[3\right]/1 = 3$ ft/sec
from 1 to 1.1 seconds	$\left[.21\right]/0.1 = 2.1$ ft/sec
from 1 to 1.01 seconds	$\left[.0201\right]/0.01 = 2.01$ ft/sec
from 1 to 1.001 seconds	$\left[.002001\right]/0.001 = 2.001$ ft/sec
from 1 to 1 + h seconds	$\left[2h + h^2\right]/h = 2 + h$ ft/sec

In any one time period we obtain only one average velocity and as the time intervals grow shorter we can see the average velocity values getting closer to 2 ft/sec. By using h as the time elapsed after 1 second we get a general expression for the average velocity of 2 + h ft/sec. If h is a very short time interval (say one-billionth of a second) the value of 2 + h does not differ from 2 by very much. This value *2* (which it should be noted is $\lim_{h \to o} (2 + h)$ is called the *instantaneous velocity* or the *derivative* of distance with respect to time at the instant t = 1 second. See [6].

The student's familiarity with automobiles in general, and drag races in particular may be utilized here. The speedometer measures instantaneous velocity, while the time traps at the end of the quarter-mile attempt to electronically approximate the instantaneous velocity by determining precisely the average velocity of a car while traveling through a distance of at most a few feet.

After the work on limits, the above example can be generalized to

$$\lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{(t^2 + 2th + h^2) - (t^2 - 1)}{h} =$$

$\lim_{h \to 0} (2t + h) = 2t$ (which gives a value of 2 when $t = 1$), which is precisely our intuitive guess from the average velocity table

If we have a function $y = f(x)$, then $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) = f(x)}{x}$, if it is a real number, will, by analogy, give the instantaneous rate of change of the ordinate withe respect to the abscissa, or the derivative of y with respect to x.

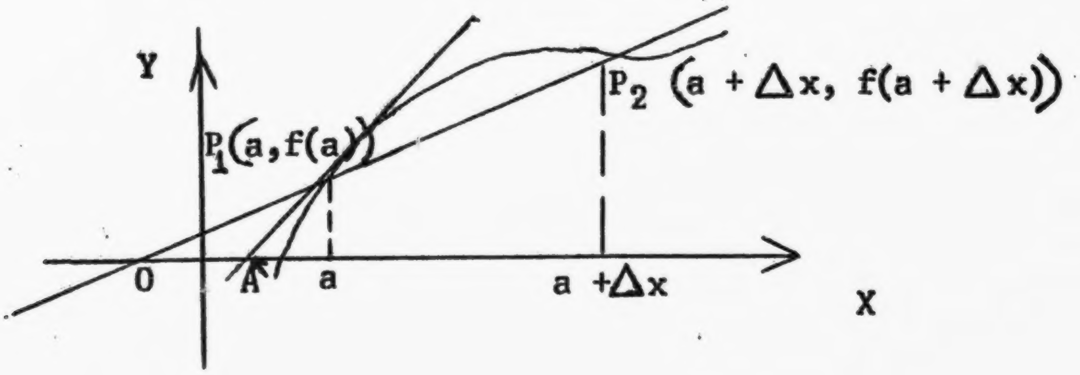
The numerical example in 1 should be linked to this discussion, as it appears that $P_1 \to P_2$, $(x_2 - x_1) \to 0$ and this problem has the pattern of $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, which is the ordinary definition of a derivative for a function $f(x)$; and for a circle, at least, this limiting value at a point also gives the slope of the tangent at this point.

A graph of $s = t^2 - 1$ will illustrate the connection between instantaneous velocity and the slope of a tangent line to a parabola at a point. It can be noted that the table of values for the average velocity in B approximates the slope.

C. Derivatives of polynomials by the Δ-process

The method of computing derivatives of polynomials by the Δ-process can not be demonstrated using the pattern established in the previous numerical examples.

1. (See A and B) The following geometric interpretation of the derivative may be used to motivate the definition of the tangent to a smooth continuous curve $y = f(x)$. P_1 is taken as fixed point, while P_2 moves along the curve



The slope of the line through P_1 and P_2 is given by

$$\frac{f(a + \Delta x) - f(a)}{(a + \Delta x) - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

When Δx is very small, P_2 is close to P_1 and the line through P_1 and P_2 is close to the line through A and P_1. This line through A and P_1 is called the tangent line to the curve at P_1 and its slope is given by the $\lim\limits_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$, if this limit exists. If the function is bounded and continuous, but $\frac{f(a + \Delta x) - f(a)}{\Delta x}$ is unbounded, then $x = a$ will be called the tangent to the curve at P_1. See [6].

This approach enables the tangent line to a curve to be defined, through the definition of the derivative, whereas many books use the reverse order (See [23] for a good development). The numerical examples listed here can still be useful. If the $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists at a point P, the function is said to possess a derivative at P or to be differentiable at P; if the limit exists for all points on the curve f(x) we say f(x) is differentiable. Similar comments can be made for intervals such as $a < x < b$ (a & b constants). Alternate forms, such as $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$ may prove useful, and can be introduced here.

III. TECHNIQUES OF DIFFERENTIATION

A. Formulas for derivatives of sums, products, quotients, rational functions and x^n (n rational) (u^n is covered in B)

These should be proven in class and students should be expected to memorize and be able to use these formulas, but to derive only the derivative of a sum. A suggested motivation for the need of such formulas, using distance and velocity, can be developed from an equality giving the distance as s, a function of time t; $s = At^2 + Bt + C$ (A, B and C non-zero constants). An application of the Δ-process gives the instantaneous velocity $v = 2At + B$ and the use of this as a formula to find velocity will quickly show the advantage of this shortcut over the application of the Δ-process to each individual $s = f(t)$ equation. The acceleration as the instantaneous rate of change of the velocity can be generalized here from the patterns for instantaneous rates of change.

It should also be proved that if a function possesses a derivative at a point, it is continuous at that point. The student could reasonably be expected to reproduce this proof on a test.

B. Chain Rule

After gaining some experience differentiating $y = x^n$, the student is usually tempted to say the derivative, with respect to x, of $y = u^n$, where u is a differentiable function of x, is $dy/dx = nu^{n-1}$. Thus, he would be led to say that the derivative of $y = (x^2 + 2)^2$ is $y^1 = 2(x^2 + 2)$. However, an application of the Δ-process, or a simple geometric agrument, based on a graph of $y = (x^2 + 2)^2$, shows this to be incorrect, since y has its minimum value at $x = 0$, and the tangent is horizontal there, so y' must be zero. Emphasis on this and other examples may cure some students of one common error.

At this point, the student should see the necessity for a new technique to handle functions such as $y = (x^2 + 2)^2$ and $y = x^2 + 1$. It should be pointed out that such functions can be considered composite functions. The idea of a "function of a function" can be introduced by suggesting the student make a substitution, say $u = x^2 + 2$ in the function $y = (x^2 + 2)^2$ so that $y = u^2$. The student will see that many functions may actually be composite functions, so that there is a need for the chain rule. This may be introduced as follows:

> If $F(u)$ is a composite function, with $u = f(x)$, and with F and f differentiable, by the definition of derivative,
>
> $$\frac{dF}{dx} = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \text{ if } \Delta u \text{ is not zero}$$
>
> for values of x close to zero. Thus, $u \to 0$ as $x \to 0$, and vice-versa, since $u = f(x_1 + x) - f(x_1)$. It follows that
>
> $$\lim_{\Delta x - 0} \frac{\Delta F}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta F}{\Delta u}, \text{ so that } \frac{dF}{dx} = \lim_{\Delta x - 0} \frac{\Delta F}{\Delta u} \lim_{x - 0} \frac{\Delta u}{\Delta x}$$
>
> $$= \lim_{\Delta x - 0} \frac{\Delta F}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$
>
> $$= \frac{dF}{du} \cdot \frac{du}{dx}$$

It should be carefully pointed out that the condition that Δu is not zero for Δx close to zero is important in this development, but does not actually restrict the chain rule.

The derivative with respect to x, of u^n, with $u = f(x)$, for f a differentiable function of x, should be derived, using the chain rule. Other examples should also be included. Note that it should be stressed that dy/dx, dy/du, and du/dx are single quantities, not ratios, if this notation is used.

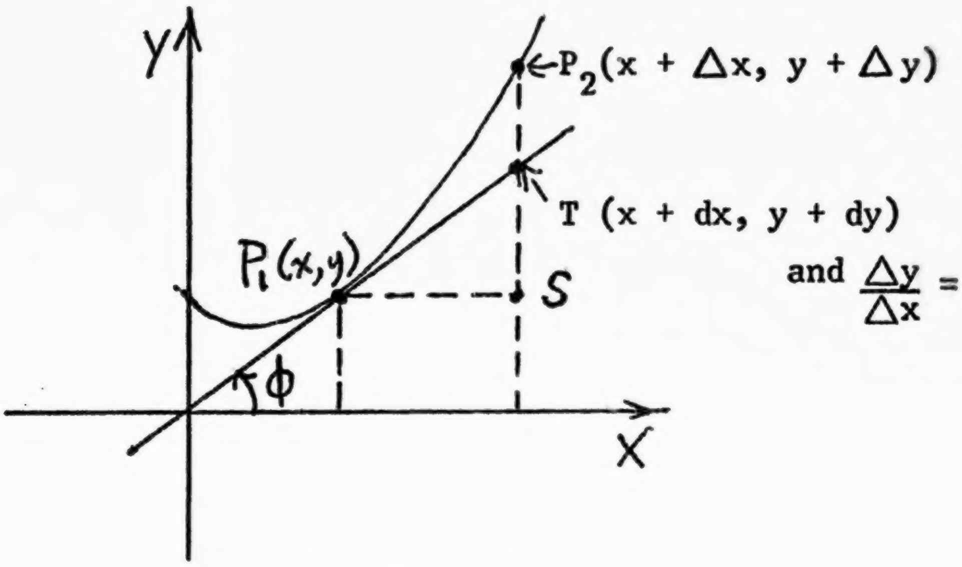
C. Derivatives of implicit functions

The difficulty of solving for y explicitly in terms of x explicitly in terms of y) in a relation such as $x^5y^2 + x^{\frac{1}{2}}y^{\frac{4}{3}} = 4$ should provide enough motivation for the student to use the rules for derivatives of sums, products and quotients from A and the chain rule of B to find dy/dx. However, it should be made clear that this is only valid under the proper conditions; if y cannot be a real-valued function of x, as in $x^2y^2 = -1$, then the student should be able to see that there are difficulties. The requirement that y be such a function may be emphasized by replacing y by f(x) in each problem, before differentiating.

D. Definitions of higher order derivatives

As soon as students have memorized the formulas for derivatives and the chain rule, it is a short step to enlarge upon the interpretation of dy/dx as a rate of change of y with respect to x to the rate of a "rate of change." Thus, the higher order derivatives may be introduced.

E. Definition of differentials and applications:



$\Delta y = \overline{SP_2}$; $\Delta x = \overline{P_1S}$

$\Delta y = f(x) + \Delta x) - f(x)$

and $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

since $\lim\limits_{x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$, $\left(f'(x) = \dfrac{dy}{dx}\right.$, the derivative$\left.\right)$ we can write $\dfrac{\Delta y}{\Delta x} = f'(x) + \epsilon$ (where $\epsilon \to 0$ as $\Delta x \to 0$; leave the proof for the third semester) if we multiply by Δx we have $\Delta y = f'(x)\Delta x + \epsilon\Delta x$ (where $\epsilon\ \Delta x \to 0$).

Now if Δx is very small, intuitively we can see that ϵ is also small and $\epsilon\ \Delta x$ will be even smaller. The arithmetic analogy of multiplying two ratios, such as, 1/10 1/100 = 1/1000 will serve to convince the class. We can then write $f'(x)\Delta x$ for a reasonable approximation to the value of Δy. Illustrative problems can be found in all texts and are therefore omitted here.

The definition of dx as the differential of x, where dx can be any real number (in the picture above $dx = \Delta x = \overline{P_1S}$), is followed by the definition of differential of y as $dy = f'(x)\,dx$, which says that $dy = \Delta y$. In the diagram above,

$$\tan \phi = f'(x) \quad \text{and} \quad f'(x) = \frac{\text{differential of } y}{\text{differential of } x}$$

(since $dy = f'(x)\,dx$),

but also $\tan \phi = \frac{\overline{ST}}{P_1S}$, which gives the coordinates of T shown in the diagram as $(x + dx, y + dy)$. Tan ϕ is by definition the slope of $\overline{P_1T}$, the tangent at P_1, and thus $\tan \phi = f'(x)$. The quotient of the differentials is defined so that it "behaves" like the derivative dy/dx. It should be noted that another way of regarding the differential is to recognize it as a linear function which approximates a given function at a point.

Formulas for sums, products, quotients and u^n (u a differentiable function of x) should be given in terms of differentials. See [2] and [23].

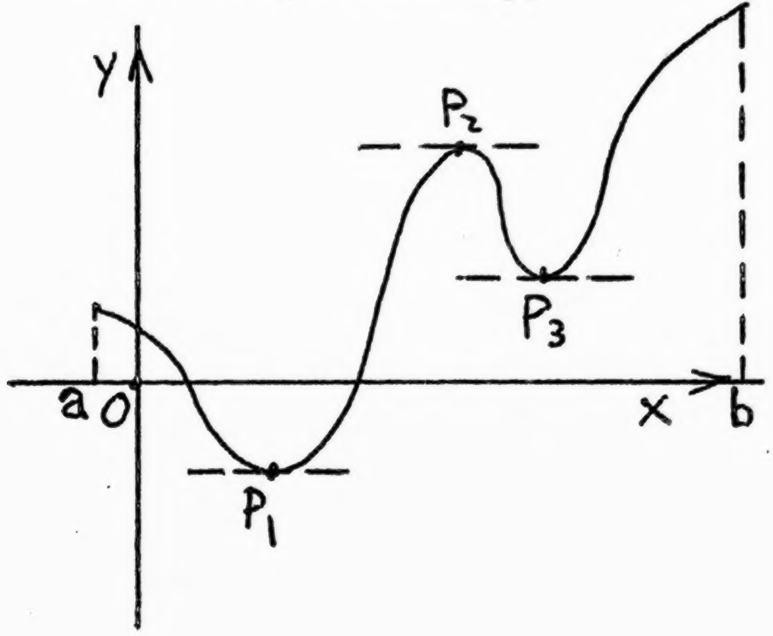
IV. APPLICATIONS OF DERIVATIVES

$f'(x) = dy/dx$ has been defined so that any point $P(a, f(a))$, where $f'(a)$ is defined, $f'(a)$ gives the slope of the tangent to the curve at the point P. This leads to several applications of the derivative in geometric considerations, physics, engineering and economics.

A. Intuitive approach to extrema of continuous functions in an open interval (restricted to "smooth" curves)

See I, C, for statements of the extreme and intermediate value theorems.

Considering the graph of a differentiable function $f(x)$, for $a < x < b$, intuitively,



It is easily noted that if a curve has a "high point" (or a bottom point) such as P_1 and P_3 the tangents at these points are parallel to the horizontal axis and their slopes are therefore zero.

Such points are called extrema; P_2 is at a local (or relative) maximum and P_1 and P_3 are at local (or relative) minimum. The real zeros of $f'(x)$ are the abscissas of these points. If the minimum value of the function in the interval (a, b) were desired we would choose the ordinate for P_1. If the maximum value of the function was requested for the interval (a, b) we would be unable to find one. (See IV, B)

Many problems from geometry, physics, economics, etc., can be expressed as continuous functions, whose graphs would be "smooth" curves, and whose relatively largest and smallest values, under certain restrictions, are of interest.

1. Mean Value Theorem and Rolle's Theorem

A standard proof for Rolle's theorem is usually included in elementary calculus texts and is therefore omitted here. The proof of the Mean Value Theorem (or the Law of the Mean) customarily rests on Rolle's Theorem, or is deferred until advanced calculus. (See II, C)

These theorems should be well illustrated, both pictorially and by examples. It can be pointed out that Rolle's Theorem justifies the previous discussion. The Extreme Value Theorem presented in I, C, with $f(x) > 0$ can be re-examined in the light of Rolle's Theorem and the Mean Value Theorem.

B. Extrema at end points of a closed interval, or where derivatives fail to exist

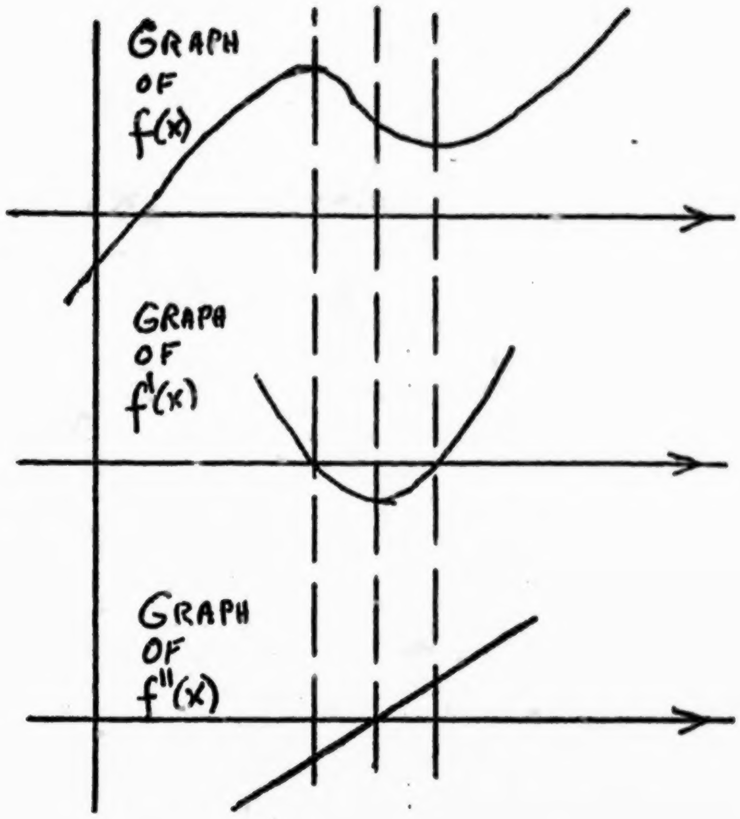
Referring to the figure in IV, A, consider that the interval is $a \leq x \leq b$. Now if the maximum value of the function on the interval $[a,b]$ is desired the answer is f(b). The minimum value would still be at P_1, and the statements about local maximum and minimum values would be unchanged.

Examples of extrema where the derivative fails to exist usually include $f(x) = |x|$, $f(x) = \begin{cases} x \ ; \ x \leq 1 \\ 2 - x \ ; \ x < 1 \end{cases}$, etc., on appropriate intervals.

Points in an open interval (x,b) whose abscissas are the real zeros of $f'(x)$ may be local extreme values. Counterexamples such as $f(x) = x^3 = 3x^2 + 3x = (x - 1)^3 + 1$ where $f'(x) = 3(x - 1)^2$ is non-negative for all x, should be graphed to illustrate the limitations of the techniques of finding extrema.

C. Interpretations and applications of first and second derivatives for extrema for continuous functions and to curve sketching; horizontal and vertical tangents.

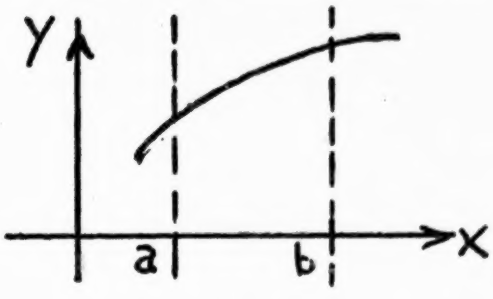
For all functions whose graphs, in the interval under consideration are "smooth" and continuous, whether the functions are simple, polynomials, or arise from physical applications, the following remarks apply:



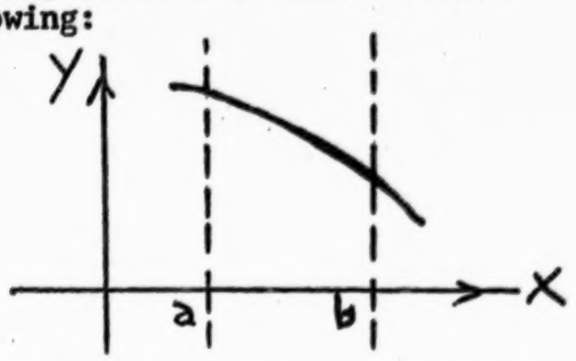
A graph of a function with a graph of $f'(x)$ and $f''(x)$ immediately below provides a good illustration of the relationship between a curve and its first and second derivatives, as each graphically describes the changes in the slopes of the tangents to the curve above it.

The real zeros of $f'(x)$ are observed to the abscissas of the points where the tangents to $f(x)$ are parallel to the horizontal axis, and the real zeros of $f''(x)$ are observed to be the abscissas of the points of inflection on $f(x)$; where the curvature changes from concave down to concave up (or vice versa). Note that some texts use convex up and convex down, others use face up and face down. These results should be derived.

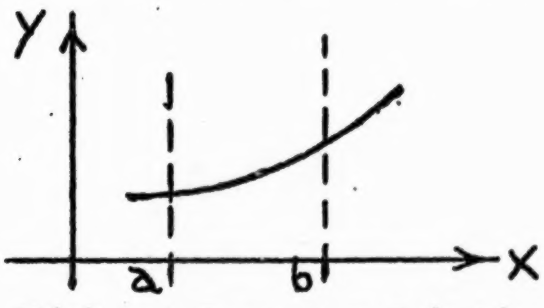
Some suggested to assist students in using first and second derivatives in sketching curves are the following:



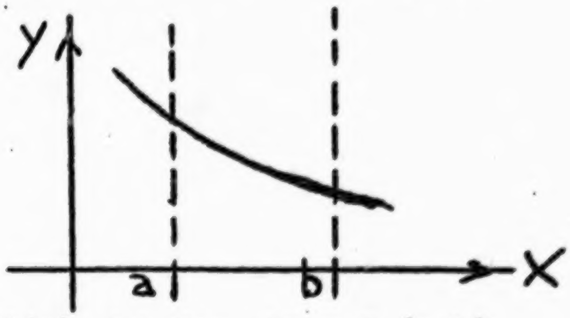
$f'(x) > 0$ for all $x \in (a,b)$
$f''(x) < 0$ for all $x \in (a,b)$



$f'(x) < 0$ for all $x \in (a,b)$
$f''(x) < 0$ for all $x \in (a,b)$

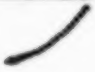
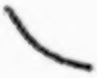
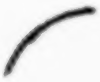
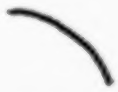


$f'(x) > 0$ for all $x \in (a,b)$
$f''(x) > 0$ for all $x \in (a,b)$



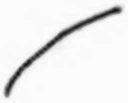
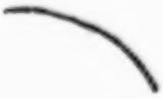
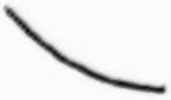
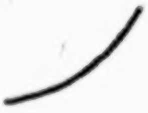
$f'(x) < 0$ for all $x \in (a,b)$
$f''(x) > 0$ for all $x \in (a,b)$

The following compact table is another convenient device to assist the student:

	$f'(x)$	$f''(x)$	Shape of graph
Signs	+	+	
	-	+	
	+	-	
	-	-	

See [6] and [23].

Illustrating the use of the table on $f(x) = x^3 - 3x$ to aid in drawing its graph: (See [6])

X-axis	. . . .						
		-1		0		1	
$f'(x)$	+	0	-	-	-	0	+
$f''(x)$	-	-	-	0	+	+	+
f							

The real zeros of $f(x)$ are 0 and $\pm\sqrt{3}$. Therefore the graph crosses (or touches) the x-axis at (0,0), $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. The real zeros of $f'(x)$ are ± 1. Therefore, the curve has horizontal tangents at (-1,2) and (1,-2). The +, 0, - pattern when $x = -1$ for $f'(x)$ shows a local maximum (as does the bottom line of the table). Similarly the table shows a local minimum at (1,-2).

The real zero of $f''(x)$ gives the inflection point as (0,0). To graph the function only a couple of other values would need to be added to the table. The real zeros of $f'(x)$ and $f''(x)$ are often called critical values."

Vertical tangents can be easily introduced by an analogy with horizontal tangents as occuring where $dx/dy = 0$; or by a use of the definition of the angle of inclination of the vertical line being 90^o is 0. If φ is the angle of inclination then the slope is $\tan \varphi = dy/dx$, $\cot \varphi = \frac{1}{dy/dx} = dx/dy$ for all $0 < \varphi < 90^o$ and $90^o < \varphi < 180^o$.

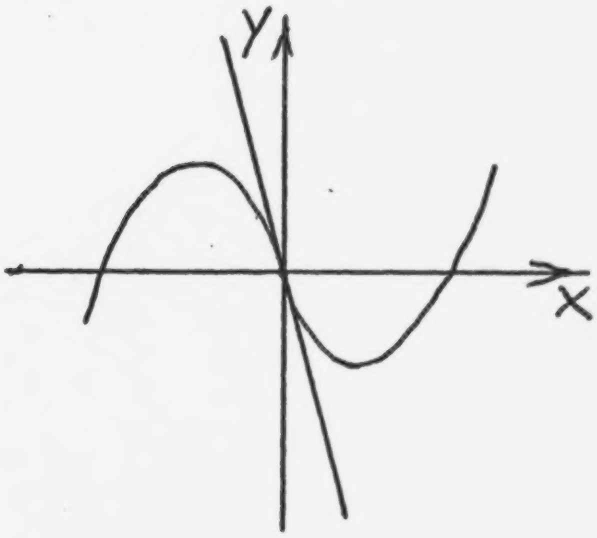
Functions where $f'(x)$ has real zeros, and no local extrema occur should also be illustrated.

It should be noted that values for which $f''(x)$ is not continuous may determine points of inflection on the curve. It can also be seen that the tangent line to a curve at the point of inflection always crosses the curve.

For example:

$$f(x) = x^3 - 3x;\ f'(x) = 3x^2 - 3 \text{ and } f''(x) = 6x$$

at the point (0,0); $f''(0) = 0$ denoting that (0,0) is an inflection point; $f'(0) = -3$ and $f(0) = 0$. The equation of the line through (0,0) with slope of -3 is $y = -3x$.



For $x < 0$ the curve must be below the tangent and be concave down; for $x > 0$ the curve must be above the tangent and be concave up.

D. Maxima and minima problems

(See IV, A, and IV, B, for a discussion of determining extreme points on graphs). Most texts include a section of problems arising from geometric, physical and economic situations where it is desirable to find a maximum or minimum value. Motivation will depend on the particular textbook being used as problems in this area differ mainly in wording from book to book.

E. Velocity and acceleration applied to rate problems (See II,B for instantaneous velocity and acceleration)

If $y = f(t)$ then the rate of change of y with respect to t is given at once by dy/dt. But if $y = f(x)$ and both x and y are arbitrary functions of t we can find dy/dt by using the chain rule. (See III,B)

F. An example of the practical use of the tangent to a curve is Newton's method for approximating the real zeros of a function

Newton's method is one example of an iterative technique. Probably one or two problems computing first and second approximations would be sufficient since the arithmetic is burdensome without a computer.

V. ANTIDIFFERENTIATION

This section may be motivated by recalling for the students the techniques of finding velocity given acceleration, or of finding distance given velocity. This should lead the student automatically to the concept of antidifferentiation. As illustrations of the technique a few functions should be differentiated to enable students to consider what is required to work backward to determine the original function.

A. Antidifferentiation formulas

These should include the formulas for $\int (f(s) + g(x))\,dx$ and $\int a\,f(x)\,dx$, (where a is a constant) as well as the antidifferentiation formulas for appropriate functions covered in the previous section (III,D). The student could be expected to derive all of these. The problem solving nature of antidifferentiation should be stressed rather than the memorization of the various formulas. Thus, a problem like $\int x^3\,dx$ might be followed by $\int \frac{5}{2} x^3\,dx$ or $\int (2x)^3\,dx$ and $\int \sin x\,dx$ could be followed by $\int 3 \sin 2x^2\,dx$. The idea of a substitution to assist in the determining of an antidifferential may be introduced later, for those who need a recipe, but the problem solving approach should be of value to the student in much the same way that trigonometric identities are. Checking the results of antidifferentiation leads to a clearer understanding by the student.

B. Applications to acceleration, velocity and distance

After the introduction to antiderivatives, the student should be ready to apply them at once.

VI. THE DEFINITE INTEGRAL

The student may not be acquainted with the summation notation, so it might be necessary to include a brief discussion as to its meaning and basic properties, such as $\sum_{i=1}^{n} (a_i + b_i)$ and $\sum_{i=1}^{n} \alpha a_i$. However, it should be kept in mind that the sigma notation is only a tool which makes a more compact expression, and is not at all necessary for understanding the concept of the definite integral; it only helps the student understand the notation. Therefore, little time need be spent on the summation at this point.

A. Intuitive definition of the definite integral

This section should be introduced by approximating the area under a curve by means of rectangles of the same width. The idea of a limit of an abstract sum is too difficult for students to understand at first. It is important, when introducing the definite integral in this way, that time not be wasted in determining exact closed form values as done in [4] and several others. The techniques used, while interesting and valuable, tend to obscure the basic nature of the definite integral, hence are postponed until the third semester of calculus. The time is better spent by actually calculating several approximations for an area, by means of increasing numbers of rectangles. If Darboux sums are used, an example might be to calculate the area bounded by $y = x^2$, $x = 1$ and $y = 0$. The calculations result as follows:

First approximation, $\Delta x_i = \frac{1}{2}$ $i = 1, 2$

Lower sum $= 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$\frac{1}{8} < A < \frac{5}{8}$

Upper sum $= \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{5}{8}$

Second approximation, $\Delta x_i = 0.1$, $i = 1, 2, \ldots, 10$

$$\text{Lower sum} = (0)(.1) + (.01)(.1) + (.04)(.1) + (.09)(.1) + (.16)(.1) + (.25)(.1) + (.36)(.1) + (.49)(.1) + (.81)(.1) = 0.285$$

$$\text{Upper sum} = (.01)(.1) + (.04)(.1) + (.09)(.1) + (.16)(.1) + (.25)(.1) + (.36)(.1) + (.49)(.1) + (.64)(.1) + (.81)(.1) + (1)(.1) = 0.385$$

$$0.285 < A < 0.385$$

If the students have learned to program a computer, they may be encouraged to carry this further. The experience should be valuable to them. If Riemann sums are used, a similar example could be given.

It should be pointed out that other numerical techniques exist, such as Simpson's rule, which will be covered later.

After the above introduction, the more abstract definition of the definite integral may be easily introduced. From this point on, no matter what a definite integral may represent in a particular prolem, it should be emphasized that it is the limit of a sum. It should be pointed out that under certain conditions, the integral may not exist. Illustrations should be included to indicate how this is possible, such as $\int_{-1}^{1} \frac{1}{x^2}\, dx$. At this point a note may be made of Bliss' theorem, which says briefly that for f and g continuous functions on [a,b],

$$\lim_{\delta \to 0} \sum_{i=1}^{n} f(\xi_i) g(\theta_i) \Delta x_i = \int_a^b f(x) g(x)\, dx$$

where δ is the maximum of the Δx_i.

B. Fundamental theorem of integral calculus

In this section, we may state the fumdamental theorem, and illustrate it by means of several sample problems so that the students can see its value, and begin to use it.

The calculus texts [6] and [23] offer two different approaches that may be of value for introducing the fundamental theorem in an intuitive manner. The derivative has been introduced from the intuitive approach of velocity, and has been defined as an abstract limit; the definite integral has been introduced from the intuitive standpoint of area, and defined abstractly as the limit of a sum. An intuitive development of the fundamental theorem which combines some of these should be of value to the student. See [6] and [23] for further suggestions. For a further connection between area under a curve and velocity, see Physics, Physical Science Committee, D.C. Heath & Co., 1960.

After this introduction, the theorem should be proved. Problems similar to those in the material on antiderivatives should be reassigned as definite integrals. It may be noted that integration tends to "smooth" a function. Examples include the step function, $[x]$.

C. Differentiation under the integral

At this point, the idea of differentiating the integral by differentiating the integrand, and then integrating may be discussed, for definite integrals. The theorem concerning this should be discussed, with careful emphasis on the condiditons under which it is valid. See [9].

VII. APPLICATIONS OF THE DEFINITE INTEGRAL

In all of the material covered, students should be required to sketch figures for each problem.

A. Area

The idea that an area may be found by means of integration has been introduced, so it should not be hard for the student to move into this section. However, emphasis must be placed on the techniques for setting up the integral. It should be noted at this point, if it hasn't been covered already, that the expression under the integral is simply an idealization of the expression under the summation in the definition of the definite integral. From this, it is an easy step to show the students how to set up the integral for an area by finding the formula for the area of a typical rectangle (or rectangles), which is the differential of area. This should make it easier for the student to set up the integral for problems such as the area enclosed by $x = y^2$ and $y = x - 2$ where the area must be split up.

1. Area between curves and coordinate axes. The material here should include areas strictly above, strictly below, and both above and below the x-axis. Also, similar problems involving areas bounded by the y-axis should be included.

2. Area between two curves. This section should be covered in such a way that the student can set up the integrals by means of a problem solving approach. For this, a typical rectangle (differential of area) should be drawn on the figure, and its length, width and area determined for each individual problem. This is in opposition to the cookbook approach, in which the student simply memorizes "Upper curve minus lower curve times dx," or something similar. The sketching and finding dimensions of the typical rectangles will be used again in the section on moments.

B. Volumes of solids of revolution, by disk, washer and shell methods

Once again the student should not attempt to memorize techniques, although it will be necessary to memorize the formulas for the increments of volume in the various methods. Perhaps the best way to approach this material is to first draw the region to be revolved, and include in the figure the typical rectangle, just as done in A. Then, when the student visualizes the solid of rotation, he can visualize also the disk, washer, etc., which is described by the rectangle. He can also easily determine the limits of integration from the area problem.

Intuitive justifications for all of the increments of volume should be included.

It should be demonstrated how a solid region might be approximated by means of solid disks, or washers and shells, and it should be pointed out again that an integral represents the limit of the sum that is thus determined. From this, the student should be shown how to set up the integral for a volume, as indicated above.

D. First moments and centroids

Note that the centroids are independent of the coordinate system.

1. Moments of plane regions. First the definition should be given for the moment of a point mass in the plane, with respect to both coordinate axes. Some problems should be defined for a plane area with constant density. Motivation for this definition might include the example of a balanced seesaw. At this point, the centroid of a narrow rectangular region with constant density should be determined, either by intuition or, preferably, by means of integration. After this, it is an easy step for the student to see how to find the moments, in the same manner as the area problems, by taking the limit of a sum of moments of individual rectangular strips. For practice, the student should

return to the areas calculated previously for the plane regions and determine their centroids. For good students, the idea of variable density might even be included. A problem should perhaps be included verifying that the centroid of a circle is at its center.

2. Moments of volumes. The ideas of moments and centroid of a solid region may be motivated by point masses in 3-space. Then, using the known centroid of a disk, washer or shell, the moments and center of gravity of a solid of revoluation may be introduced. The role of the axis of revolution should be carefully pointed out to the students. They should realize that the centroid of a disk, washer, or shell, is on the axis of revolution, so that, when calculating the moment with respect to the axis, the integrand must be zero. A more detailed treatment of moments and centroids will be included in the third semester.

   Optional material

   It is recommended in this report that the material on arc length and surface area in three space be covered from the standpoint of vectors. However, it may be desired to cover the material on plane curves and surfaces of rotation at this stage instead. Note that due to the nature of the differential of arc length, the problems which may be solved at this stage are very limited.

D. Arc length of a plane curve

The student should by now be used to the idea that the limit of a sum may be expressed as an integral. The arc length of a curve may be approximated by

$$\sum_{i=1}^{n} \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2}$$, the limit of which is

$$\int_{t_1}^{t_2} \sqrt{dy^2 + dx^2} \;, \int_a^b \sqrt{(dy/dx)^2 + 1\, dx}, \int_c^d \sqrt{1 + (dx/dy)^2}\, dy,,$$

or $$\int_{t=\alpha}^{t=\beta} \sqrt{(dy/dt)^2 + (dx/dt)^2}\; dt.$$

Since paremetric representations of curves have not been discussed, if it is desired to use the parametric form, some time will need to be spent discussing the pair of equations $y = y(t)$, $x = x(t)$ as an extension of Section IV,D. Now such representations as $x = a \cos t$, $y = a \sin t$, for a circle should be considered.

E. Area of a surface of revolution

Using the idea of approximating a curve by means of a polygonal arc, it is easy to develop an approximation of a surface of revolution by conical sections, as is done in [21] and [29]. Areas of other surfaces will be determined later.

VIII. TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS AND THEIR INVERSES (10 hours)

A. Derivatives of the trigonometric functions and corresponding integrals. In attempting to find the derivative of $\sin x$, various approaches may be used. Two very common approaches using basic trigonometric identities involving only sines and cosines are those given in [14], [21], and [37]. Some authors will define arctangent x as a function whose derivative is $\frac{1}{1+x^2}$, which will enable them to use quite a different approach.

See [21]. Another approach is to use infinite power series definitions (Taylor's series) for $\sin x$ and $\cos x$ to obtain the desired results. See [34]. But in the approaches used in some books there is a need to find $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$ and $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$.

1. These two limits are developed in a great variety of ways. Usually $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ is done first, from which the second limit follows readily. Some discussions of $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$ are approached by using inequalities of three areas involving regions associated with the unit circle and a given angle θ, measured in readians [37], But for other approaches, see the following: [3], [6], [23], and [5].

2. The derivatives of the remaining trigonometric functions may be done most easily by using various trigonometric identities and the chain rule. The Δ-process may be used to reinforce the important concept of a derivative.

B. The derivatives of the inverse trigonometric functions and the corresponding integrals. Some review of the graphs and properties of these functions may well be needed. The principal values, in particular will probably need to be mentioned.

The difficulties in defining the principal value ranges for $\sec^{-1}x$ and $\csc^{-1} x$ should be mentioned.

1. The student should be able to evaluate the indefinite integrals $\int \frac{dx}{a^2-x^2}$ and $\int \frac{dx}{a^2+x^2}$.

C. $\int_1^x \frac{dt}{t}$; derivative of $\log_e x$. Some authors prefer to define $\log_e x$ as $\int_1^x \frac{dt}{t}$. By doing so, the "question of continuity causes no difficulty", according to [10], "Since the logarithm is defined as an integral and therefore as a continuous and differentiable function, whose inverse function is also continuous". Then we immediately have or can quickly obtain: $\frac{d(\log_e x)}{dx} = \frac{1}{x}$ [29]. For a more traditional treatment, that of building directly upon the elementary definition of $\log_e x$ as the inverse of the exponential function, see [6], [14], and [23]. For another approach, see [34], which first uses an infinite series approach for e^x and its derivative. Also, see [2], [21], and [31]. The approach should be geared to the text being used. However, the committee feels that each of the first two approaches above have merit and should be discussed if time allows. Whatever approach to the logarithm is presented, its various properties should be considered, such as $\ln xy = \ln x + \ln y$ and $\ln x^a = a \ln x$. Also, the graph should be emphasized. This could be introduced by plotting points, but its derivatives should be used also, to give the students a review of curve sketching.

1. $\int \tan x\, dx$, $\int \cot x\, dx$, $\int \sec x\, dx$, and $\int \csc x\, dx$ can then be developed.

2. Derivatives and integrals of a^x and e^x. These derivatives developed rather easily after the derivative of $\log_e x$

has been determined. The uniqueness of e^x as the only function equal to its own derivative and integral should be pointed out. Different methods are employed by [2], [6], [23], and [31].

The derivatives for $\log_a x$ and a^x may be developed next by using standard change of base formulas.

IX. HYPERBOLIC FUNCTIONS

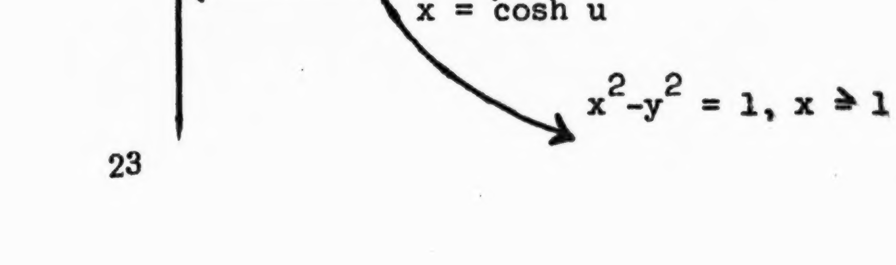
A. The ideas about hyperbolic functions which are necessary for further work can be covered in 3 or 4 hours of class work.

The sinh and cosh should be defined in terms of e and then the tanh, coth, sech, and csch can be defined in terms of the sinh and cosh. Analyze the sinh and cosh functions, including the graphs, using methods of calculus where applicable. If time permits the properties of the other functions can be studied in a similar fashion.

Comparisons and contrasts should be made with the properties and identities of the so-called circular functions. In particular the counterparts of the Pythagorean identities should be mentioned. See [23] for a fairly complete list of properties and identities.

The students must know the formulas for the derivatives and corresponding antiderivatives of all six hyperbolic functions and in fact should be able to derive them.

B. The Hyperbolic Radian



For the unit half-hyperbola, $x^2 - y^2 = 1 \quad x \geq 1$, we find an interpretation analogous to the concept of radian in the unit circle.

Let $x = \cosh u$, $y = \sinh u$ and calculate the area of the sector AOP in the above figure. This area is equal to the area of triangle OQP minus the area of AQP.

$$\text{Area of AQP} = \int_A^P y\,dx = \int_A^P \sinh u \sinh u\,du = \int_A^P \sinh^2 u\,du$$

$$= \frac{1}{2}\int_A^P (\cosh 2u-1)\,du = \frac{1}{2}\left[\frac{1}{2}\sinh 2u-u\right]_{A(u=0)}^{P(u=u)}$$

$$= \frac{1}{4}\sinh 2u - \frac{1}{2}u = \frac{1}{2}\sinh u\cosh u - \frac{1}{2}u$$

Therefore, area of sector AOP = area of OQP - area of AOP

$$= \frac{1}{2}\sinh u\cosh u - \left(\frac{1}{2}\sinh u\cosh u-\frac{1}{2}u\right)$$

$$= \frac{1}{2}u$$

or, u = twice the area of the sector AOP

The term hyperbolic radian is sometimes used in connection with the variable u.

C. The inverse functions may be defined along with a brief analysis of their properties including their graphs. The derivatives of these functions should be included but the corresponding anti-derivatives are not absolutely essential since all integrals of this type can be handled by a regular trig substitution. The natural log form of the inverse hyperbolic cosine could be derived and the corresponding forms for the other inverse functions mentioned.

D. It is suggested that if students have difficulty integrating certain expressions containing hyperbolic functions, that the hyperbolics be changed to exponential form and the resultant expression hopefully integrated.

X. TECHNIQUES OF INTEGRATION.

A. Methods of substitution. It should be recalled from Section V that the process of antidifferentiation is the inverse process of differentiation. This process can be expanded by giving the student some methods to integrate certain forms. The first of these methods is that of substitution. The basic idea behind this method is, if we know that $\int f(u)\,du = F(u) + C$ and $u = g(x)$ then $\int f(g(x)) \cdot g'(x))\,dx = F(g(x)) + C$.

1. Algebraic substitutions. An example will illustrate this method. Since $\int x\,dx = \frac{x^2}{2} + C$ and $\int \sin\theta \cos\theta \, d\theta = \frac{\sin^2\theta}{2} + C$

then we might show the similarity between these examples with

$$\begin{tikzcd}
\int x \cdot dx \arrow[d, "x\to\sin\theta"] & = & \frac{1}{2} x^2 + C \arrow[d, "x\to\sin\theta"] \\
\sin\theta\, d(\sin\theta) \arrow[d] & = & \frac{1}{2} \sin^2\theta + C \arrow[d] \\
\sin\theta \cos\theta\, d\theta & = & \frac{1}{2} \sin^2\theta + C
\end{tikzcd}$$

Another example of integration by substitution is the evaluation of $\int x\sqrt{1-x}\,dx$. Students have difficulty in this type of integration because they invent a new rule, "the integral of a product is the product of the integrals". So we try to eliminate this difficulty by a substitution. Suppose we let $u = \sqrt{1-x}$. Then $x = 1-u^2$ and $dx = -2u\,du$. Thus

$$\int \underset{\uparrow}{x}\,\underset{\uparrow}{\sqrt{1-x}}\;\underset{\uparrow}{dx}$$

$$= \int (1-u^2)\;u(-2u\,du)$$

$$= -2\int (u^2-u^4)\,du$$

$$= -2\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= -2\left[\frac{(1-x)^{3/2}}{3} - \frac{(1-x)^{5/2}}{5}\right] + C$$

2. Partial fractions. This section is concerned with methods of integration of rational functions. That is, integrals of the form $\int \frac{P(t)}{Q(t)}\,dt$, $Q(t) \neq 0$, where P and Q are polynomials in t. The importance of factorization in integration can be motivated by an example such as $\int \frac{dx}{x^2-3x+2}$.

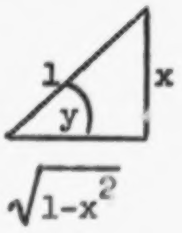
$$\frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$$, which implies that

$$\int \frac{1}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$$, which can be integrated.

What is needed then, is a systematic method of separating a proper rational fraction which does not come under any of the basic forms into a sum of fractions which can be integrated. We call this the method of partial fractions. See [6].

3. Trigonometric substitution. An integrand, which contains one of the forms $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$, a^2+u^2, or a^2-u^2, where a is a constant and u is a differentiable function,

may be transformed into an integrand involving trigonometric functions. If the u substitution is tried on the above forms we seem to get nowhere. However, by use of a trigonometric substitution we are usually able to succeed. The student should take care to see that the choice of substitution leaves the domain unaltered. In working problems with trigonometric substitution, one step in the substitution that might be troublesome to the student is: Find cos y if sin y = x. A handy device to solve this is to draw the figure:



hence,

$\cos y = \sqrt{1-x^2}$, assuming proper domain.

B. Integration by parts. This section could be motivated by taking an easy differential and then writing in antidifferential form. Thus, by working backwards, the solution of an integration problem is found. For example:

$$d(x \sin x) = (x \cos x + \sin x)\,dx$$

Since we know $d(\cos x) = -\sin x\,dx$, we can then write,

$$d(x \sin x + \cos x) = (x \cos x + \sin x - \sin x)\,dx$$

$$d(x \sin x + \cos x) = (x \cos x)\,dx. \quad \text{Hence}$$

$$\int x \cos x\,dx = x \sin x + \cos x + C$$

On the other hand we could have approached this example using the following method. Compute $\int x \cos x\,dx$. We begin be looking for two functions, say f and g, such that $d(f \cdot g) = x \cos x\,dx$; try $f = x$ and $g = \sin x$. Then $d(x \sin x) = (x \cos x + \sin x)dx$. So our first guess failed. However, from this failure we can see that $d(x \sin x) - \sin x\,dx = x \cos x\,dx$. Therefore, $\int x \cos x\,dx = x \sin x + \cos x + C$. The above guessing method can be formalized into the integration by parts technique. In the development of this technique some steps should be emphasized. When the technique is formalized we state that

$d(uv) = u\,dv + v\,du$, where u and v are differentiable functions, and

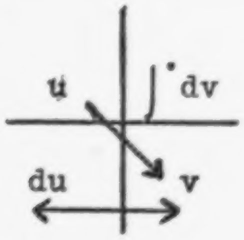
$$uv + C = \int u\,dv + \int v\,du$$

$$uv - \int v\,du = \int u\,dv.$$

It is important to note that the constant is dropped (in some textbooks) because the equal sign in the last expression means that the two sides of the equation represent the same class of functions.

1. Inverse trigonometric functions. The formula for integration by parts enables us to shift the problem from integrating one form to that of integrating another, which may be easier to handle. This form is especially useful for integrating the inverse trigonometric forms like $\int \cot^{-1} x\,dx$.

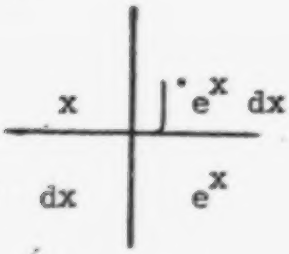
2. Tabular device. A tabular device for integration by parts is to write the formula in the following manner:



indicates $uv - \int v\,du$

Compute: $\int x\,e^x\,dx$

Hence:



indicates

$$\int x\,e^x\,dx = x\,e^x \quad \int e^x\,dx$$

$$\int x\,e^x\,dx = x\,e^x - \int e^x dx + C$$

C. Approximations. The teacher should emphasize that we have functions whose integrals are not elementary functions. For example, using a finite number of functions already studied, we cannot write an expression whose derivative is e^{-x^2}. See [37].

Approximation methods are necessary when ordinary integration is difficult, when the indefinite integral cannot be expressed in terms of elementary functions, or when the integrand is defined by a table of values.

1. Simpson's Rule and Trapezoidal Rule. Since not all integrals are elementary functions, a numerical method is needed to compute definite integrals. Two of these numerical methods are Simpson's rule and the trapezoidal rule. The application of Simpson's rule has the advantage that it does not require a graph of the curve nor a determination of the approximating parabolic arcs. See [37].

In the trapezoidal rule the points in which successive ordinates meet the graph are connected by straight line segments; in Simpson's rule the points are connected by segments of parabolas.

A student might ask at this point, if the area under the curve can be approximated by rectangles then why bother with Simpson's rule and the trapezoidal rule? In the problems mentioned above it is highly desirable to be able to estimate how much error is made by calculating the integral by an approximate method. Simpson's rule and the trapezoidal rule are used since error under these methods has a greater rate of convergence. See [21].

2. Other numerical methods. As might be expected, better approximations to the area of a region can be obtained by using other methods, but the resulting formulas are usually more complex than those of either the trapezoidal rule or Simpson's rule. It is generally more practical in computational work to use one of these two rules with a larger n than a more complex rule

with a smaller n. One of these methods is to use Chebyshev polynomials to approximate the function.

If a computer is to be used the method will often depend on the available memory space and machine time.

XI. VECTORS

A. Introduction. Since the abstract idea of a vector should be covered in depth in the linear algebra course, the development here should proceed from two to three space on an intuitive basis. Theorems should be illustrated geometrically, as most texts do. Proofs should be motivated and proven geometrically.

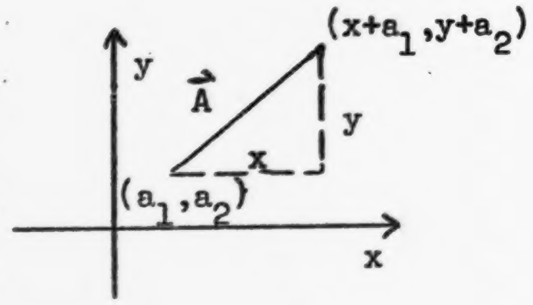
As many precalculus courses do not contain the algebra of vectors the committee recommends such a development in this first year calculus.

The geometric approach through applied vectors in two space is probably the easiest way to introduce the student to the concepts, as previous experience in vectors would probably be in physics or trigonometry. However the committee recommends proceeding into an analytic description of vectors and their components unconnected with applications, since the main advantages of a vector approach appear in spaces of higher dimensions. Most books appear to have the same sort of analytic description, so this need not be repeated here. See [4],[7],[8], and [23]. For a good description of free vectors see [23].

Several things should be noted: (1) Ordered pairs of numbers are used to describe many mathematical quantities as well as two dimensional vectors determined by the origin and the ordered pair (x,y); (2) The same mathematical quantity can be described in different ways by ordered pairs (rectangular coordinates versus polar for example); (3) vectors can be pictured in different ways - as arrows with tails at the origin or with an arbitrary initial point; and (4) vector ideas appear in many different forms which must not be confused with one another.

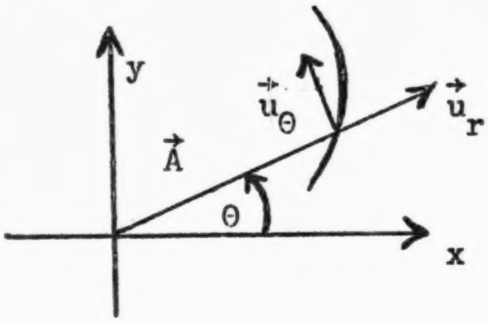
Also, since the elementary treatment of vectors defines the inner product, it should be noted that a vector space is usually required to have laws of addition and scalar multiplication only and is not ordinarily required to have an inner product.

B. Vectors in the Plane. A two dimensional vector can be defined as an ordered pair of numbers (x,y) with the numbers x and y defined as its components, or as a line segment whose initial point is at any point (a_1,a_2) in the Cartesian plane and whose terminal point is the point $(x + a_1, y + a_2)$. The magnitude of the vector is then determined by the metric for the space determined by the set of points (x,y). This space of points can then also be regarded as a set of vectors called a vector space involving the concepts of the sum of two vectors and the scalar product of a vector, concepts not usually associated with points. See [29] and [8]. The importance of the applications of vectors in engineering and the physical science should be stressed. The introduction of coordinates yields a correspondence between vectors and sets of numbers, which permits the use of vector methods in the study of linear equations. This in turn leads to the concept of matrix which is used in a variety of fields. See [6].



If the vector $\vec{A} = (a_1,a_2)$ is represented by a line segment whose initial point is the origin, then $\vec{A}$ is called a position vector. The first and second derivatives of the position vector are usually derived by the delta process with respect to time using $\vec{i}$ and $\vec{j}$ as the orthonormal basis. The first derivative of the position vector using arc length as parameter is also customarily covered. Some texts include an orthonormal basis in the polar coordinate system, defining unit vectors as follows: $\vec{u}_r = \vec{i}\cos\Theta + \vec{j}\sin\Theta$; $\vec{u}_\Theta = -\vec{i}\sin\Theta + \vec{j}\cos\Theta$.

The derivatives of the position vector can then be expressed in terms of these unit vectors. It must be emphasized that only the lengths of these vectors are constant, since their direction changes from point to point.



C. Vectors in Three Dimensions.

1. Three dimensional vectors can be defined as an ordered triple (x,y,z) with the numbers x, y and z defined as its components or as a directed line segment in three dimensions. The magnitude of the vector is then determined by the metric for the space. As in E_2 we have the concept of addition of vectors and scalar multiplication. A vector $\vec{A} = (a_1,a_2,a_3)$ may be interpreted as a line segment with its initial point at (x,y,z) in E_3 and whose terminal point is at $(x + a_1, y + a_2, z + a_3)$. If the initial point is the origin then $\vec{A}$ is defined to be a position vector. It is desirable to treat directed quantities such as force or velocity (which are independent of coordinate systems) without reference to a set of coordinate axes. For example, Newton's Law $\vec{F} = m\vec{a}$. Such a coordinate free treatment is made possible by vector analysis. See [4′] for an interesting section on escape velocities.

2. At this point the various forms of the equation of a line in E_3 are usually introduced; the vector form, the parametric form, the symmetric form, and the equations of two intersecting planes. Both vector and normal forms of the equation of a plane should be developed.

3. Scalar and vector products should have a careful geometric interpretation given. Note that the associative and distributive properties of scalar multiplication; and the commutative, associative and distributive properties of the inner product need no reference to the dimensionality of space.

4. Motion on space curves is best discussed in terms of the orthonormal frame of $\vec{i}$, $\vec{j}$ and $\vec{k}$. If time permits the orthonormal frames for both cylindrical and spherical coordinates may also be used. If this is done, an interesting proof of Kepler's three laws of planetary motion may be derived from Newton's second law using vectors. See [4].

5. Differentiation of vectors provides a good section to allow the students to do some "guessing" about the formulas for derivatives of dot and cross products. Derivatives of triple products may be deferred until the third semester of calculus. Most texts include some applied problems on the motion of a projectile.

6. Curvature is one topic which is treated somewhat differently in various texts. The definition is deceptively easy; the difficulty is in using the formula to compute the curvature. Some texts simplify the business of figuring out if arc length is increasing, if it is increasing as its parameter increases, or whether the curve is to your left or right as you walk along it, by putting an absolute value symbol on the formulas for the curvature. See [4], [31], and [37].

   This is usually followed by a development of the unit tangent and unit normal vectors which form a new orthonormal basis. Some good problems in this area can be found in [14]. Arc length as a parameter follows naturally here. See [3]. The tangential and normal components of acceleration of a particle should also be covered here.

XII. PARTIAL DIFFERENTIATION

A. Introduction. One may approach the study of the calculus of functions of several independent variables in a variety of ways.

A popular approach at present is that essentially used in pp. 657-729 [37] or in pp. 540-588 [14]. This approach first develops the idea of the directional derivative or its special case, the partial derivative, in two and three dimensions. This is done through a Δ-process development similar to that used for ordinary differentiation. Another approach, used in pp. 156-218 [4,II], for example, first develops the derivative of a scalar function of a vector with respect to another arbitrary (direction) vector. Coverage here should contain many geometric illustrations.

B. Functions of several independent variables. This should probably be restricted to two or three independent variables in this first-year course. Examples of one, two, three and more variables should be given, i.e. $A = \pi r^2$, $P = 2l + 2w$, $V = lwh$, etc., and then generalized to $y = f(x)$, $z = g(x,y)$, etc. Finally, the informal introduction should be concluded with the formal definition appropriate to the textbook being used.

1. Scalar fields. By replacing each point in the plane or in three space by its position vector, we may write, for example, $f(x)$ in place of $f(x_1,x_2,x_3)$, where $x = (x_1,x_2,x_3) = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$.

This notation has the real advantage of conciseness and generality. Regardless of the number of dimensions, the scalar-valued function at point $\vec{x}$ is then simply $f(\vec{x})$. This function of a vector variable defines a scalar field, as in [4], II.

2. Neighborhoods and open sets. A neighborhood of a point may now be defined as an open circular disk in the plane and an open spherical solid in three-space. The open set may then be easily

and naturally defined, as in [4], II, or [14]. Also, the neighborhood of a vector, $\{\vec{x} \mid |\vec{x} - \vec{A}| < r,\ r > 0\}$, should be included.

C. Derivative of a function with respect to a vector. See [4],II. If we follow this approach, we may now define the derivative of f at the point $\vec{x}$ with respect to $\vec{y}$, denoted by

$$f'(\vec{x};\vec{y}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{y}) - f(\vec{x})}{h} .$$

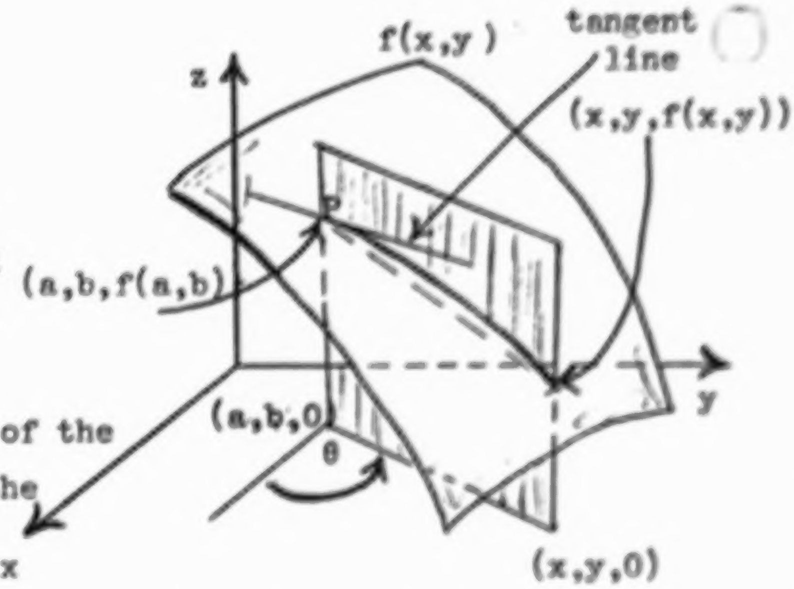
1. The directional derivative. If $\vec{y}$ is a unit vector, $f'(\vec{x};\vec{y})$ is called the directional derivative of f at $\vec{x}$ in the direction $\vec{y}$. In case $\vec{x} = (x)$ this gives us the ordinary derivative: $\frac{df}{dx} = f'(\vec{x};\vec{1})$. It should be pointed out that, as defined above, these are still two-sided derivatives. That is $h \to 0$ and $h > 0$ or $h < 0$. However, [37] defines

$$\frac{df}{ds} = \lim_{\Delta S \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x,y)}{\Delta S}$$ for a function of two variables, where $\Delta S = \sqrt{\Delta x^2 + \Delta y^2} > 0$. But we have

$$\begin{aligned}
f'(\vec{x};\vec{y}) &= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{y}) - f(\vec{x})}{h} \\
&= \lim_{h \to 0} \frac{f(\vec{x} - h\vec{y}) - f(\vec{x})}{-h} \\
&= -\lim_{h \to 0} \frac{f(\vec{x} + h(-\vec{y})) - f(\vec{x})}{h} \\
&= -f(\vec{x};-\vec{y})
\end{aligned}$$

This would not be true, in general, for the one-sided derivative defined in [37].

If a more standard approach to the directional derivative is desired, point out that it is nothing more than a generalization of the partial derivatives. That is, the directional derivative is the slope of the tangent line at P to the curve formed by the intersection of the plane perpendicular to the xy axes and at an inclination of θ from the x axis. A diagram can then be used to formalize the definition of the directional derivative by using the following argument:



The rate of change of $f(x,y)$ with respect to its projection on the xy plane, call it x will be given by:

$$\lim_{s \to 0} \frac{f(x,y) - f(a,b)}{s}$$

which, if the limit exists we call the directional derivative of the function at the point $(a,b,f(a,b))$ in the direction θ.

The discussion should be carried to a logical conclusion by deriving the formula for finding the directional derivative. That is, substituting for x and y above the parametric equations $x = a + s \cos \theta$, $y = b + s \sin \theta$ and applying the chain rule to obtain:

$$D_\theta f(a,b) = f_1(a,b) \cos \theta + f_2(a,b) \sin \theta.$$

2. The partial derivative. When $\vec{y}$ (in the discussion above) is $\vec{i}$, $\vec{j}$, or $\vec{k}$, we have a special case of the directional derivative: the partial derivative. For example, if $\vec{x} = (x_1,x_2,x_3) = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$, and $\vec{y} = \vec{j}$, then $f'(\vec{x};\vec{y}) = f'(\vec{x};\vec{j})$ is known as the partial derivative with respect to x_2 or the derivative in the $\vec{j}$ direction. Notations prevalent here include:

$\dfrac{\partial f(x_1,x_2,x_3)}{\partial x_2}$, $D_2 f(x_1,x_2,x_3)$, $f'_{x_2}(x_1,x_2,x_3)$, or f_{x_2}. See [4],II.

(Often, we write $\vec{x} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ in 3-space. Then we would have above $\frac{\partial f}{\partial y}$ of f_y).

3. Continuity and limits of a scalar field. When we differentiate a scalar field f, we obtain a new scalar field f'. Continuity and limits in such fields can be briefly defined, as in [37], [14], or [4]. Then it may be briefly mentioned that many limit theorems which are true in one dimension remain valid in two, three, or n dimensions. However, an important difference should be mentioned: The existence of the derivative of f at a point x implies continuity of f at x, in one-dimension. But this does not hold in higher dimensions. A scalar field may be differentiable in all directions at some point and yet be discontinuous at that same point. See [4],II.

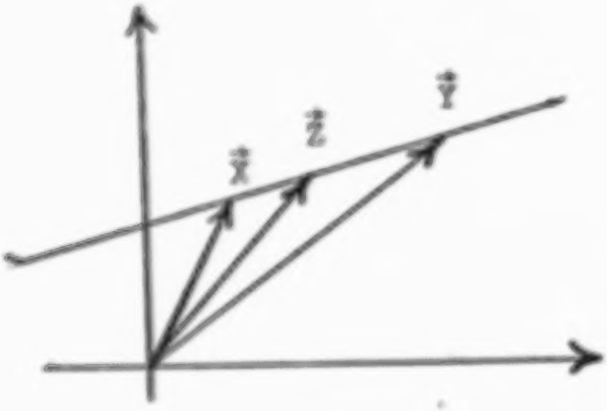
4. The mean-value theorem for scalar fields. The student is already familiar with this theorem in the one-dimensional derivative form: $f(b) - f(a) = f'(c) \cdot (b - a)$, where $a < c < b$. This can now be

   $f(\vec{x} + \vec{y}) - f(\vec{x}) = f'(\vec{z};\vec{y})$, where $\vec{z} = \vec{x} + \theta\vec{y}$, $0 < \theta < 1$.

   Letting $\vec{A} = \vec{x}$ and $\vec{B} = \vec{x} + \vec{y}$, we obtain

   $f(\vec{B}) - f(\vec{A}) = f'(\vec{z};\vec{B}-\vec{A})$.

   The full requirements of the theorem may be found in [4],II. But the analogy is clear. Geometrically, we have



5. Linearity of the derivative. It may now be shown that $f'(\vec{A};\vec{y})$ is a linear function of y. That is $f'(a;\, a\vec{y} + b\vec{z}) = af'(\vec{A};\vec{y}) + bf'(A;\vec{z})$. As in [4],II, this may be derived in two stages: one showing the homogeneous property $f'(\vec{A};c\vec{y}) = cf'(\vec{A};\vec{y})$, the other the additive property $f'(\vec{A};\vec{y} + \vec{z}) = f'(\vec{A};\vec{y}) + f'(\vec{A};\vec{z})$. Extending this immediately to three vectors, we obtain $f'(\vec{x};\sum_{k=1}^{3} y_k\vec{A}_k) = \sum_{k=1}^{3} y_k\, f'(\vec{x};\vec{A}_k)$. In particular, if $\vec{A}_1 = \vec{i}$, $\vec{A}_2 = \vec{j}$, and $\vec{A}_3 = \vec{k}$, with $\vec{y} = (y_1,y_2,y_3) = y_1\vec{i} + y_2\vec{j} + y_3\vec{k}$,
and $\vec{x} = (x_1,x_2.x_3) = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$,
then we have

$$f'(\vec{x};\vec{y}) = \sum_{k=1}^{3} y_k f'(\vec{x};\vec{A}_k)$$

$$= y_1\, f_{x_1}(\vec{x}) + y_2 f_{x_2}(\vec{x}) + y_3 f_{x_3}(\vec{x})$$

Thus we may express the directional derivative as a linear combination of the partial derivatives of $f(\vec{x})$.

D. The gradient of a scalar field. We have been considering the scalar field $f(\vec{A})$ and the derivative $f'(\vec{A})$ in some direction $\vec{y}$. The directional derivative of any scalar in each direction is a scalar. If we single out the direction that corresponds to the maximum derivative for each scalar, max $f'(\vec{A})$ for each $\vec{A}$, we have a vector field. This vector field finds use in such a variety of applications of vectors that it has been given a special name "the gradient."

1. Definition. The gradient of a scalar field is a vector that represents the direction and magnitude of the greatest change. Two common symbolic representations for the gradient of a scalar field $f(\vec{A})$ are "grad f" and ∇f (pronounced "del f").

2. If we recall, from the previous work with directional derivatives, that $f'(\vec{x};\vec{y}) = \nabla f(\vec{x})\cdot(\vec{y}) = |f(\vec{x})||\vec{y}|\cos\theta$ where θ is the angle between $\nabla f(\vec{x})$ and $\vec{y}$ we may deduce:

   a. $f'(\vec{x};\vec{y})$ is the component of $\nabla f(\vec{x})$ in the direction of the unit vector y.

   b. the directional derivative $f'(\vec{x};\vec{y})$ has a maximum value when $\theta = 0$; that is when $\vec{y}$ has the same direction as $\nabla f(x)$

   c. max $f'(\vec{x};\vec{y}) = |\nabla f(\vec{x})|$ in this case.

   Using α, β, and γ for the direction angles of $\vec{y}$ in 3-space we may write $\vec{y} = \vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma$ and then $f'(\vec{x};\vec{y}) = f_{x_1}(\vec{x})\cos\alpha + f_{x_2}(\vec{x})\cos\beta + f_{x_3}(\vec{x})\cos\gamma$ or if $\vec{x} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ $\quad f'(\vec{x};\vec{y}) = f_x\cos\alpha + f_y\cos\beta + f_z\cos\gamma$.

   Armed with this, we now may turn our attention to the mean-value theorem for a scalar field.

   If we consider two scalars $f(\vec{A})$ and $f(\vec{B})$ the mean-value theorem has the form;

   $f(\vec{A}) - f(\vec{B}) = f(\vec{z})\cdot(\vec{A} - \vec{B})$ where $\vec{z}$ lies on the line segment connecting $\vec{A}$ and $\vec{B}$. See p. 170 [4],II. An excellent physical motivation of the gradient may be found in [ ].

3. The tangent plane and the normal line. If $L(c) = \{\vec{x}|\vec{x} \in S$ and $f(\vec{x}) = c\}$, then $L(c)$ is called the level set of f. In two space $L(c)$ is called a level curve, in three space $L(c)$ is called a level surface. If f is defined in 3-space and $f(\vec{x}) = c$ it is quite easy to show that at any point $\vec{x} = (x_1,y_1,z_1)$ where $f(\vec{x}) \neq 0$, $f(x)$ is normal to $L(c)$. See [4],II.

   If the concept of direction numbers for a line has been developed, and the student has found that the direction numbers of the normal to a surface are given by $f_x : f_y : f_z$ then the fact that the direction numbers and the components of the gradient are the same is easily seen. The equation of the tangent plane to $f(x) = c$ at $X = (x_1,y_1,z_1)$ is $A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$ where $A = f_x$, $B = f_y$, $C = f_z$ all evaluated at $x = (x_1,y_1,z_1)$. The normal line may be seen to be $\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$

where A, B, and C are as above and non-zero. The alternate forms for the normal line:

$$\left.\begin{aligned} x - x_1 &= A \cdot t \\ y - y_1 &= B \cdot t \\ z - z_1 &= C \cdot t \end{aligned}\right\} \quad \text{or} \quad (x_1 + A\,t,\ y_1 + B\,t,\ z_1 + C\,t),$$

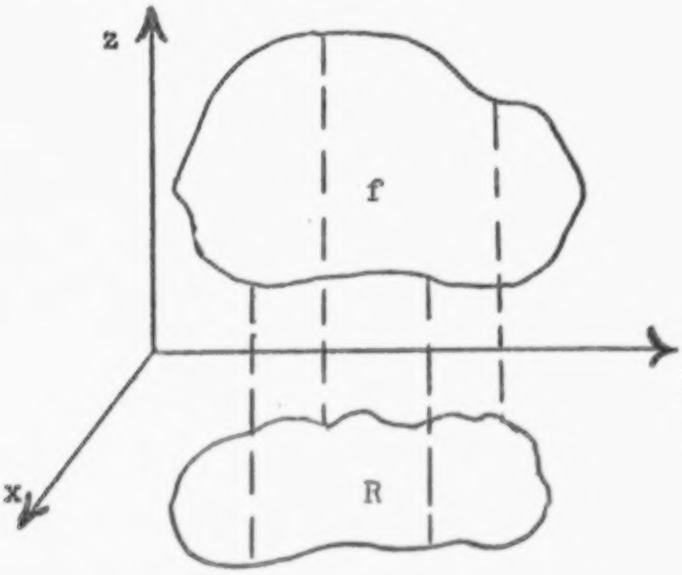
where t is a parameter, leads to a more general form of a line in space $(f(t), g(t), \phi(t))$. This may be used as a review by developing all of the concepts, previously developed by vectors, using parametric forms. The instructor may wish to develop the concepts first by parametric form and then consider vectors, or develop by parametric forms and leave out vectors altogether, depending upon the nature of the class in any particular semester.

4. Maxima and Minima. One of the principal applications of differentiation of functions of one variable occurs in the study of maxima and minima. In that study we derive various tests using first and second derivatives which enable us to determine relative maxima and minima of functions of a single variable.

The study of maxima and minima for functions of two, three, or more variables has its basis in the following theorem (or its extension), usually stated without proof, at this level.

Theorem. Let R be a region in the xy-plane with the boundary curve of R considered as part of R also (see figure). If f is a function of two variables defined and continuous in R, then there is (at least) one point in R where f takes on a maximum value and there is (at least) one point in R where f takes on a minimum value. [28].

Definition. A function $z = f(x,y)$ is said to have a relative maximum at (x_o,y_o) if there is some region containing (x_o,y_o) in its interior such that $f(x,y) \leq f(x_o,y_o)$ for all (x,y) in this region i.e. For any $\varepsilon > 0$ there exists a ε such that whenever $|x - x_o| < \delta$, and $|y - y_o| < \delta$ then $|f(x_o,y_o) - f_9x,y)| < \varepsilon$. [7],[24],[29],[37],[4],II.



Alternate Definition. Whatever the domain R of f, it may happen that there exists a point $p \in R$ for which there exists a neighborhood $N(p,\delta)$ such that for every $x \in R \quad N(p,\delta)$, $f(x) \leq f(p)$. One says that at p, f(x) has a relative maximum. [35],[9′].

Similar definitions hold for relative minimum and are easily extended to functions of three, four, or more variables.

Theorem. Suppose that $z = f(x,y)$ is defined in a region R containing (x_o,y_o) in its interior. Suppose that $f_x(x_o,y_o)$ and $f_y(x_o,y_o)$ are defined and that $f(x,y) \leq f(x_o,y_o)$ for all (x,y) in R; that is, $z_o = f(x_o,y_o)$ is a relative maximum. Then $f_x(x_o,y_o) = f_y(x_o,y_o) = 0$.

The same result holds at a relative minimum. The proof of one of these results usually occurs in the text.

Definition. A value (x_o, y_o) at which both f_x and f_y vanish is called a critical point of f (stationary point of the surface $z = f(x,y)$).

This is a point where the tangent plane is horizontal. [4],II.

The conditions that f_x and f_y vanish at a point are necessary conditions for a relative maximum or a relative minimum. It is easy to find a function for which f_x and f_y vanish at a point, with the function having neither a relative maximum nor a relative minimum at that point. A critical point (stationary point) at which f is neither a maximum nor a minimum may be a saddle point. A simple example of a function which has such a point is given by $f(x,y) = x^2 - y^2$. [28], Maxima, minima, and saddle points may be defined in a geometric manner, as in [14], [37], [4],II, [17].

The following examples illustrate several types of stationary points. In each case the stationary point in question is at the origin.

Example 1: Relative maximum. $z = f(x,y) = 2 - x^2 - y^2$. This surface is a paraboloid of revolution. In the vicinity of the origin it has the shape of a nose cone of a rocket. Cross sections parallel to the xy-plane intersecting the z-axis between $z = 0$ and $z = f(0,0)$ form concentric circles. Since $f(x,y) \leq 2 = f(0,0)$ for all (x,y), it follows that f not only has a relative maximum at $(0,0)$ but also an absolute maximum on any set containing the origin. Both partial derivatives f_x and f_y vanish at the origin.

Example 2: Relative minimum. $z = f(x,y) = x^2 + y^2$. This example, another paraboloid of revolution, is essentially the same as Example 1, except that there is a minimum at the origin rather than a maximum.

Example 3: Saddle Point. $x = f(x,y) = xy$. This surface is a hyperbolic paraboloid. Near the origin the surface is saddle shaped. Both partial derivatives f_x and f_y are zero at the origin but there is neither a relative maximum nor a relative minimum there. In fact, for points (x,y) in the first or third quadrants, x and y have the same sign, giving us $f(x,y) > 0 = f(0,0)$, whereas for points in the second and fourth quadrants x and y have opposite signs, giving us $f(x,y) < 0$. Therefore, in every neighborhood of the origin there are points at which the function exceeds $f(0,0)$, so the origin is a saddle point. Cross sections parallel to the xy-plane form a family of hyperbolas having the x- and y-axes as asymptotes. [4],II.

The basic criterion for finding maxima and minima for functions of two variables is the Second Derivative Test, which follows.

Theorem. Suppose that f and its partial derivatives up to and including the third order are continuous near the point (a,b), and suppose that $f_x(a,b) = f_y(a,b) = 0$; that is, (a,b) is a critical point. Then we have

i) a local minimum if

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0 \quad \text{and} \quad f_{xx}(a,b) > 0$$

ii) a local maximum if

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0 \quad \text{and} \quad f_{xx}(a,b) < 0$$

iii) a saddle point if

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0$$

iv) no information if

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 0$$

Proof. [4],II, [17], [24], [26],II, [28], [29], [35].

E. The Increment of a Scalar Field: Δf. [29], [4],II.

Definition. Let f be a function of two variables and let $z = f(x,y)$; let Δx and Δy be increments of the independent variables, x and y; then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$$

is called the increment of the dependent variable z which corresponds to the increments Δx and Δy of the independent variables, at the point (x,y).

We now have four independent variables, x, y, Δx and Δy, and in general Δz depends on all of them.

The next theorem enables us to express the increment of a function of two variables in a useful form.

Theorem. Let f be a function of two variables with f_x and f_y continuous in a rectangular region R of the xy-plane, and let (x,y) be a point inside R; then for each point $(x + \Delta x, y + \Delta y)$ in R,

$$f(x + \Delta x, y + \Delta y) - f(x,y) = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \Theta_1\Delta x + \Theta_2\Delta y,$$

where Θ_1 and Θ_2 are functions of both Δx and Δy such that $\Theta_1 \to 0$ and $\Theta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. [37], [35].

1. Chain Rule for derivatives of scalar fields. [18], [23],II, [24], [26], [28], [37], [41]. Let $z = f(x,y)$, while $x = g(r,s)$, $y = h(r,s)$. If s changes to some value $s + \Delta s$, while r remains fixed, then x changes to $x + \Delta x$ and y to $y + \Delta y$. The z changes to $f(x + \Delta x, y + \Delta y)$. Consequently

$$\frac{\Delta z}{\Delta s} = \frac{f(x + \Delta x, y + \Delta y) - f(x,y)}{\Delta s}$$

$$= \frac{f(x + \Delta x, y + \Delta y) - f(x,y+\Delta y) + f(x,y+\Delta y) - f(x,y)}{\Delta s}$$

Applying the mean value theorem for functions of one variable,

$$\frac{\Delta z}{\Delta s} = f_x(x_1, y + \Delta y)\frac{\Delta x}{\Delta s} + f_y(x, y_1)\frac{\Delta y}{\Delta s}$$

where x_1 lies between x and $x + \Delta x$ and y_1 between y and $y + \Delta y$. As Δs approaches 0, Δx and Δy approach 0. Moreover, if f_x is continuous, and we do suppose that it is, $f_x(x_1, y + \Delta y)$ approaches $f_x(x,y)$. The analogous statement applies to f_y. Hence

$$z_s = \lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s} = f_x(x,y)\, x_s + f_y(x,y)\, y_s.$$

We write z_s or $\frac{\partial z}{\partial s}$ rather than $\frac{dz}{ds}$ because z is a function of s and r and we are considering the rate of change of z with respect to s only. Similarly,

$$z_r = f_x(x,y)\, x_r + f_y(x,y)\, y_r.$$

The last two formulas are the chain rule for functions of two variables. The same type of argument leads to a number of variations of the chain rule which fit different situations. For some examples see [4′]. See [17] for another approach using the Total Differential. Also see [1′], for the Chain Rule in both gradient form and matrix form. Yet another approach using vector notation is found in [4],II and [2′].

Let $\vec{F}$ be a vector field defined on an open set S, say

$$\vec{F}(x,y) = P(x,y)\,\vec{i} + Q((,y)\,\vec{j}.$$

where P and Q are continuously differentiable on S. Let u and v be two real functions that are differentiable over (a,b) and assume that $(u(t),v(t)) \in S$ for each $t \in (a,b)$. Let

$$\vec{r}(t) = \vec{F}[u(t), v(t)].$$

Then $\vec{r}'(t) = D_u \vec{F}[u(t),v(t)]\, u'(t) + D_v \vec{F}[u(t),v(t)]\, v'(t)$

where $D_u\vec{F} = D_uP\,\vec{i} + D_uQ\,\vec{j}$ and $D_v\vec{F} = D_vP\,\vec{i} + D_vQ\,\vec{j}$.

(1) may be differentiated to give $\vec{r}'(t) = \frac{\partial F}{\partial x} u'(t) + \frac{\partial F}{\partial y} v'(t)$.

2. The Total Differential. We now define $\Delta x = dx, \Delta y = dy$, and $dz = f_x dx + f_y dy$, where x and y are independent variables. It is called the total differential of f. [37],[29].

We then note that $\Delta z = dz + \theta_1 x + \theta_2 y$. $\theta_1 \to 0$ and $\theta_2 \to$ as $x \to 0$ and $y \to 0$, thus dz is a good approximation to z for "small" x, y. [5] Also $dz = f_x dx + f_y dy$ is valid even when x and y are not independent. [4'], [37] Suppose $x = x(r,s)$ and $y = y(r,s)$, then $z = f[x(r,s),y(r,s)] = F(r,s)$ If we consider the above as a function of r and s, then $dz = F_r dr + F_s ds$. $F_r dr + F_s ds = f_x dx + f_y dy$ as a consequence of the chain rule for derivatives. i.e.

$$dz = f_x(x_r dr + x_s ds) + f_y(y_r dr + y_s ds) = (f_x x_r + f_y y_r)dr$$
$$+ (f_x x_s + f_y y_s)ds$$
$$= F_r dr + F_s ds.$$

Note that r, s, dr, and ds are independent variables, but dx and dy are not. [24], [26], [28], [17].

F. Mixed Partial Derivatives. The partial derivatives of $f(x,y)$ are themselves functions of x and y which may have partial derivatives, called the second partial derivatives of $f(x,y)$. The partial derivatives of $\frac{\partial f(x,y)}{\partial x}$ are defined by the formulas:

$$\frac{\partial f(x,y)}{\partial x^2} = \lim_{h \to 0} \frac{\frac{\partial f(x + h,y)}{\partial x} - \frac{\partial f(x,y)}{\partial x}}{h}$$

$$\frac{\partial f(x,y)}{\partial x\, \partial y} = \lim_{h \to 0} \frac{\frac{\partial f(x,y + h)}{\partial x} - \frac{\partial f(x,y)}{\partial x}}{h}$$

If $w = f(x,y)$, then the second partial derivative of the function with respect to x is variously designated:

$$\frac{\partial^2 f}{\partial x\, \partial y} ; \frac{\partial^2 w}{\partial x^2} ; f_{xx} ; w_{xx} ; f_{1,1}$$

Other partial derivatives are designated as:

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 w}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = f_{xy} = w_{xy} = f_{1,2}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 w}{\partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = f_{yx} = w_{yx} = f_{2,1}$$

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y} ; \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)$$

It should be pointed out that the order of partial differentiation is immaterial, provided the function and its various partial derivatives are continuous; $\frac{\partial^{p+q} f}{\partial x^p\, \partial y^q}$ signifies p differentiations with respect to x, q with respect to y. Extension to more variables should be discussed.

G. **Implicit Functions**. An equation involving x, y, and z establishes a relation among the variables. A solution for z in terms of x and y may have one or more functions determined by the relation. Partial derivatives can be computed implicity from the original equation in a way similar to the methods used for ordinary derivatives. (x and y are considered fixed and differentiating the original equation with respect to x, a $\frac{\partial z}{\partial x}$ may be computed).

If $f(x,y,z) = 0$, then the $\frac{\partial z}{\partial x}$ can be found by finding the quantity $-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$. In similar fashion other partial derivatives could be found.

H. Line Integrals.

1. Line Integrals in the plane.

Definition: Suppose there is a number L with the following property: for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta)(x_i - x_{i-1}) - L\right| < \epsilon \qquad \text{for every subdivision}$$

with $||\Delta||<\delta$ and for any choices of the (ξ_i,η_i) on the plane curve C from A to B. Then the line integral of f with respect to x along the curve C exists and its value is L. Symbols representing this line integral include:

$$\int_C f(x,y)\,dx \quad \text{and} \quad (C)\int_A^B f(x,y)\,dx.$$

$\int_C f(x,y)\,ds$ is the line integral of $f(x,y)$ along C with respect to the arc length s.

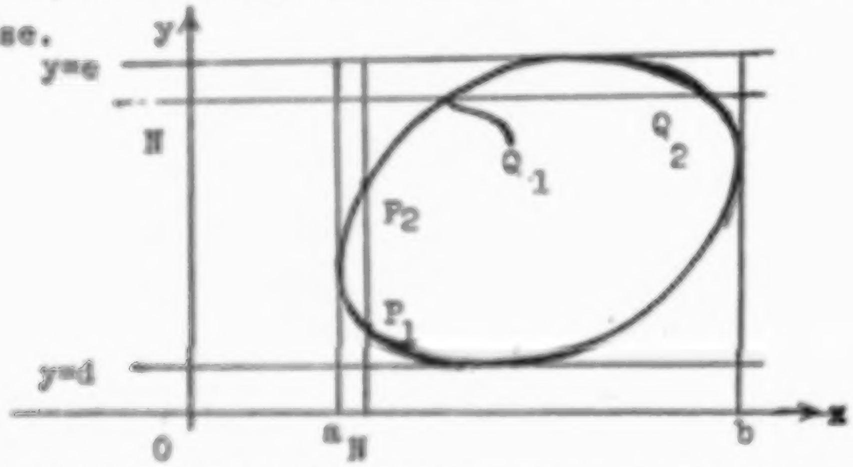
Attention should be called to the fact that if the arc C is traversed in the opposite direction, the line integral changes sign.

$$\int_{\substack{C \\ A \text{ to } B}} f(x,y)\,dx = - \int_{\substack{C \\ B \text{ to } A}} f(x,y)\,ds$$

2. A Geometric Interpretation. Consider a closed curve C tangent to the striaght lines $x = a$, $x = b$, $y = d$, $y = e$, and of such shape that a straight line parallel to either of the coordinate axes intersects it in not more than two points. Let the vertical line through any point M intersect C in P_1 and P_2, where P_1 and P_2 have y coordinates $y_1(x)$ and $y_2(x)$, respectively. Then if A is the area enclosed by the curve,

$$A = \int_a^b y_2 dx - \int_a^b y_1 dx = -\int_b^a y_2 dx - \int_a^b y_1 dx = -\int_{(C)} y\, dx$$

The last integral being taken around C in a direction counter clock-wise.



Similarly, if the line NQ_2 intersects C in Q_1 and Q_2, with x coordinates $x_1(y)$ and $x_2(y)$, respectively, then

$$A = \int_d^e x_2 dy - \int_d^e x_1 dy = \int_d^e x_2\, dy + \int_e^d x_1 dy = \int_{(C)} x\, dy$$

the last integral being taken also in the direction opposite to the motion of the hands of a clock. By adding the two values of A, we have

$$2A = \int_{(C)} (-y\, dx + x\, dy) \quad \text{whereby} \quad A = 1/2 \int_{(C)} (-y\, dx + x dy)$$

3. Path Independent Line Integrals. Suppose that $P(x,y)\, dx + Q(x,y) dy$ is an exact differential. That is, there is a function $f(x,y)$ with

$df = P\,dx + Q\,dy$. Then the line integral $\int_C P\,dx + Q\,dy$ depends only on the end points and not on the arc C joining them. This may be extended to more variables. The integral of an exact differential taken around a closed path is, in general, zero; while the line integrals of other differentials around a closed path are not zero.

4. Work. When $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ is at least a piece-wise continuous vector function specified along a smooth curve C given by the vector equation $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = \vec{r}(s)$, $0 \leq s \leq 1$, then the line integral $\int_C \vec{F}\,d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$ represents work when F is the force acting on a particle moving along C.

5. Evaluation of a line integral. If the integrand can be easily expressed in parametric form, the integration may be performed with respect to only that parameter. If one of the variables can be easily and explicitly expressed in terms of another, the integral may be changed to one involving only one variable. In cases where parts of the integral would vanish, such as along a polygonal path with segments parallel to the coordinate axes, the line integral might best be taken in parts, degenerating to integrals of single variables. In the case of a path-independent line integral, when an exact differential is noticed as the integrand, the best method may be to take the difference of functional values at upper and lower limits. See b and c.

a. Evaluate $\int_C [xy\,dy - y\,dx]$ where C is the arc $x = t^2$, $y = 6t + 3$, $0 < t < 1$ $\int_0^1 t^2(6t+3)6\,dt - (6t+3)2t\,dt = 8$ noting $dx = 2t\,dt$, $dy = 6\,dt$.

b. Evaluate $\int_C [(x + y)\,dx + (x - y)\,dy]$ where C is the straight line segment from (0,0) to (2,1). Since this segment is $y = \frac{1}{2}x$ or $x = 2y$, $dx = 2\,dy$, and we get

$$\int_0^1 [(2y + y)(2\,dy) + (2y - y)dy] = 7/2$$

c. Evaluate $\int_C [(x + y)\,dx + (x - y)\,dy]$ where C consists of the straight line segment from (0,0) to (2,0) followed by that from (2,0) to (2,1). Noting that along the first segment $y = 0$ and $dy = 0$, and along the second segment $x = 2$ and $dx = 0$, we get

$$\int_0^1 (x + 0)dx + (x - 0)(0) + \int_0^1 (2 + y)(0) + (2 - y)dy = 7/2$$

d. In examples b and c above, the integrand is an exact differential. It can be shown that in this situation the line integral is path-independent and can be evaluated by finding the function for which it is the exact differential and evaluating at the limit. Evaluate $\int_C [(x + y)dx + (x - y)dy]$ where C is any from (0,0) to (2,1)

$$\left. \frac{x^2}{2} + xy - \frac{y^2}{2} \right]_{0,0}^{2,1} = 7/2$$

e. Find work if a force, $F = (x^2-y)i + (y^2-x)j + (z^2-x)k$ moves a particle from the origin to (1,1,1) along a straight line. Since, along this line, $x = y = z$, $dx = dy = dz$, we substitute

$$\text{Work} = \int_C [(x^2-y)dx + (y^2-x)dy + (z^2-x)dz] = \int_0^1 3(x^2-x)dx = -\frac{1}{2}$$

Find the work if the same force moves the same particle from (0,0,0) to (1,1,1) along $x = t$, $y = t^2$, $z = t^3$. Here it might be best to substitute $dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$

$$\text{Work} = \int_0^1 0\,dt + \int_0^1 (t^4 - t^3)2t\,dt + \int_0^1 (t^6 - t)3t^2\,dt = -\frac{29}{60}$$

XIII. MULTIPLE INTEGRATION

Some authors develop the concept of multiple integration from a general n-dimensional point of view and introduce the double and triple integrals as special cases (See Bell, et al, *Modern University Calculus*). In this syllabus the double integral is taken up separately from the triple integral with the triple integral as a logical extension of the double integral.

A. Double Integral

1. *Motivational Examples*. A novel way to introduce the double or triple integral is to consider the problem of finding a function of several variables if some partial derivative is known (See [8]). For example, find $f(x,y)$ if $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x$. The students should have no trouble in seeing that $\frac{\partial f}{\partial x} = 2xy + C$ and that $f(x,y) = x^2y + Cx + D$

Most authors give examples of finding the area of plane regions bounded by a simple closed curve and approximating the volume of a three dimensional region bounded above by a surface and below by the xy-plane.

2. Formal Properties of the Double Integral

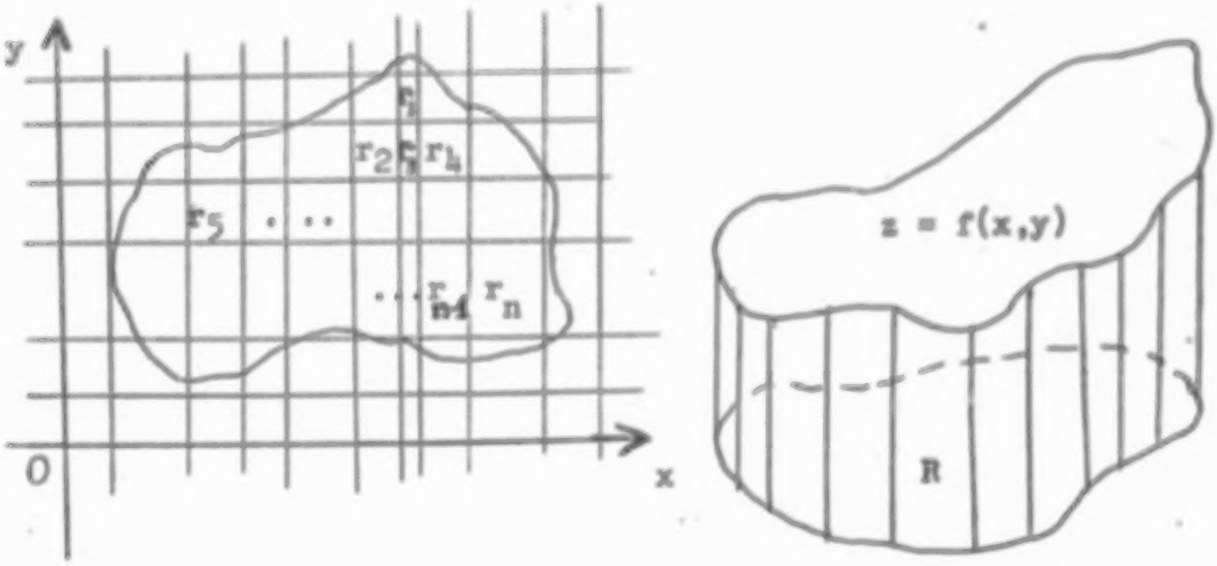
Definition. (The double integral is defined as the volume of the three dimensional region bounded above the surface $z = f(x,y)$ and below by a region of the xy plane). Let R be a region of the xy-plane bounded by a simple closed curve. Let Δ be a set of n non-overlapping adjacent rectangles formed by two sets of lines respectively parallel to the x-axis and y-axis which lie completely within R, and denote by $||\Delta||$ the length of the longest diagonal of all rectangles in Δ. Denote the rectangles in Δ by $r_1, r_2, r_3, \ldots, r_n$, and denote their areas by $A(r_1)$, $A(r_2), \ldots, A(r_n)$. If f is a function defined for all (x,y) in the region R, and if (a_i, b_i) is any point in r_i, then the function f is integrable over R if and only if the following limit exists:

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(a_i, b_i) \cdot A(r_i) = L.$$

The limit L is called the double integral of f over R and is written

$$\iint_R f(x,y)\, dA$$

The following figure gives a geometric interpretation of the definition.



Theorem. If f is continuous for all (x,y) in a plane closed region R, then f is integrable over R.

For proof, see (24).

Theorem. If c is any number and if f is integrable over R, then $c\,f$ is integrable over R and

$$\iint_R c\ f(x,y)\ dA = c \iint_R f(x,y)\ dA.$$

The proof follows from the definition above and a property of limits.

Theorem. If f and g are integrable over R, then $f + g$ is integrable over R, and

$$\iint_R (f + g) = \iint_R f + \iint_R g.$$

The proof follows from the definition above and a property of limits.

Theorem. If R is decomposed into two subregions R_1 and R_2 such that $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \emptyset$, and if f is continuous over R, then

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f.$$

3. Evaluation of Double Integrals. An intuitively appealing way to introduce the student to the iterative process is given by the example on "anti-partialdifferentiation" given in 2 above.

The teacher should state clearly the theorem on evaluation of double integrals by the iterative process, and he should show by examples that the student should be able to use both orders of iteration. The following example might be used:

Evaluate $\iint_R x^2y^2\ dA$ where R is the region bounded by $y = 1$, $y = 2$, $x = 0$, and $x = y$.

First, note that $R = \{(x,y) \mid 1 \leq y \leq 2,\ 0 \leq x \leq y\}$. Hence

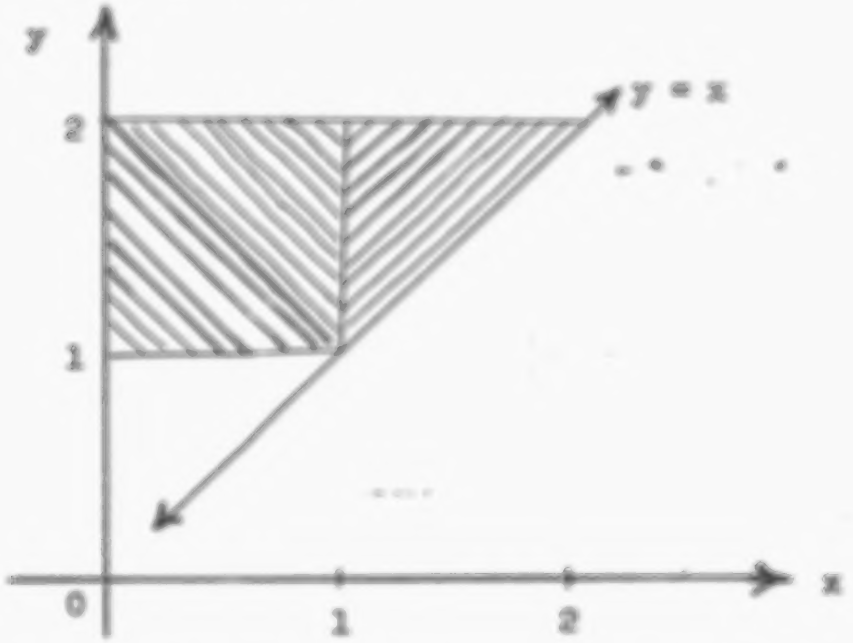
$$\iint_R x^2y^2\,dA = \int_1^2 \left[\int_0^y x^2y^2\,dx\right] dy = \int_1^2 \left[\frac{1}{3}\,x^3y^2\right]\Big|_0^y\,dy$$

$$= \frac{1}{3}\int_1^2 y^5\,dy = 7/2$$

The above method is less involved than the following:
Note that $R = \{(x,y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2$ if $0 \leq x \leq 1$, and $x \leq y \leq 2$ if $1 < x \leq 2\}$. Therefore

$$\int_R x^2y^2\,dA = \int_0^1\left[\int_1^2 x^2y^2\,dy\right]dx + \int_1^2\left[\int_x^2 x^2y^2\,dy\right]dx.$$

See the figure below.



4. Counterexample. The following example is given to show that the iterated integral may exist while the double integral may fail.
Evaluate $\int_R f(x,y)\,dA$ if $f(x,y) = \begin{cases} \sin y & \text{if } x \text{ is rational} \\ -\sin y & \text{if } x \text{ is irrational} \end{cases}$

and $R = \{(x,y) \mid 0 \leq x \leq 2\pi,\ 0 \leq y \leq 2\pi\}$.

First, note that $\int_0^{2\pi}\int_0^{2\pi} f(x,y)\,dy\,dx = 0$, since $\int_0^2 f(x,y)dy = 0$

for x rational or irrational.

Observe as follows that $\iint_R f(x,y)\, dA$ does not exist.

Subdivide R into two subregions $R_1 = \{(x,y) \mid 0 \leq x \leq 2\pi, 0 \leq y \leq \pi$, and $R_2 = \{(x,y) \mid 0 \leq x \leq 2\pi, \pi < y \leq 2\pi\}$, then the defining limit may be written,

$$\lim_{\|\Delta\| \to 0} \sum_{R} f(a_i,b_i)\, A(r_i) = \lim_{\|\Delta\| \to 0} \left[\sum_{R_1} f(a_i,b_i)\, A(r_i) + \sum_{R_2} f(a_i,b_i)\, A(r_i) \right].$$

where (a_i,b_i) is any point in r_i.

First, evaluate the limit by choosing a_i rational if $r_i \subseteq R_1$ and a_i irrational if $r_i \subseteq R_2$.

Then $$\lim_{\|\Delta\| \to 0} \sum_{R} f(a_i,b_i)\, A(r_i) = \lim_{\|\Delta\| \to 0} \left[\sum_{R_1} \sin b_i\, A(r_i) + \sum_{R_2} -\sin b_i\, A(r_i) \right] = 8\pi.$$

Now, evaluate the same limit by choosing a_i irrational if $r_i \subseteq R_1$ and a_i rational if $r_i \subseteq R_2$. Then the limit becomes

$$\lim_{\|\Delta\| \to 0} \left[\sum_{R_1} -\sin b_i\, A(r_i) + \sum_{R_2} \sin b_i\, A(r_i) \right] = -8\pi.$$

For the limit to exist its value must be the same for all choices of a_i. Hence, the double integral fails. [1′].

5. Applications of Double Integrals.

a. Area. Referring to the definition of the double integral, if $f(x,y) = 1$, the student should see that the integrand becomes the increment of area for a plane region. It should be carefully demonstrated that the limits of integration must be chosen in such a way that the first integration in the iteration can be described by a narrow rectangular strip, after which the second is just the standard single integral for area. This will show the close relationship between the two methods for determining area, and will demonstrate that one is no real improvement on the other.

b. Moments. After having used single integrals for finding the various moments, the student will appreciate the use of the double integral. For the increments for determining M_x and M_y are respectively $y\ dydx$ and $x\ dydx$, for either order of integration. This eliminates the difficulty of always measuring from the centroid of a rectangle. The same simplification results for higher moments.

c. Volumes. The main difficulty with problems in this section is sketching the boundaries of the solid region accurately enough to be able to determine the limits of integration correctly.

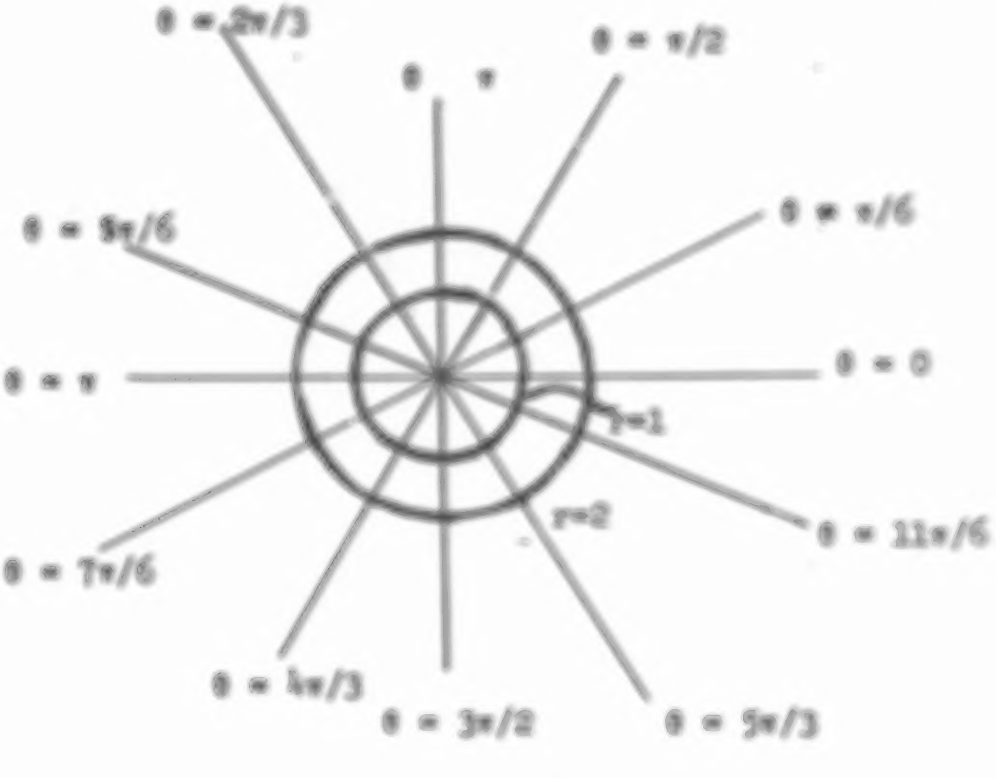
This should be reviewed.

B. Change of Variables in a Double Integral. This material can be introduced by recalling the substitutions used in integrals of functions of one variable. The desirability of substitutions in integrals of functions of two variables should be evident to the students.

Assume that continuous functions, possessing continuous first partial derivatives with respect to x and y, $u = f_1(x,y)$ and $v = f_2(x,y)$ have been determined. Also, assume that for some set of values of u and v, f_1 and f_2 have inverse functions, so that $x = \phi_1(u,v)$ and $y = \phi_2(u,v)$.

If $u = u_0$ is held fixed, $x = \phi_1(u_0,v)$ and $y = \phi_2(u_0,v)$ are parametric equations for a curve in the xy-plane. Similarly $v = v_0$, a constant, defines a curve whose parametric equations are $x = \phi_1(u,v_0)$ and $y = \phi_2(u,v_0)$. Such a procedure defines a parameterization of the xy-plane.

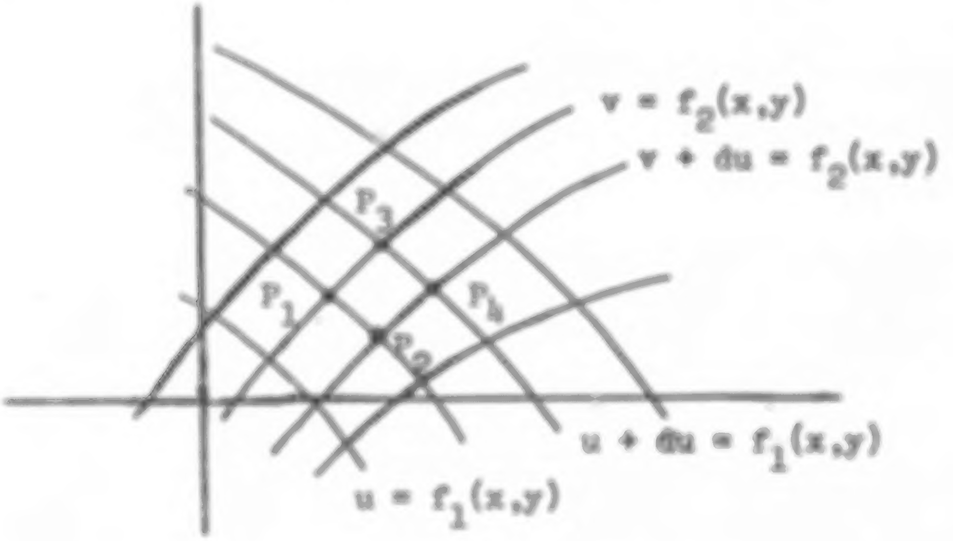
As an example, let $x = r \cos \theta$, $y = r \sin \theta$, $r \epsilon (0,\infty)$, $\theta \epsilon (0,2\pi)$ with inverse functions $r = \sqrt{x^2 + y^2}$, $\theta = \text{arc tan } y/x$, with proper restrictions on θ so that its terminal side falls in the quadrant containing (x,y). This gives the following parameterization:



Next, consider the increment of area in the new parameterization Let $x_1 = \phi_1(u,v)$, $y_1 = \phi_2(u,v)$, $x_2 = \phi_1(u,v + dv)$, $y_2 = \phi_2(u,v + dv)$ $x_3 = \phi_1(u + du, v)$, $y_3 = \phi_2(u + du, v)$, $x_4 = \phi_1(u + du,v + dv)$, and $y_4 = \phi_2(u + du, v + dv)$, where du and dv are positive, and small.

Then $P_1(x_1,y_1)$, $P_2(x_2,y_2)$ $P_3(x_3,y_3)$, and $P_4(x_4,y_4)$ are points in the xy-plane.

The uv-coordinate curves which pass through these points are determined by $u = f_1(x,y)$, $v = f_2(x,y)$, $u + du = f_1(x,y)$, $v + dv = f_2(x,y)$ (here the u,v, du, and dv are fixed, and the x and y then give the coordinates of the points on the coordinate curves). These appear as follows:



The area of the region defined by the curved quadrilateral then is the increment of area, dA, which is to be determined.

The mean value theorem gives the following:

$$\phi_1(u,v + dv) - \phi_1(u.v) = \frac{\partial\phi_1}{\partial v}\, dv$$

$$\phi_2(u,v + dv) - \phi_2(u,v) = \frac{\partial\phi_2}{\partial v}\, dv$$

$$\phi_1(u + du,v) - \phi_1(u,v) = \frac{\partial\phi_1}{\partial u}\, du$$

$$\phi_2(u + du,v) - \phi_2(u,v) = \frac{\partial\phi_2}{\partial u}\, du$$

Thus $x_2 = \phi_1(u,v) + \frac{\partial\phi_1}{\partial v}\, dv$, $y_2 = \phi_2(u,v) + \frac{\partial\phi_2}{\partial v}\, dv$, $x_3 = \phi_1(u,v) + \frac{\partial\phi_1}{\partial u}\, du$, and $y_3 = \phi_2(u,v) + \frac{\partial\phi_2}{\partial u}\, du$.

One formula for the area of a triangle, which may have been included in the students' analytic geometry training, is, for points (x_1,y_1), (x_2,y_2) and (x_3,y_3),

$$A = \pm 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

, with the sign chosen to make A positive. Substitution gives

$$\pm 1/2 \begin{vmatrix} \phi_1 & \phi_2 & 1 \\ \phi_1 + \frac{\partial\phi_1}{\partial v}\,dv & \phi_2 + \frac{\partial\phi_2}{\partial v}\,dv & 1 \\ \phi_1 + \frac{\partial\phi_1}{\partial u}\,du & \phi_2 + \frac{\partial\phi_2}{\partial u}\,du & 1 \end{vmatrix}$$

$$= \pm 1/2 \begin{vmatrix} \frac{\partial\phi_1}{\partial v} & \frac{\partial\phi_2}{\partial v} \\ \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_2}{\partial u} \end{vmatrix} du\,dv = \pm 1/2 \; J(u,v)\,du\,dv$$

The area dA is approximated by $|J(u,v)|\,dudv$. However, $dA = (\,|J| + \varepsilon)\,dudv$, and it is not obvious that $\varepsilon \to 0$ as du and $dv \to 0$, in all cases. The previous example in polar coordinates should be recalled to make this seem reasonable, without attempt at proof. See [5].

Then, the Jacobians for this and other transformations should be calculated.

C. The Triple Integral

1. Motivational Examples. The need for the triple integral can be shown by attempting to find the volume of a general three dimensional region bounded by sufaces. Also, and example using the "anti-partial differentiation" process of A,1 above could be devised. (See [8]).

2. Formal Properties of the Triple Integral. A definition and some theorems should be presented parallel to those given for the double integral in A,2 above.

3. Evaluation of Triple Integrals. The discussion should parallel that of section A,3 above.

4. Applications of Triple Integrals. Volumes and moments should be covered here in a manner similar to areas and moments with double integrals. The comparison between triple and double integrals is essentially the same as that between double and single integrals.

APPENDIX:

Sample Final Examination Questions
for the First Semester of Calculus

1. Approximate the area bounded by the curve $y = x^2$, the x-axis and the line $x = 2$ by dividing the area into 6 vertical rectangles whose widths are equal and whose altitudes are the average of left and right ordinates.

2. A baseball diamond is a square 90 ft. on a side. A man runs from first base to second base at 25 ft. per second. At what rate is his distance from third base decreasing when he is 30 ft. from first base?

3. A. Define what is meant by "the function f is continuous at $x = a$".
   B. Give an example of, and sketch, a discontinuous function.

4. How do continuity and differentiability relate?

5. If f is a function of the independent variable x, the derivative of f with respect to x is defined by the formula:

6. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

7. Let f be defined by $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3 \\ a, & \text{if } x = 3 \end{cases}$

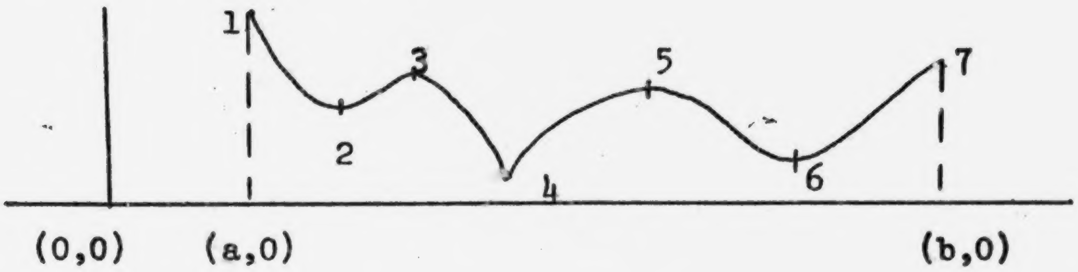
   Find the value of a so that f is continuous at $x = 3$.

8. True or False: The function F defined by $F(x) = (2x - 3)^{3/7}$ is differentiable on its entire domain.

14

Calculus - 1st Year

Questions 9 through 14 will refer to the following graph:



If f is a continuous function of the closed interval [a,b] having the graphical representation above find:

9. Which points are relative maximums?

10. Which points are relative minimums?

11. If there is an absolute maximum, which point would it be?

12. Since f is continuous, between what pairs of points must there be a point of inflection?

13. What point seems to be an absolute minimum (if one exists)?

14. What can be said about f' and f'' at each of the seven points?

Point	f'	f''
1		
2		
3		
4		
5		
6		
7		

Use answers $\left\{ \begin{array}{l} =0,\ >0,\ <0,\ ?, \\ \geq 0,\ \leq 0 \end{array} \right\}$

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LINEAR ALGEBRA OUTLINE

LINEAR ALGEBRA (4 semester hours)

- I. Introduction 1-3 hours
    - A. Sets
    - B. Vectors
- II. Vector Spaces 13-15 hours
    - A. Definition of real vector spaces
    - B. Generators
        - 1. Linear dependence and independence
        - 2. Bases and dimension of vector spaces, and subspaces
    - C. Simultaneous equations
    - D. Direct sums and intersections of vector subspaces
    - E. Inner products and orthogonal bases
    - F. Applications to E_n
- III. Determinants 2-3 hours
    - A. Properties of determinants
    - B. Applications
        - 1. Solution of simultaneous equations
        - 2. The content of a parallelotope
- IV. Linear Transformations 4-5 hours
    - A. Definition of linear transformation
    - B. Elementary properties of linear transformation
    - C. Algebraic properties of linear transformations
- V. Matrices 15 hours
    - A. Linear transformation and matrices
    - B. Algebraic operations on matrices

- C. Nonsingular matrices
- D. Special types of square matrices
- E. Inversion of matrices
- F. Change of bases of vector spaces
- G. Applications
  - 1. Solution of simultaneous linear equations
  - 2. Transformation of E_n

VI. Bilinear and Quadratic Forms and Characteristic Values 6-12 hours

- A. Bilinear forms
- B. Quadratic forms
- C. Characteristic values and vectors

LINEAR ALGEBRA

I. Introduction and Review

A. Sets

The definition of and notation for a set, with union and intersection, and map should be covered. Any other concepts introduced here should have a definite application in the course. (Math induction might be one.)

B. Vectors

A review should be made of two-and-three-dimensional vector spaces, with review of both i, j, k and ordered triple notations. The student should be familiar with vector addition, scalar multiplication, the dot and cross products, and their intrepretations when applied to geometry. These ideas need only be reviewed, and not elaborated on. However, it should be made clear to the student that vectors have many applications and interpretations, so that he will see the need for an abstract idea of a vector. Throughout the following, references should be made to the vector concepts covered in Calculus as often as possible.

II. Vector Spaces

A. Definition of real vector spaces

The abstract definition should be given, as in [10], with illustrations such as the set of real-valued functions on $[0,1]$, function spaces, as well as finite dimensional spaces.

B. Generators

1. Linear dependence and independence

Several examples of dependent and independent sets of vectors should be given, as well as theorems concerning dependence and independence, such as the theorem that says that, if $x, \ldots, x_n$ are a set of vectors, such that $x_1 \neq 0$, then they are linearly dependent if and only if some one of the vectors $x_2, \ldots, x_n$, say x_k, is a linear combination of $x_1, \ldots, x_{k-1}$. Implicit here should be the idea of generation of a vector space.

2. Bases and dimension of vector spaces and subspaces

While working with independent sets of vectors and their linear combinations, the student should develop the idea of bases and dimension, so that these definitions come as no suprise. Included in this material should be the basic theorems concerning collections of independent vectors as generators of vector spaces and subspaces, such as, if $x_1, \ldots, x_n$ are a basis for x, and $y_1, \ldots, y_m$ are a linearly independent subset of x, then $m = n$. Another would be, if x is n-dimensional, and $y_1, \ldots, y_m$ are linearly independent, with $m < n$, there is a basis of x consisting of $y_1, \ldots, y_m$, and $n - m$ other vectors.

C. Systems of simultaneous equations

While determining dependence or independence of a set of vectors, the student learns the connection between vectors and simultaneous equations. At this point, it is appropriate to formalize the connection between sets of vectors and solutions of homogeneous and non-homogeneous systems of linear equations. This may be done by a theorem such as the following: A system

$$\begin{array}{c} a_{11}c_1 + a_{12}c_2 + \ldots + a_{1k}c_k = d_1 \\ \vdots \\ a_{n1}c_1 + a_{n2}c_2 + \ldots + a_{nk}c_k = d_n \end{array}$$

has a solution if and only if the vector $[d_1, \ldots, d_n]$ is in the subspace generated by $[a_{11}, a_{21}, \ldots, a_{n1},] \ldots, [a_{1k}, a_{2k}, \ldots, a_{nk}]$.

D. Direct sums and intersections of vector subspaces

This should include as examples the direct sum and intersection of two-dimensional vector spaces as subspaces of three-dimensional vector spaces. Theorems should be included on bases and dimension of intersections and direct sums of vector spaces. Such a theorem is: If S and T are subspaces of X, then $\dim (S) + \dim (T) = (S \cap T) + \dim (S + T)$.

E. Inner products and orthogonal bases

The inner product should be defined on an abstract vector space. Then, it should be noted that there are other inner products than the standard Euclidean n-dimensional one.

Orthogonal vectors and orthogonal bases should be defined, with numerous examples from two and three dimensions. Then, the Gram-Schmidt process should be covered. (Note this is independent of any particular inner product definition.)

F. Applications to E_n. (This section may be omitted.)

The vector equations of lines and K-dimensional planes should be covered, as well as uses of the dot product as a metric. This material should be motivated by examples from 2- and 3-space. Of course, this is not to say that this is all that should be covered here, or that no other applications should be considered elsewhere.

Such things as simple transformations of coordinates may be introduced here when discussing K-planes, by means of the equation

$$X = P_o + t_1(X_1 - P_o) + \cdots + t_n(X_n - P_o),$$

where P_o is a given vector to a point P_o, which is desired to be the "new" origin, and $X_1 \cdots X_n$ are independent vectors. If X is a vector to a point in the old coordinate system, $(t_1, \cdots, t_n)$ represents the same point in the new coordinate system. (The alibi approach may be used also). This fits in well with the equations for lines and planes, but may be deferred until the section on linear transformations, especially if time is short.

III. Determinants

All linear algebra texts have a development of determinants. Any of these is satisfactory, but it should be kept in mind that the determinant is a tool (mainly a theoretical one), and only those properties that will be of use should be included. One development of the basic properties of determinants that is especially practical is covered in [7]. Also, [8] contains this material, in a rather abbreviated form.

Whatever approach is used, Cramer's rule should be mentioned as a method for solving systems of linear equations, but it should be pointed out that it is impractical. Its derivation is not necessary.

IV. Linear Transformations

A. Definition of linear transformations

The definition should be given in an abstract form, so that the student does not get the impression that linear transformations only apply to finite dimensional vector spaces. As familiar examples, the derivative, and definite integral with variable upper limit, defined on appropriate vector spaces, should be mentioned.

B. Elementary properties of linear transformations should be covered, especially those properties associated with the basis vectors. Included should be the theorem that the image, under a linear transformation, of a basis will generate the image space. Another would be the necessary and sufficient conditions that a linear transformation be one-one. One such condition is that the only vector carried to the zero vector is the zero vector. Examples of transformations of finite dimensional vector spaces should be given, both formal and numerical. In this way, the algebraic relationships are emphasized.

C. Algebriac properties of linear transformations

The additive group of linear transformations should be discussed first. Then, singular and nonsingular transformations should be defined, and inverses of transformations discussed. This leads into the multiplicative group of nonsingular transformations of a vector space onto itself. Also, theorems may be included here concerning singularity and nonsingularity of transformations. The fact that the existence of the inverse of a linear transformation $A : X \rightarrow Y$, as a function $A^{-1}:A(X) \rightarrow X$, is equivalent to A being one-one should be discussed.

V. Matrices

A. Linear transformations and matrices

After listing the system of linear equations which result from a linear transformation on a vector space into another vector space, it should be a short step to the rectangular array which forms a matrix. This system can be written as $T(E_1) = a_{11}E_1{}^1 + a_{12}E_2{}^1 + \dots + a_{1m}E_m{}^1$

$T(E_n) = a_{n1}E_1{}^1 + a_{n2}E_2{}^1 + \dots + a_{nm}E_m{}^1$,

where $E_1, \dots, E_n$ are basis vectors for the domain of T, and $E_1{}^1, \dots, E_n{}^1$ are basis vectors for the range of T. The aij' may be then used in matrix form, with the usual multiplication definition, to show the effects of T on any n-dimensional vector. It should be noted that the matrix representation of a transformation is dependent on the bases used, and various bases should be used as examples. Also, for different bases, a given matrix represents different transformations.

B. Algebriac operations on matrices

Equality and operations on matrices should be covered from the standpoint of a linear transformation. Thus, if matrices A and B represent linear transformations T_1 and T_2, then $C = A + B$ represents $T_1 + T_2$ where addition of A and B is defined as it is because it fits with the operator $T_1 + T_2$. The more formal approach to matrix theory, in which a matrix is regarded as a function, or as a rectangular array of numbers, with appropriate definitions, seems too abstract for the students at this level. They need to see that a matrix is a natural outgrowth of the application of a linear transformation to a finite dimensional vector space.

C. Nonsingular matrices

Nonsingular matrices should be defined and related to nonsingular linear transformations, by showing that one determines the other, for a given basis. Then, various necessary and sufficient theorems for singularity and nonsingularity should be proved, as in [10].

D. Special types of square matrices

The transpose of a matrix, and symmetric, triangular, orthogonal and diagonal matrices should be discussed, along with the properties of the associated transformations. Similar matrices should be included here also.

E. Inversion of nonsingular matrices

Methods of inverting matrices should be given including the method of using elementary matrices. These should include numerical techniques, which should be applied if the students have had a course in computer programming. See [4].

F. Change of bases of vector spaces

The use of matrices to change from any given basis of a vector space to any other should be covered here. Included here should be the theorem that says that, if A and B are matrices representing the linear transformation $T : X \longrightarrow Y$, with respect to bases $\{X_i\}$ and $\{Y_j\}$, and $\{X_i^1\}$ and $\{Y_j^1\}$, respectively, and if P and Q are matrices representing the change of basis from $\{X_i\}$ to $\{X_i^1\}$ and $\{Y_j\}$ to $\{Y_j^1\}$, respectively, then $B = PAQ^{-1}$.

G. Applications of matrices

1. Solution of simultaneous linear equations

Gaussian elimination should be used here. The relationship between the rank and the number of solutions should be covered.

2. Transformations of E_n

Familiar transformations, such as rotation of axes, should be included as examples.

VI. Bilinear and quadratic forms and characteristic values

A. Bilinear forms

Bilinear operators and forms should be defined and their properties briefly discussed.

B. Quadratic forms

The quadratic form should be defined in terms of the bilinear form, and its representation by a symmetric matrix should be discussed. It should be proved that two real quadratic forms are equivalent (there is a linear change of variables transforming one into the other) if and only if their symmetric matrices A and B are such that $B = PAP^t$, for P some nonsingular matrix. The simplification of quadratic forms by means of a change of basis should then be discussed. This requires a brief introduction to characteristic values and vectors. For one application, requiring calculus, see [2].

C. Characteristic values and vectors

The material on characteristic values and vectors, at least should be covered so as to reduce a symmetric matrix to an equivalent diagonal matrix, with characteristic values on the diagonal. See [10]. If time is available, more material on the various canonical forms may be covered. The Cayley-Hamilton theorem may be stated, but there would probably be no time available for a proof.

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CALCULUS - 4th SEMESTER - OUTLINE

CALCULUS - 4th SEMESTER	(4 semester hours)
I. Sequences and Series	15 hours
A. Improper integrals	
B. Infinite sequences	
C. Series; finite and infinite	
1. Definition	
2. Convergence tests	
D. L'Hospital's rule.	
II. Vector-Valued Functions on a E_1	6 hours
A. Limits and continuity	(1 hr)
B. Derivatives of vectors	(3 hrs)
III. Functions on a Vector Space	24 hours
A. Limits and continuity	(2 hrs)
B. Directional derivative and gradient	(3 hrs)
C. Chain rule	(2 hrs)
D. Exact differential	(2 hrs)
E. Divergence and curl of vector valued functions	(3 hrs)
1. Definitions	
2. Applications	
F. Line integrals	(6 hrs)
G. Transformation of variables and multiple integrals	(2 hrs)
H. Surfaces and solids	(5 hrs)
IV. Differential Equations	15 hours
A. First order equations	
1. Variables separable	
2. Homogeneous	
3. Exact	
4. Linear, integrating factor	
5. Series Solutions	
B. Second order equations	
1. Special forms	
2. Operators	
3. Linear with constant coefficients	
4. Linear nonhomogeneous	
5. Equations with variable coefficients	

CALCULUS - 4th Semester 4 Semester Hours

I. SEQUENCES AND INFINITE SERIES

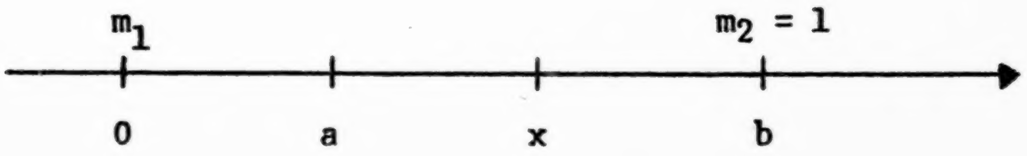
A. Improper integrals

Improper integrals are introduced here so that they may be used for the integral test for infinite series. There are many analogies between $\int_0^\infty f(x)\,dx$ and $\sum_{n=0}^{\infty} a_n$. Improper integrals involve functions with discontinuities, at one or more points in the interval under consideration, or at one or the other of its end-points, or at both endpoints. Their value is obtained, by definition, by a limit process and if this limit exists we say the integral converges; otherwise we say it diverges. Formation of the indefinite integral is the first step in finding the convergence or divergence of the integral; after this the limiting process takes over. If the function involved is an elementary function, the limit is easily obtained.

Various types of indefinite integrals are listed in all textbooks so are omitted here. Integrals like $\int_0^2 \frac{dx}{(x-1)^2}$, which is discontinuous in the interval, are perhaps the most difficult for students to see. Good references are found in [12], [14], and [35].

Improper integrals can arise from problems involving probability and statistics, work, force, area of regions under curve, and Laplace transforms.

As an example, consider a particle of mass m_1 and let it be situated at the origin, and let a test particle of mass $m_2 = 1$, move on the x axis



in a positive direction as a result of a force f(x) acting on the test particle. The total work moving the particle from a to b is:

$$Wab = \int_a^b f(x)\,dx$$

If f is the gravitational force acting of m_1 due to m_2, then by Newton's Law:

$$Wab = \int_a^b \frac{km}{2^1}\,dx = km_1\,(1/a - 1/b).$$

The gravitational potential at a due to m_2 is:

$$P = \lim_{b\to\infty} Wab = km_1 \frac{1}{a}, \text{ or } P = \int_a^{\infty} km_1/x^2\, dx$$

This is an example of an improper integral.

1. DEFINITION. Improper integrals of the first kind.

If f is continuous for all $x > a$,

$$\int_a^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

If the limit exists we say that the improper integral converges. If not, we say it diverges. For example,

$$\int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \int_0^b e^{-x}\, dx = \lim_{b\to\infty} (-e^{-x}) = \lim_{b\to+\infty} (-e^{-b}+1) = 1$$

2. DEFINITION. Improper integrals of the second kind.

If f is continuous for $a < x \le b$ but is not continuous at a

$$\int_a^b f(x)\, dx = \lim_{e\to 0+} \int_{a+e}^b f(x)\, dx.$$

If the limit exists, we say the improper integral converges. If not, we say it diverges. For example,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{\epsilon\to 0^+} \int_{a+e}^1 \frac{1}{\sqrt{x}}\, dx = \lim_{\epsilon\to 0^+} 2x^{\frac{1}{2}} \Big|_e^1 = 2 - 0 = 2$$

$$\int_0^1 \frac{1}{x}\, dx = \lim_{\epsilon\to 0^+} \int_{a+e}^1 \frac{1}{x}\, dx = \lim_{\epsilon\to 0^+} \text{lu } x \Big|_e^1 = \infty \text{ (undefined)}$$

These two basic definitions apply also to variations such as

$\int_{-\infty}^0 f(x)\, dx$; $\int_{-\infty}^{+\infty} f(x)\, dx$ for the first kind (1) and

$\int_a^b f(x)dx$ where f is discontinuous at b or at some intermediate point, for the second kind (2).

In summary, we present the following.

DEFINITION. If the function f becomes infinite at several points of the interval, then we can choose convenient points and integrate the function by integrating its subdivisions and then adding (see fig. A). The total improper integral converge if and only if it converges over each smaller interval. Its value will equal the sum of the values of the individual integrals.

If, for example, we want to integrate the function in the the figure as $\int_{-\infty}^{+\infty} f(x)dx$ we select points c, d, e and proceed to evaluate

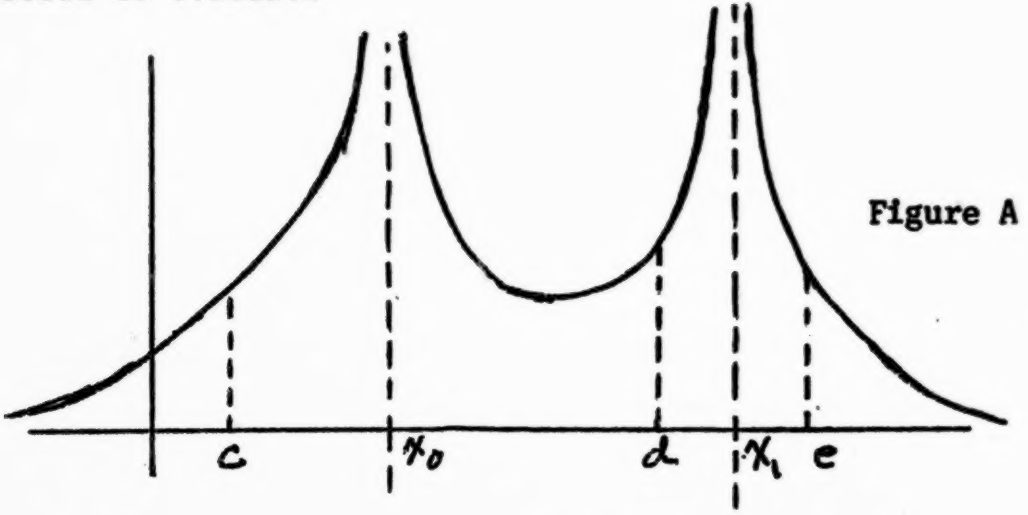


Figure A

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a \to -\infty} \int_a^c f(x)dx + \lim_{\epsilon_1 \to 0^+} \int_c^{x_0 - \epsilon_1} f(x)dx + \lim_{\epsilon_2 \to 0^+} \int_{x_0 + \epsilon_2}^{d} f(x)dx + \lim_{\epsilon_3 \to 0^+} \int_d^{x_1 - \epsilon_3} f(x)dx + \lim_{\epsilon + 0^+} \int_{x_1 + \epsilon_4}^{l} f(x)dx + \lim_{b \to \infty} \int_e^b f(x)dx$$

provided all the above limits exist.

After these basic definitions and examples, a natural extension would be to give some tests of convergence or divergence such as the comparison test, quotient test, etc. depending on the degree of proficiency the instructor desires his students to achieve. One need not go into detailed proofs of these tests as is done in advanced calculus. For example, to test the convergence of

$$\int_1^{\infty} \frac{\sin x}{x^{3/2}} dx$$

we employ the fact that $\int_a^{\infty} x^{-p} dx$ converges for $p > 1$ and write

$$\left| \frac{\sin x}{x^{3/2}} \right| \leq \frac{1}{x^{3/2}} \quad \text{where } p = 3/2. \text{ Therefore}$$

$\int_1^{\infty} \frac{\sin x}{x^{3/2}} dx$ converges.

One improper integral which may be mentioned, if time permits, is the gamma function. It should be pointed out that the gamma function, denoted by $\Gamma(x)$ and defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad x > 0$$

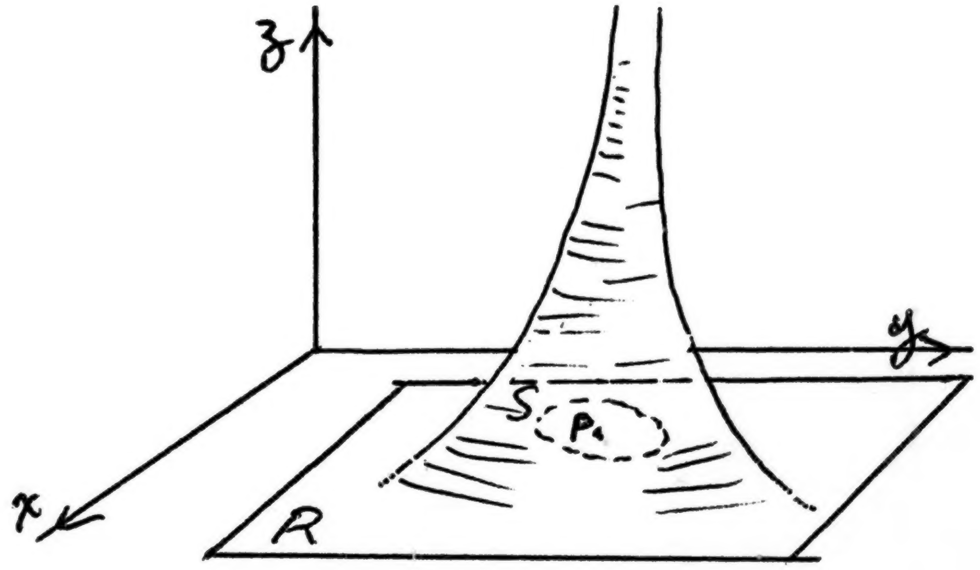
should be treated not only as a mathematically interesting function, but one that proves to be extremely useful in practice. One mathematical, interesting point is the fact that $\Gamma(x+1) = x\,\Gamma(x)$ which can be proven easily by integrating $\Gamma(x)$ by parts and using L'Hospital's Rule. Also, by using the recursion formula and mathematical induction students can derive the interesting fact that $\Gamma(n + 1) = n!$

3. MULTIPLE IMPROPER INTEGRALS

This topic is easily extended to functions of two independent variables, but somewhat intricate for more than two since we are not able to demonstrate by means of geometric models. Integrating a function as in the figure on the right poses a similar problem as in two dimensions. However, a few adjustments in notation need to be made. We now consider the function over the rectangular region of the xy-plane R and let a sequence of circular regions "close down" on the point where the function becomes discontinuous. If the limit exists we define the integral as

$$\lim_{S \to 0} \iint_{R \cap S'} f(x, y)\, dA_{xy}.$$

where S is the sequence $\{S_n\}$ of circles with decreasing diameters.



For example, to evaluate the integral

$$\iint_R \frac{1}{\sqrt{x^2 + y^2}^{\,p}} \, dA_{xy} \quad p > 0 \quad \text{where R is the region}$$

$x^2 + y^2 < 1$, we notice that the function becomes infinite at the origin. We choose a sequence of circles closing down on the origin by choosing the radius of each $1/n \leq r \leq 1$ where $r = \sqrt{x^2 + y^2}$. We change to polar coordinates,

$$\iint_{S_m} \frac{1}{r^p} \, dA = \int_0^{2\pi} \int_{\frac{1}{n}}^{1} r^{-p} r \, dr \, d\theta = \left\{ \frac{2\pi}{2-p} \left(1 - \frac{1}{n^{2-p}}\right) \right.$$

$$\left. \text{if } p < 2 \right\} \cup \left\{ 2\pi \ln n \text{ if } p = 2 \right\} \cup \left\{ \frac{2\pi}{p-2} (n^{p-2} - 1) \text{ if } p > 2 \right\}$$

Then, by a previous theorem of convergence it can be shown that the integral converges for $p < 2$ but not for $p > 2$.

B. INFINITE SEQUENCES

Sequences can be introduced as functions whose domain is the set of all positive integers and range in the set of real numbers. Sequences can be finite or infinite. Examples of sequences are:

$f(n) = a_n = 2, 7, 12, 17, \ldots, 5n-3, \ldots$ where $n = 1, 2, 3, 4, 5, \ldots$

$g(n) = b_n = 1/2, 1/3, 1/4, 1/5, 1/6, \ldots, 1/n, \ldots$ $n = 2, 3, 4, 5, \ldots$

Sequences are said to be convergent or divergent accordingly as $u_n \to L$ as $n \to \infty$ or n does not approach any particular real number respectively, More specifically, if for any positive number e we can find a positive number N, dependent on e, such that $|u_n - L| < e$ for all $n > N$, then we say that the sequence $\{u_n\}$ converges and we write $\lim_{n \to \infty} u_n = L$

At this stage, the following theorems should be given attention. We list them here without proof: If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, then

(a) $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = A + B$

(b) $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n = A + B$

(c) $\lim_{n \to \infty} (a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n = A \cdot B$

(d) $\lim_{n \to \infty} a_n/b_n = \lim_{n \to \infty} a_n / \lim_{n \to \infty} b_n = A/B$ if $B \neq 0$

(e) $\lim_{n \to \infty} a_n^p \quad = \quad (\lim_{n \to \infty} a_n)^p = A^p$

(f) $\lim_{n \to \infty} p^{a_n} \quad = \quad p^{\lim_{n \to \infty} a_n} = p^A$

C. SERIES: FINITE OR INFINITE

1. Definition

There is a lack of agreement as to the definition of a series. Some authers (see [5], [23], [33], [34]) define a series as an indicated sum. In this case the interest is in the form $a_1 + a_2 + a_3 + \ldots + a_n$ for the finite series and for the infinite series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$. There is a rather impressive group of authors (see [4], [9]) that define the sequence of partial sums as the series. In this case, the interest is centered on the result of the summation.

When there is danger of confusion, it would be advisable to use S_n for the calculated sum of the first n terms and $\sum_{k=1}^{n} a_k$ for the series (form). A major problem concerns the use of $\sum_{k=1}^{\infty} a_k$ and the attempt to assign a value to S_∞. Here we need to define the sum of an infinite sequence, which is the problem that presented itself earlier with the second definition of a series. The desired definition may be patterned after that found in [19], [34], [35], and [37], i.e., $S_n = \sum_{k=1}^{n} a_k$,

$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$. It should be pointed out

that $S_\infty = \lim_{n \to \infty} \{S_n\}$ and therefore the evaluation of S_∞ becomes a problem in finding the limit of an infinite sequence.

At this stage, properties of the summation notation should be reviewed.

A few examples of finite series should reveal to the student that the sum of a finite series of finite terms is finite. It should be noted that infinite decimals are infinite series in disguise. The student may recall, after prodding, the converging geometric series ($|r| < 1$). This may be used to find the common fraction form for a repeating decimal.

Some numerical approximation by Taylor, and/or Maclaurin (Sterling) series, with the remainder term, should be covered, as well as power series. The theorems should be carefully stated and rigorously proved (see [2],[4],[12]).

2. Convergence Tests

A convenient listing of convergence tests may be found in [33].

Some rather interesting series can be created by the instructor based upon those found in [37]. Other interesting series may be found in 13 .

The nature of the tests used will depend upon the text, instructor, and class. Some attention should be given to empirical methods, and perhaps "educated guesses," based upon observations from graphical representations.

The use of the integral test will give the student a chance to use the Integral Tables. The instructor will be able to assist the student in polishing his skill in using these tables.

D. L'HOPITAL'S RULE

This useful rule for finding limits first appeared in print as part of the first text on differential calculus. Published in 1696 by the Marquis de l'Hopital, based upon notes of his teacher Johann Bernoulli, a friend and supporter of G. W. Leibnitz. The student should be aware of the proof of this rule for this cases $\frac{0}{0}$ and $\frac{\infty}{\infty}$. See [12]. The proof of other forms will depend upon the time that is available. If the choice is one of proof or practice, the rule should be presented without proof and the time spent in practice in the correct application of the various forms of the rule.

Every effort must be made to impress upon the student the necessity of the expression being in the required indeterminate form. Example: $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan x}$.
The application of the rule here will result in the original expression. The student must learn to simplify the ratio first.

Examples of the type:

$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ and $\lim_{x \to \infty} \frac{2x^2 + 5x + 6}{x^2 + 2}$ where the evaluation is easier without the rule should encourage the student to use the quickest way and not to use a tractor to pull a "kiddie car." Examples such as $\lim_{x \to 0+} x \ln x$

illustrate the need for simplification for each application.

L'Hopital's rule may also be applied to expressions of the type: $(0 - 0)$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0.

As an example of 1^∞ type consider $\lim_{x \to 0} (\cos x)^{\frac{1}{x}}$, $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$, $\lim_{x \to 0} (x^2 + 5x + 1)^{\frac{2}{5x}}$.

Some interesting examples of additional cases where a guess does not pay off are; for the form 0^0: $\lim_{x \to 0} x^x$; and for ∞^0: $\lim_{x \to \infty} (x)^{\frac{1}{x}}$.

Some additional frustrating examples such as $\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x}$ and $\lim_{x \to \infty} \frac{4^x}{x^4}$ will challenge the student if the textbook has provided only routine examples in the exercise.

II. VECTOR-VALUED FUNCTIONS ON E_1

It will be assumed that the students have covered the material on vectors included in the outline for the first year of calculus. The linear algebra course is not necessary, but give the student a decided advantage in this course.

Must of the following material was covered more intuitively in Chapter XI. Therefore, it is not repeated here. However, the level of rigor should be raised during this treatment, and all of the geometric aspects should be reviewed.

A. Limits and continuity

This material, covered intuitively in the first year, should now be covered by the δ-ϵ approach, as done in [8]. Also, [7], and [12] have good discussions in pre-press editions. Neighborhoods should be defined and discussed thoroughly. Prove, among other theorems, that the composition of continuous functions is continuous.

B. Derivatives of vectors

A δ-ϵ treatment should be covered here, showing the similarity of these to ordinary derivatives. Then, the applications of these derivatives to differential geometry should be discussed, with the Frenet-Serret formulas included, as in [12]. This includes the tangent, normal, and binormal vectors, curvature, and the osculating plane.

III. FUNCTIONS ON A VECTOR SPACE

(This may include functions on a vector space into a E_1 or into a vector space, depending on the approach the particular text uses. Some would refer to them simply as multivariable functions.)

As an introduction to this section, a discussion of surfaces should be included, as in [12].

A. Limits and continuity.

The material in IA. should be extended her, but using the metric topology with its ϵ-neighborhoods, as in [4] [7], [8], and [12]. [7] contains perhaps the most rigorous coverage.

B. Directional derivative and gradient.

Since the student is familiar with partial differentiation, from the previous calculus course, directional derivatives may be motivated by considering what a partial derivative is, i.e., a derivative in the direction of a coordinate axis. Then, the directional derivative in the direction of V, if the limits exist, may be defined as either

$$\lim_{h \to 0} \frac{f(x_0 + \lambda h, y_0 + \mu h) - f(x_0, y_0)}{h}, \text{ where } \sqrt{\lambda^2 + \mu^2} = 1,$$

or $$f^1(\vec{v}_0, \vec{v}) = \lim_{h \to 0} \frac{f(\vec{v}_0 + h\vec{v}) - f(\vec{v}_1)}{h}, \text{ where } \vec{v} = (\lambda, \mu),$$

$|\vec{v}| = 1$, $\vec{v}_0 = (x_0, y_0)$, and h approaches 0 from both sides. This last should be carefully noted. Prove the linear properties of this derivative.

A scalar function $f(\vec{x})$ is said to be continuously differentiable at $\vec{v}_0$ if, for every $e > 0$ and for every $\vec{v}$ such that $|\vec{v}| = 1$, there is a $\delta > 0$ such that $|f^1(\vec{x};\vec{v}) - f^1(v_0, \vec{v})| < e$ if $|\vec{x} - \vec{v}_0| < \delta$.

With this definition, it is possible to prove, as in [4], that if $f(\vec{v}) = f(x,y,z)$, is continuously differentiable in some neighborhood of $\vec{v}_0$, then f is continuous at $\vec{v}_0$.

The gradient, $\nabla f = \frac{2f}{2x}\vec{i} + \frac{2f}{2y}\vec{j} + \frac{2f}{2z}\vec{k}$, should be introduced, and the relationship between it and the directional derivative explained. It should be noted that it is a vector, and is independent of the coordinatization of space that is chosen. See [18]

Also, the fact that the gradient of a differentiable function of R^3 to R or R^2 to R at a point P_o is normal to the level surface or curve, respectively, of f at P_0, should be discussed.

[12] has a good treatment of this material, although in a slightly different order. [20] has an unusual approach that is also worthwhile.

C. Chain Rule

This should be proved by means of material that has been carefully developed already. It says in vector form that if $f(\vec{v})$ is a scalar field which is continuously differentiable in an open set of vectors, S and $\vec{U}(\vec{x})$ is a vector field differentiable on an open set of vectors T, such that, if $\vec{v} \in T$, $\vec{U}(\vec{v}) \in S$, and if $\phi(\vec{x}) = f(\vec{U}(\vec{v}))$, then $\phi'(\vec{V};\vec{y}) = \nabla f(\vec{U}(\vec{v})) \cdot \vec{U}'(\vec{v};\vec{y})$.

Such proofs, for various forms of the chain rule, are given in [11], [1], [20], and [29]. Then is should be applied to implicit differentiation as in [4] or [8]

In particular, if two surfaces given by $F(x,y,z) = 0 = G(x,y,z)$, intersect to form a curve, and their equations are solved simultaneously to give x and y as functions of z, say $x = X(z)$, $y = Y(z)$, then

$$\frac{dx}{dz} = \frac{\begin{matrix} \frac{\partial F}{z} & \frac{\partial F}{y} \\ \frac{\partial G}{z} & \frac{\partial G}{y} \end{matrix}}{\begin{matrix} \frac{\partial F}{x} & \frac{\partial F}{y} \\ \frac{\partial G}{x} & \frac{\partial G}{y} \end{matrix}}, \text{ and similarly for } \frac{dy}{dz}.$$

These are determined implicitly. This is covered in [4] and [8].

D. Exact differential.

[11] has a good discussion of the exact differential at a point as a linear transformation, with matrices, which leads directly into the Jacobian. [9] also uses the same approach. For students who have had linear algebra, this should be a good approach. For others, a more traditional approach may be better, without matrices, but still regarding the differential as a linear transformation

E. Divergence and curl of vector valued functions.

1. Definitions

The definitions are as follows: $\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ and

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial y}\right)k,$$

for $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, where P, Q, and R are functions of x, y, and z. These may be written respectively as ∇F and $\nabla \times F$.

2. Applications

In fluid flow, if $\vec{F}(x,y,z)$ is the scalar product of density$\delta(x,y,z)$ and velocity of fluid flow $\vec{v}(x,y,z)$ for steady state conditions, then F represents flux density of the flow. Then div F is the time rate of change of mass per unit volume at a point (x,y,z). Also, curl $\vec{v}$ measures the local rotational effect of the fluid flow. See [4].

F. Line integrals

In the first year course integration was discussed, in an intuitive manner, for particularly nice functions, and the line integral was briefly discussed. At this stage, the student should be familiar enough with the definitions of limit and continuity that a review of the pertinent material from the first year can be made, filling in the details for continuous functions, as in [23], including the Appendix. At this stage, this material might even be assigned as outside reading. Now, the line integral may be redefined, perhaps motivating it by means of work as in [4] or potential functions, as in [20]. The basic properties should be covered, including the invariance under change of parameter. Then, the independence of the line integral with respect to path, under appropriate conditions, should be proved. (This amountsto the generalization of $\int_a^b f(x) = F(b) - F(a)$, for $\frac{dF(x)}{dx} = f(x)$. Applications to both potential functions and work may be found in [20], [4], and [29].

At this point, Green's theorem for "nice" plane regions may be proved, after a short review of double integrals. Such a proof is in [4]. Then, the theorem should be extended to multiply connected regions, after which it is easy to prove that, under the proper conditions, line integrals are independent of deformation of Jordan curves. Vector forms of Green's theorem should be mentioned, if not used throughout.

G. Transformation of variables and multiple integrals.

[35] has a good development of this topic for double integrals using only material discussed so far, if it is assumed that the mapping of the uv-plane, given by $x = f(u, v)$, $y = g(u, v)$, has an inverse mapping which may be determined explicitly as $u = h(x, y)$, $v = k(x, y)$. For the problems considered here, this is not an undue restriction.

This use of the Jacobian may be extended intuitively to to transformations of integrals of order n. See [10] for proof that is valid for this situation. A short discussion of the application to n-dimensional solids or surfaces could be included, in conjunction with the next section. See [10].

H. Surfaces and solids

The definition for the area of a surface and its motivation should be discussed first, as in [12]. Then, the general surface integral may be defined in a manner analogous to that used for the line integral above. If $S = r(u, v)$ is a vector which describes a surface S, for a region T in the uv-plane, then area of $S = \iint_T \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| dudv$. See [14].

Surface integrals may be applied to surfaces to calculate area, moments, fluid flow, etc., in much the same way as done for ordinary double integrals over regions in the plane. See [4].

Now Stoke's theorem may be proved, by transforming the surface into the plane, in order to use Green's theorem, as is done in [4].

Gauss' Theorem can now be considered, through an application of the fact that, in this case, the line integral is independent of its path, as in [4].

In may be noted that Green's, Stokes, and Gauss' theorems are all generalizations of the fundamental theorem of integration, in that a multiple integral over a region may be expressed as an integral of one lower order, around the "edge" of the region.

IV. DIFFERENTIAL EQUATIONS

A. First Order Equations

A differential equation is an equation which contains derivatives or differentials. An equation with derivatives can be changed into one which contains differentials, and hence its name. An ordinary differential equation is an equation which involves one unknown function, say y, and one or more derivatives of y taken with respect to an independent variable, say x. A partial differential equation is one which contains partial derivatives. A differential equation containing x, y, and derivatives of y with respect to x, is said to be solved or integrated when a relation between x and y, but not containing the derivatives, has been found which, if substituted in the differential equation, reduces it to an identity.

The order of a differential equation is the order of the derivative of highest order appearing in the equation. Order is not to be confused with degree which is the power of the highest order derivative appearing. a differential equation in the form

$$Co(x)\frac{d^ny}{dx^n} + c_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + c_{n-1}(x)\frac{dy}{dx} + c_n(x)y = f(x) \qquad (1)$$

is said to be a linear differential equation. The functions may be arbitrary functions of x and their character does not affect the linearity of the equation. Equation (1) is of the n^{th} order. Examples of linear differential equations are:

$$x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - x^3y = \tan x, \quad \frac{d^3y}{dx^3} + 2x^4\frac{dy}{dx} + 8y = 6x$$

Non-linear differential equations are:

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 1 \text{ and } \left(\frac{dy}{dx}\right)^2 + 2y = 8x.$$

Note that the degree of the latter equation is two, but the order is one.

1. Variables Separable

The equation $Mdx + Ndy = 0$, where $M = G(x, y)$ and $N = F(x, y)$, is said to have its variables separated when it is expressed as $\theta(y)dy + \phi(x)dx = 0$

The solution is then $\int \theta(y)dy + \int \phi(x)dx = c$, where c is an arbitrary constant. If a first-order differential equation is or the form $\frac{dy}{dx} = f(x) \cdot g(y)$, $g(y) \neq 0$, we may write $\frac{dx}{g(y)} = f(x)dx$, and the solution is $\int \frac{dy}{g(y)} = \int f(x)dx + c$.

Example: $\sqrt{1 - y^2}\ dx + \sqrt{1 - x^2}\ dy = 0$, $\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} = 0$,

from which $\sin^{-1}x + \sin^{-1}y = c$.

2. Homogeneous

A polynomial in x and y is said to be homogeneous when the sum of the exponents of those letters in each term is the same. Thus, $ax^2 + bxy + cy^2$ is homogeneous of the second degree, and $ax^3 + bx^2y + cxy^2 + ey^3$ is homogeneous of the third degree. If, in such a polynomial, we place $y = vx$, it becomes $x^nf(v)$, where n is the degree of the polynomial. Thus $ax^2 + bxy + cy^2 + x^2(a + bv + cv^2) = x^m f(v)$.

This property enables us to extend the idea of homogeneity to functions which are not polynomials. Representing a function of x and y by $f(x, y)$, then $f(x, y)$ is a homogeneous function of x and y of the nth degree, if when we place $y = vx$, $f(x, y) = x^nF(v)$. When M and N are homogeneous functions of the same degree the equation $Mdx + Ndy = 0$ is said to be homogeneous and can be solved as follows: Place $y = vx$, then $dy = vdx + xdv$ and the differential equation becomes

$$x^n\,\theta(v)dx + x^n\,\phi(v)(vdx + xdv) = 0 \quad \text{or}$$

$$\left[\theta(v) + v\right]\phi(v)\quad dx + x\,\phi\,(v)dv = 0 \quad \text{If}$$

$$\theta(v) + v\,\phi(v) \neq 0, \text{ then } \frac{dx}{x} + \frac{\phi(v)dv}{\theta(v) + v\,\phi(v)} = 0$$

where the variables are now separated and the equation may now be solved.

3. Exact

If the left member of the equation Mdx + Ndy = 0 is an exact differential, df(x, y), that is, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then Mdx + Ndy = 0 may be written df(x,y) = 0, having f(x, y) = c as its solution.

4. Linear; integrating factor

The equation $\frac{dy}{dx} + \theta(x)y = \phi(x)(1)$, where $\theta(x)$ and and $\phi(x)$ may reduce to constants but cannot contain y is called a linear equation of the first order. An equation of the form Mdx + Ndy = 0 may be put in form (1) if, after transforming it to $\frac{dy}{dx} + \frac{M}{N} = 0$, $\frac{M}{N}$ can be expressed as $\theta(x)y - \phi(x)$, that is, as the difference of two terms one of which is y multiplied by a function of x and the other of which is a function of x only.

To solve (1) let y = uv, where u and v are unknown functions of x to be determined later in any way which may be advantageous. Then (1) becomes $u\frac{dv}{dx} + v\frac{du}{dx} + \theta(x)uv = \phi(x)$ or $v\left[\frac{du}{dx} + \theta(x)u\right] + u\frac{dv}{dv} = \phi(x)(z)$.

Let us now determine u so that the coefficient of v shall be zero. $\frac{du}{dx} + \theta(x)u = \phi$ or $\frac{dw}{u} + \theta(x)dx = 0$.
The general solution is: $\log u + \int \theta(x)dx = c$.
Since, all we need is a particular function which will make the coefficient of v equal to zero, we take c = 0.
Then, $\log u = -\int \theta(x)dx$ or $u = e^{-\int \theta(x)dx}$.

With this value of u, (2) becomes $e^{-\int \theta(x)dx}\frac{dv}{dx} = \phi(x)$ or $\frac{dv}{dx} = e^{\int \theta(x)dx}\phi(x)$, where v may be found by integration.
Substituting the values of u and v in (2), we have solution (1), where y = uv.

If the equation Mdx + Ndy = 0 is not exact, i.e., if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then there exists an infinite number of functions of x and y such that if Mdx ≠ Ndy = 0 is multiplied by any one of them, it is made an exact equation. Such a function is called an integrating factor. Since no general method is known for finding integrating factors, (lists can be found in treatises on differential equations) sometimes an integrating factor can be found by inspection. Common differentials frequently encountered in practice are:

$d(uv) = vdu + udv$, $d(\frac{u}{v}) = \frac{vdu - udv}{v^2}$,

$d \tan^{-1}(\frac{u}{v}) = \frac{vdu - udv}{u^2 + v^2}$, $d \log \frac{u}{v} = \frac{vdu - udv}{uv}$ and

$d(u^2 + v^2) = 2(udu + vdv)$

An inspection of the differention problems found earlier in the course may prove quite useful.

5. Power Series

The use of power series in solving ordinary differential equations should be noted. This is, in fact, an important and powerful method of investigating the function defined by the equation. The method consists in assuming a series of the form $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots)$ where m and the coefficients $a_0, a_1, a_2, \ldots$ are undetermined. This series is then substituted in the differential equation, and m and coefficients are so determined that the equation is identically satisfied.

B. Second Order Equations

The general second-order differential equation has the form $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$. No attempt will be made to develop the theory for such general differential equations. If we assume that the Implicit Function Theorem holds with respect to the last argument of the function F above, then $\frac{d^2y}{dx^2} = 0$.

1. Special Forms

Certain equations of the second order, occuring frequently in practice, which are readily integrated include:

a. $\frac{dy^2}{dx^2} = f(x)$, which by direct integration gives: $\frac{dy}{dx} = F(x) + C$, and $y = G(x) + c_1x + c_2$ where F is a particular antiderivative of f and G is one of F. This method is equally applicable to the equation, $\frac{d^ny}{dx^n} = f(x)$. Some discussion should relate to the number of constants of integration which will appear.

b. $\frac{d^2y}{dx^2} = f(x, \frac{dy}{dx})$, substituting $\frac{dy}{dx} = p$, then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ and the equation becomes $\frac{dp}{dx} = f(x, p)$ which is a differential equation of the first order in which p and x are the variables. If they can be solved so that $p = G(x, c_1)$, than a second integration yields $y = F(x, c_1) + c_2$ where F is an antiderivative of G.

c. $\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx})$, again, place $\frac{dy}{dx} = p$, but now write $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} p$ so that the equation becomes $p \frac{dp}{dy} = f(y, p)$ which is a differential equation of the first order in which p and y are the variables. If solvable, $p = F(y, c_1)$ from which $x = \frac{dy}{F(y, c_1)} + c_2$.

d. $\frac{d^2y}{dx^2} = f(y)$. By multiplying both sides of this equation by $2\frac{dy}{dx}$ we have $2 \frac{d^2y}{dx^2} \frac{dy}{dx} dx = 2f(y) \frac{dy}{dx} dx$ or $d \left[(\frac{dy}{dx})^2 \right] = 2f(y)dy$. Integrating $(\frac{dy}{dx})^2 = 2 f(y)dy + c_1$; whence by solving for $\frac{dy}{dx}$ and then separating we have

$$\int \pm \frac{dy}{2\sqrt{f(y)dy + c}} = x + c_2.$$

Another approach to the solution of this form again involves the substitution $p = \frac{dy}{dx}$ so that the equation becomes $\frac{dp}{dx} = f(y)$. But $\frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} p$. Therefore, $p\frac{dp}{dy} = f(y)$ which is an equation with variables separable. Integrating once, we obtain $\frac{p^2}{2} = F(y) + c_1$, where F in an antiderivative of f. Then we have $p = \frac{dy}{dx} \pm \sqrt{2F(y) + 2c_1}$, which again is a variables separable equation. Therefore, $x \pm \int \frac{dy}{\sqrt{2c_1 + 2F(y)}} + c_2$.

Examples of applications using these forms should be presented here. (Equation of curve formed by uniform flexible cable suspended from its ends; equation of motion for free falling body and missile problems; bending of beams; equation of motion of simple pendulum; etc.) See [3], [4], [30], [36], [8]

2. Operators

If this third semester of calculus comes after the linear algebra, a good discussion on linear operators and linear transformations may be found in [27]. See also [3], [30], and [36]. If no linear algebra is in the students' background, a straight-forward approach through definitions can still enable him to use operators, since the resulting answers can be easily verified. See [29] and [21] for this type of development. Since many books do not discuss Operators, we include the latter approach. [21]

Let F be the set of all functions with one argument; then any mapping of F into F is said to be an operator. Essentially, an operator is a rule or relation which associates a function with a given function. For example, we can associate f^2 with f, $\int_0^x f dx$ with f, or F with f.

Since we want to solve differential equations, let us concentrate on the operator which associates f' with f. It is helpful to give this operator a name, say "D." Thus Df = f' whenever f is a function.

Definitions: (1) $D^o f = f$ whenever f is a function

(2) $D^{k+1} = D(D^k f)$ whenever f is a function

(3) $D_1 + D_2 = \{(f, g) \mid f \in F \text{ and } g = D_1 f + D_2 f\}$.

(4) $D_1 \cdot D_2 = \{(F, g) \mid f \in F \text{ and } g = D_1(D_2 f)\}$.

Theorems: (1) $D^n f = f^{(n)}$ whenever n is a non-negative integer.

(2) Let $a \in R$ and let n be any non-negative integer; then $aD^n = \{(f, g) \mid f \in F$ and $g = af^{(n)}\}$.

(3) $D_1 + D_2 = D_2 + D_1$ (Commutative Law)

(4) $D_1 + (D_2 + D_3) = (D_1 + D_2) + D_3$ (Associative Law)

(5) $D_1 \cdot (D_2 \cdot D_3) = (D_1 \cdot D_2) \cdot D_3$ (Associative Law)

(6) $(D_1 + D_2) \cdot D_3 = D_1 \cdot D_3 + D_2 \cdot D_3$ (Distributive Law)

We point out that multiplication of operators is not commutative. For example, $x^2 \cdot D \neq D \cdot x^2$ since the operator $x^2 \cdot D$ associates $x^2 f'$ with f whenever $f \in F$ whereas the operator $D \cdot x^2$ associates $2xf + x^2 f'$ with f whenever $f \in F$.

Next, let $a_o, a_1, a_2, \ldots, a_n$ be real numbers where $a_n \neq 0$, and consider the operator $a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$. We need a name for this operator. Let "P" denote the corresponding polynomial function $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$; then we shall take "P(D)" to be a name of our operator. Thus $P(D) = \sum_{i=0}^{n} a_i D^i$ whenever $P = \sum_{i=0}^{n} a_i x^i$. We shall call P(D) a polynomial operator. Let us establish the basic properties of polynomial operators. Notice that in the following theorems we use "P" as a place-holder, or variable, for polynomial operators.

Theorems:

(1) Let P be any polynomial operator and let $k \in R$; then $P \cdot k = k \cdot P$.

(2) Let P be any polynomial operator, let $f \in F$ let $g \in F$, then $P(f + g) = Pf + Pg$.

(3) Let P be any polynomial operator, let $f \in F$, $f_2 \in F$, $k_1 \in R$, and $k_2 \in R$; then $P(k_1 f_1 + k_2 f_2) = [k_1 P] f_1 + [k_2 P] f_2$.

(4) Let P be any polynomial operator, let $f_i \in F$, and let $k_i \in R$ whenever $1 \leq i \leq m$; then
$$P\left(\sum_{i=i}^{m} k_i f_i\right) = \sum_{i=i}^{m} [k_i P] f_i.$$

(5) Let P be any polynomial operator; then $DP = PD$.

(6) Let P be any polynomial operator and let $n \in N$; then $D^n P + PD^n$.

(7) $P_1 P_2 = P_2 P_1$ whenever P_1 and P_2 are polynomial operators.

Notice that any linear differential equation with constant coefficients can be represented by means of a polynomial operator.

3. Linear with constant coefficients.

Equations of the form $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$ may be solved by operators.

The general solution may also be introduced by simply substituting $y = e^{rx}$ to get an auxiliary equation $r^2 + a_1 r + a_2 = 0$ which may be solved for r_1 and r_2. The general solution resulting, $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ can easily be checked by the student if r_1 and r_2 are real. The cases when $r_1 = r_2$, and when the roots are complex conjugates should be covered. Proofs should be given wherever possible. See [14] and [29] for good developments.

4. Linear nonhomogeneous

   For lengthy discussion of various cases, see [28]. Also see [3], [29], [30], and [36]. The solution of nonhomogeneous linear equations may be divided into two major attacks.

   a. If the given equation has constant coefficients, the solution may be shown to be the sum of the general solution of the corresponding homogeneous equation (complementary function) and a particular solution of the nonhomogeneous equation. This particular solution may be found by the method of undetermined coefficients--"guessing" at the form of the answer. For functions such as ae^{bx}, $a \sin x + b \cos x$, and any polynomial in x, it may be of value to develop the general formula for the constants in that particular solution. Another approach is to use the substitution $y = v(x)w(x)$, where v is a solution of the corresponding homogeneous equation.

   b. If the given equation has variable coefficients, use of one of the solutions of the corresponding homogeneous equation as above, substituting $y = v(x)w(x)$ may be of value.

      If two solutions of the corresponding homogeneous equation are known, the method of variation of parameters may be employed.

5. Equations with variable coefficients.

   Inclusion of equations with variable coefficients will depend on the time available and the text being used. For a complete discussion, suitable for this level, see [3], [28], [30] and [36]